SOME STABILITY RESULTS IN BRUNN-MINKOWSKI THEORY

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Declaration

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Abstract

This thesis deals with some stability results in Brunn-Minskowski theory, a rich and active field within Convex Geometry and Geometric Functional Analysis. The classical Brunn-Minkowski inequality, the cornerstone of Brunn-Minkowski theory is intimately related to the Minkowski problem and form a core theme of various areas of convex geometry, partial differential equations, probability, additive combinatorics, calculus of variations and others. Firey's and subsequently Lutwak's extension to L_p Minkowski theory naturally gave rise to L_p analogues of the problems, and the geometric and functional inequalities found in the classical theory.

Here we focus on obtaining stability results in certain cases for the Prékopa-Leindler inequality (a generalization and functional form of the Brunn-Minkowski inequality), the log-Brunn-Minkowski and log-Minkowski inequalities (which correspond to the L_0 versions of the corresponding classical inequalities), and finally, stability of solution of the logarithmic (L_0) Minkowski problem. Particularly, we establish:

(i) a stability version of the Prekopa-Leindler inequality at least for log-concave functions in $\mathbb{R}^n$ in Chapter 3,

(ii) stability versions of the logarithmic Brunn-Minkowski inequality and the Logarithmic Minkowski inequality for convex bodies in $\mathbb{R}^n$ which are symmetric with respect to linear reflections through n independent hyperplanes in Chapter 4, and

(iii) stability of solution of the Logarithmic Minkowski Problem on S^{n-1} in the case of symmetries with respect to a Coxeter group $G \subset O(n)$ acting without non-zero fixed points on $\mathbb{R}^n$ in Chapter 5.

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Chapter 1

Notations, definitions and basics

We will be working in the space $\mathbb{R}^n$ equipped with the usual Euclidean structure, that is, the usual inner-product denoted $\langle \cdot, \cdot \rangle$, and the Euclidean norm, $\|x\|_2 = \sqrt{\langle x, x \rangle}$. We'll use $\|x\|$ without the subscript to denote the Euclidean norm unless otherwise specified. $\|x\|_p$ would denote the L_p norm.

We interchangeably use $V(X)$ or $|X|$ to denote Lebesgue measure of a measurable subset $X \subset \mathbb{R}^n$ (with $V(\emptyset) = |\emptyset| = 0$), and $\mathcal{H}$ to denote the $(n-1)$-dimensional Hausdorff measure normalized in a way such that it coincides with the $(n-1)$-dimensional Lebesgue measure on $n-1$-dimensional affine subspaces. The k-dimensional Hausdorff measure will be denoted $\mathcal{H}^k$, wherever we might need to specify the dimension.

We denote by o the origin in $\mathbb{R}^n$. We write B^n and S^{n-1} for the Euclidean ball and sphere centered at the origin respectively.

$C(S^{n-1})$ denotes the set of continuous functions on the sphere S^{n-1} and equip it with the max-norm metric, that is, for $f, g \in C(S^{n-1})$, we write

$$\|f - g\|_\infty = \max_{u \in S^{n-1}} |f(u) - g(u)|.$$

A measure μ on S^{n-1} is even if $\mu(-\omega) = \mu(\omega)$ for any Borel set $\omega \subset S^{n-1}$

$\mathrm{GL}(n, \mathbb{R})$ denotes the group of non-singular linear transformations on R^n, while $\mathrm{SL}(n, \mathbb{R})$ denotes those transformations with determinant 1, and $O(n)$ denotes the group of orthogonal transformations.

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conv(X) denotes the convex hull of $X \subset \mathbb{R}^n$.

For $X, Y \subset \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, the Minkowski linear combination is given by $\alpha X + \beta Y = \{\alpha x + \beta y : x \in X,\ y \in Y\}$. If X and Y are convex compact, then so is $\alpha X + \beta Y$.

A compact convex set in $\mathbb{R}^n$ with non-empty interior is called a convex body. We denote by $\mathcal{K}^n$ the family of convex bodies in $\mathbb{R}^n$. Further, we denote by $\mathcal{K}^n_o$ the family of convex bodies in $\mathbb{R}^n$ containing the origin o, and by $\mathcal{K}^n_{(o)}$ those convex bodies that contain the origin in their interior ($o \in \mathrm{int} K$). The family of origin symmetric convex bodies ($K = -K$) is denoted by $\mathcal{K}^n_e$.

Convex bodies K and C are said to be homothetic if if $K = \gamma C + z$ for some $\gamma > 0$ and $z \in \mathbb{R}^n$, translates of each other if $K = C + z$ for some $z \in \mathbb{R}^n$, and dilates of each other if $K = \gamma C$ for some $\gamma > 0$.

For a linear subspace $L \subset \mathbb{R}^n$, $K|L$ or $P_L K$ denotes the orthogonal projection of K onto L.

The support function of a compact convex set K is given by $h_K(u) = \max_{x \in K} \langle u, x \rangle$ for $u \in \mathbb{R}^n$. h_K is subadditive and positively homogeneous of degree one ($h_K(\lambda u) = \lambda h_K(u)$ for $\lambda \geq 0$)). $h_K \leq h_C$ if and only if $K \subset C$. Given any subadditive and positively homogeneous function h on $\mathbb{R}^n$, there exists a unique compact convex set K such that $h = h_K$. For $\Phi \in \mathrm{GL}(n, \mathbb{R})$, the support function of ΦK satisfies

$$h_{\Phi K}(u) = h_K(\Phi^t u).$$

The Haussdorff distance between compact convex sets K and C is denoted $d_H(K, C)$ or $d_\infty(K, C)$ and is given by

$$d_H(K, C) = \min\{r \geq 0 :\ K \subset C + r\, B^n \text{ and } C \subset K + r\, B^n\}.$$

Equivalently, we write $d_H(K, C) = \|h_K - h_C\|_\infty$.

We equip the set $\mathcal{K}^n$ with the topology induced by the Hausdorff metric. That is, for a sequence of convex bodies $\{K_i\}$ in $\mathcal{K}^n$, we say $K_i \to K$ in $\mathcal{K}^n$ if

$$\left\|h_{K_i} - h_K\right\|_\infty \to 0.$$

For a convex body K, $u \in S^{n-1}$ is said to be an exterior unit normal of $x \in \partial K$ if $\langle x, u \rangle = h_K(u)$. We call a boundary point $x \in \partial K$ smooth if x has a unique exterior

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unit normal. $\partial' K$ denotes the set of all smooth boundary points of ∂K. It is a well known fact that $\mathcal{H}(\partial K \backslash \partial' K) = 0$ (see Schneider [141]) and $\partial' K$ is a Borel set.

The spherical Gauss map $\nu_K : \partial' K \to S^{n-1}$ assigns to each $x \in \partial' K$ its unique exterior normal. ν_K is continuous on $\partial' K$. For a Borel set $\omega \subset S^{n-1}$, $\nu_K^{-1}(\omega)$ is the set of all points of $\partial' K$ that have exterior unit normal in ω.

The supporting distance at ∂K of any $x \in \partial' K$ is given by $\langle x, \nu_K(x) \rangle = h_K(\nu_K(x))$ and often denoted $d_K(x)$.

For an origin symmetric convex body K in $\mathbb{R}^n$, there exists a unique ellipsoid of maximal volume contained in K, called the John ellipsoid , and a unique ellipsoid of minimal volume containing K known as the Löwner ellipsoid. The John ellipsoid E in this case satisfies

$$E \subset K \subset \sqrt{n} E.$$

We say K is in John (Löwner) position if the unit ball B^n is the John (Löwner) ellipsoid for K. For any origin symmetric K, there exists $\Phi \in \mathrm{GL}(n, \mathbb{R})$ such that ΦK is in John (Löwner) position. If K is in John's position, there exist $u_1, \ldots, u_m \in S^{n-1} \cap \partial K$ (contact points) and weights $\alpha_1, \ldots \alpha_m > 0$ such that

$$\sum_{i=1}^{m} \alpha_i u_i \otimes u_i = I_n$$

where $u \otimes u(x) = \langle x, u \rangle u$ for $x \in \mathbb{R}^n, u \in S^{n-1}$.

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Chapter 2

Introduction

In this chapter we discuss the background and context (see Böröczky [30]) of Brunn-Minkowski theory, which our results are a part of including basic ideas, definitions, tools and other relevant theory. Particularly in sections 2.3, 2.8, 2.9, we mention an overview of our results from the following papers:

- K. J. Böröczky and A. De. "Stability of the Prékopa-Leindler inequality for log-concave functions." In: *Adv. Math* 386 (2021), p. 107810. DOI: 10.1016/j.aim.2021.107810
- K. J. Böröczky and A. De. "Stability of the log-Brunn-Minkowski inequality in the case of many hyperplane symmetries". In: (2021). DOI: 10.48550/ARXIV.2101.02549. arXiv: 2101.02549 [math.MG]
- K. J. Böröczky and A. De. "Stable solution of the Logarithmic Minkowski problem in the case of hyperplane symmetries". In: *Journal of Differential Equations* 298 (Oct. 2021), pp. 298–322. DOI: 10.1016/j.jde.2021.07.002

They are presented in Chapters 3, 4 and 5 respectively.

For a comprehensive survey of the state of the art concerning L_p Brunn-Minkowski theory, we refer to Böröczky [30].

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2.1 Introduction to Brunn-Minkowski Theory

The Minkowski problem plays a central role in several fields, including fully nonlinear partial differential equations and convex geometry. The problem's influence extends to various domains (see Trudinger, Wang [146] and Schneider [141]). Lutwak [115, 116, 117] further developed it into the L_p-Minkowski theory which has become a core area of research in modern convex geometry and geometric analysis. Minkowski and Aleksandrov established Minkowski's classical existence theorem, thereby providing a characterization of the so-called surface area measure S_K for a convex body K in $\mathbb{R}^n$. Specifically, it offers a solution to the Monge-Ampére equation

$$\det(\nabla^2 h + h\,\mathrm{Id}) = f$$

on the sphere S^{n-1}. Given f, a convex body K with a C^2_+ boundary provides a solution if we set $h = h_K|_{S^{n-1}}$, where h_K is the support function of K. Here $1/f(u)$ is the Gaussian curvature at the point $x \in \partial K$, at which $u \in S^{n-1}$ is an exterior normal.

The logarithmic Minkowski problem (aka log-Minkowski problem) or the L_0 Minkowski problem, originally formulated by Firey in his groundbreaking paper [76] has the associated Monge-Ampére equation:

$$h \det(\nabla^2 h + h\,\mathrm{Id}) = nf \tag{2.1}$$

Its objective is twofold: first, to define the cone volume measure denoted as $dV_K = \frac{1}{n}, h_K dS_K$ for a convex body K that contains the origin o, and second, to investigate whether a unique solution exists when the function f is even. The uniqueness of even solutions to the above Monge-Ampére equation is the log-Minkowski conjecture, as put forward by Lutwak. The log-Minkowski conjecture is interestingly versatile and appears to play a role in various seemingly disparate topics.

The log-Minkowski problem corresponds to the case $p = 0$ of Lutwak's L_p-Minkowski problem

$$h^{1-p} \det(\nabla^2 h + h\,\mathrm{Id}) = f$$

posed in the 1990's, whereas the case $p = 1$ refers to the classical Minkowski problem.

2.2 Brunn-Minkowski inequality and stability

Closely related to the Minkowski problem crops up the Brunn-Minkowski inequality (see Gardner [80] or Schneider [141]) which states for convex bodies K and C in $\mathbb{R}^n$ and $\alpha, \beta \geq 0$, we have

$$V(\alpha K + \beta C)^{\frac{1}{n}} \geq \alpha V(K)^{\frac{1}{n}} + \beta V(C)^{\frac{1}{n}}, \tag{2.2}$$

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with equality if and only if K and C are homothetic. In fact, the uniqueness up to translation of the solution to the classical Minkowski problem follows from the equality case of the Brunn-Minkowski inequality. The Brunn-Minkowski inequality remains valid when K and C are bounded Borel subsets of $\mathbb{R}^n$. Minkowski linear combinations of measurable subsets may not themselves be measurable, and when that's the case we use outer measures.

For a subset X of $\mathbb{R}^n$, denote the convex hull of X by $\operatorname{conv} X$. We say that X is homothetic to $Y \subset \mathbb{R}^n$ if $Y = \gamma X + z$ where $\gamma > 0$ and $z \in \mathbb{R}^n$. Using $|X|$ or $V(X)$ to denote the Lebesgue measure of a measurable subset X of $\mathbb{R}^n$ (with $V(\emptyset) = 0$), the Brunn-Minkowski inequality (Schneider [141]) states that if α and $\beta > 0$, and X, Y, and Z are bounded measurable subsets of $\mathbb{R}^n$, then if $\alpha X + \beta Y \subset Z$, we have

$$V(Z)^{\frac{1}{n}} \geq \alpha V(X)^{\frac{1}{n}} + \beta V(Y)^{\frac{1}{n}}$$

In the case where $V(X)$ and $V(Y)$ are both positive, equality is achieved if and only if $\operatorname{conv} X$ and $\operatorname{conv} Y$ are homothetic convex bodies with $V((\operatorname{conv} X)\backslash X) = V((\operatorname{conv} Y)\backslash Y) = 0$, and $\operatorname{conv} Z = \alpha(\operatorname{conv} X) + \beta(\operatorname{conv} Y)$. It's important to note that even when X and Y are Lebesgue measurable, the Minkowski linear combination $\alpha X + \beta Y$ may not be a measurable set.

Because of the homogeneity of the Lebesgue measure, an equivalent form of (2.2) is the following. If $\lambda \in (0, 1)$, then

$$V(Z) \geq V(X)^{1-\lambda} V(Y)^{\lambda} \quad \text{provided} \quad (1-\lambda)X + \lambda Y \subset Z. \tag{2.3}$$

In the case $V(X), V(Y) > 0$, equality in (2.3) implies that $\operatorname{conv} X$ and $\operatorname{conv} Y$ are translates, and $V((\operatorname{conv} X)\backslash X) = V((\operatorname{conv} Y)\backslash Y) = 0$.

Stability

Geometric and functional inequalities are of paramount importance in tackling a variety of issues in fields like calculus of variations, partial differential equations, and geometry, among others. Recently, there has been a rising interest in investigating the stability of these inequalities: suppose we have an inequality with established equality cases, that is, we know the minimizers, then can we say in a quantifiable manner, that if a function or a geometric entity comes "very close" to satisfying the inequality, then it is also close (in a suitable manner) to one of the minimizers? Is the inequality sensitive to small perturbations? Naturally, the way we define closeness, that is, an appropriate notion of "distance" is crucial in this regard. Finding stability results is much more than a mere

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academic curiosity as it has important applications, for example, in determining rates of convergence among other things.

Minkowski originally established the first stability versions of the Brunn-Minkowski inequality for convex sets X and Y (see Groemer [84]). When considering the distance between these convex sets, measured in terms of the Hausdorff distance, Diskant [61] and Groemer [83] provided stability versions that are close to optimal (see Groemer [84]). However, the natural notion of distance is based on the volume of the symmetric difference, and the optimal results in this context were achieved by Figalli, Maggi, and Pratelli [69, 70]. Let $\alpha = V(K)^{\frac{-1}{n}}$, $\beta = V(C)^{\frac{-1}{n}}$, and

$$\sigma(K, C) = \max\left\{\frac{V(C)}{V(K)}, \frac{V(K)}{V(C)}\right\}.$$

Then the "homothetic distance" $A(K, C)$ of convex bodies K and C, is defined as

$$A(K, C) = \min\left\{V(\alpha K \Delta(x + \beta C)) :\, x \in \mathbb{R}^n\right\}.$$

THEOREM 2.2.1 (Figalli, Maggi, Pratelli)**.** *If K and C are convex bodies in $\mathbb{R}^n$, then for $\gamma^*(n) = (\frac{(2-2^{\frac{n-1}{n}})^{\frac{3}{2}}}{122n^7})^2$, we have*

$$V(K+C)^{\frac{1}{n}} \geq (V(K)^{\frac{1}{n}} + V(C)^{\frac{1}{n}})\left[1 + \frac{\gamma^*}{\sigma(K,C)^{\frac{1}{n}}} \cdot A(K,C)^2\right].$$

The exponent 2 isn $A(K, C)^2$ is shown to be optimal as per Figalli, Maggi, Pratelli [70]. Before this, Diskant [61] and Groemer [83] had provided the only known error term in the Brunn-Minkowski inequality and it was of the order of $A(K, C)^\eta$ with $\eta \geq n$. Esposito, Fusco, Trombetti [67] offer a more direct approach.

The factor $\gamma^*(n) = cn^{-14}$ for some absolute constant $c > 0$ provided by Figalli, Maggi, Pratelli [69] has since been improved. Segal [142] improved it to cn^{-7} and Kolesnikov, Milman [106] (Theorem 12.12) further improved it to $cn^{-5.5}$.

Kolesnikov-Milman's [106] estimate (Theorem 12.2) and Yuansi Chen's [52] bound $n^{o(1)}$ on the Cheeger constant of a convex body in isotropic position (related to the Kannan-Lovasz-Simonovits conjecture) together lead to the current best known bound for $\gamma^*(n)$ of the order of $n^{-5-o(1)}$.

Harutyunyan [89] showed that γ^* can't be of a smaller order than n^2 and conjectured that $\gamma^*(n) = cn^{-2}$ should be the optimal order of the constant, whereas Segal [142]

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showed that the choice of $\gamma^*(n) = cn^{-2}$ would follow from Dar's conjecture in [59] if it were to be true.

Dar [59] conjectured the following strengthening of the Brunn-Minkowski inequality. It states that for convex bodies K and C in $\mathbb{R}^n$, and $M = \max_{x\in\mathbb{R}^n} V(K \cap (x + C))$,

$$V(K+C)^{\frac{1}{n}} \geq M^{\frac{1}{n}} + \left(\frac{V(K)V(C)}{M}\right)^{\frac{1}{n}}. \tag{2.4}$$

Dar's conjecture has only been verified in $\mathbb{R}^2$ by Xi, Leng [151], and only in certain specific cases in higher dimensions (see Dar [59]).

Eldan, Klartag [66] explore certain "isomorphic" stability versions of the Brunn-Minkowski inequality under conditions of the type $V(\frac{1}{2}K + \frac{1}{2}C) \leq 5\sqrt{V(K)\cdot V(C)}$. They consider, for example, the L^2 Wasserstein distance of the uniform measures on suitable affine images of K and C.

When X is a measurable bounded set and Y is a convex body, Barchiesi, Julin [15] improves the estimate given in Carlen, Maggi [47] and shows that for some $\delta_n > 0$ depending on n, we have

$$|X+Y|^{\frac{1}{n}} \geq |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}} + \delta_n \min\{|X|,|Y|\}^{\frac{1}{n}} A(X,Y)^2. \tag{2.5}$$

Here, we discuss a bit about the case when X, Y, Z are bounded measurable with positive measure and $X+Y \subset Z$. So, neither X nor Y are assumed to be convex. If $X = Y$ and $n \geq 1$, Hintum, Spink, Tiba [94] obtained estimates similar to (2.5), whereas Hintum, Spink, Tiba [93] considered the case when $n = 2$ and X, Y are any bounded measurable sets. If $n = 1$, Freiman obtained an even better error term of the order of $A(X,Y)$ (see Christ [54]). However, in $\mathbb{R}^3$, when X, Y, Z are any bounded measurable sets with $X + Y \subset Z$, only a much weaker estimate is known, as shown by Figalli, Jerison [72, 73]: if

$$\big||X|-1\big| + \big||Y|-1\big| + \big||Z|-1\big| < \varepsilon \quad\text{and}\quad \tfrac{1}{2}X + \tfrac{1}{2}Y \subset Z$$

for small $\varepsilon > 0$, then there exist a convex body K and $z \in \mathbb{R}^n$ such that

$$X \subset K,\ \ Y + z \subset K \quad\text{and}\quad |K\backslash X| + |K\backslash(Y+z)| < c_n\varepsilon^{\eta}$$

where $c_n, \eta > 0$ depend on n and $\eta < n^{-3^n}$.

2.3 Prékopa-Leindler inequality and stability

For a convex subset $\Gamma \subset \mathbb{R}^n$, we say that a function $f : \Gamma \to [0,\infty)$ is *log-concave*, if for any $x, y \in \Gamma$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, we have $f(\alpha x + \beta y) \geq f(x)^{\alpha} g(y)^{\beta}$.

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For a convex function $W : \mathbb{R}^n \to (-\infty, \infty]$, the function $\varphi = e^{-W}$ is log-concave with $e^{-\infty} = 0$, that is, $\log \varphi$ is concave for $\varphi : \mathbb{R}^n \to [0, \infty))$.

The Prékopa-Leindler inequality is a classical functional form and a generalization of the Brunn-Minkowski inequality. The inequality itself, was initially introduced by Prékopa [131] and Leindler [108] in one dimension, and later extended to higher dimensions and generalized by Prékopa [130, 132] and Borell [29] (*cf.* also Marsiglietti [120], Bueno, Pivovarov [129], Brascamp, Lieb [42], Kolesnikov, Werner [107], Bobkov, Colesanti, Fragalà [27]). A recent variant can be found in Artstein-Avidan, Florentin, and Segal [9]. Various applications are explored and summarized in Ball [10],Barthe [17], Fradelizi, Meyer [77] and Gardner [80]. Dubuc [63] has characterized the case of equality. The following multiplicative version presented in [10] is particularly well-suited for geometric applications.

THEOREM 2.3.1 (Prékopa-Leindler, Dubuc)**.** *If $\lambda \in (0,1)$ and h, f, g are non-negative integrable functions on $\mathbb{R}^n$ satisfying $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for $x, y \in \mathbb{R}^n$, then*
$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \cdot \left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{2.6}$$
with equality implying that setting $a = \int_{\mathbb{R}^n} f / \int_{\mathbb{R}^n} g$, there exists $w \in \mathbb{R}^n$, and a log-concave function $\tilde{h}$ such that $h = \tilde{h}$, $f(x) = a^\lambda \tilde{h}(x - \lambda w)$ and $g(y) = a^{-(1-\lambda)} \tilde{h}(y + (1-\lambda)w)$ almost everywhere.

Our main result here is the following stability version of Prékopa-Leindler inequality for log-concave functions in terms of the L_1 distance.

THEOREM 2.3.2. *For some absolute constant $c > 1$, if $\tau \in (0, \frac{1}{2}]$, $\tau \leq \lambda \leq 1 - \tau$, $h, f, g : \mathbb{R}^n \to [0, \infty)$ are integrable such that $h((1-\lambda)x + \lambda\, y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for $x, y \in \mathbb{R}^n$, h is log-concave and*
$$\int_{\mathbb{R}^n} h \leq (1+\varepsilon) \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda}$$
for $\varepsilon \in (0,1]$, then there exists $w \in \mathbb{R}^n$ such that setting $a = \int_{\mathbb{R}^n} g / \int_{\mathbb{R}^n} f$, we have
$$\begin{aligned} \int_{\mathbb{R}^n} |f(x) - a^{-\lambda} h(x - \lambda\, w)|\, dx &\leq c^n n^n \sqrt[19]{\frac{\varepsilon}{\tau}} \cdot \int_{\mathbb{R}^n} f \\ \int_{\mathbb{R}^n} |g(x) - a^{1-\lambda} h(x + (1-\lambda)w)|\, dx &\leq c^n n^n \sqrt[19]{\frac{\varepsilon}{\tau}} \cdot \int_{\mathbb{R}^n} g. \end{aligned}$$

Remark If f and g are log-concave, then so is
$$h(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda$$

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and Theorem 2.3.2 applies.

Ball, Böröczky [13] provided a similar result to Theorem 2.3.2 for the case of even log-concave functions and $\tau = \frac{1}{2}$, where the error term is of the order of $\varepsilon^{\frac{1}{6}}|\log \varepsilon|^{\frac{2}{3}}$ instead of the $\varepsilon^{\frac{1}{19}}$ term in Theorem 2.3.2. Instead of considering the "translative" L_1 distance, Bucur, Fragalà [45] showed a nice stability result in terms of the weaker notion of bounding the (translative) distance of all one-dimensional projections.

Eldan [65], Lemma 5.2 attained an "isomorphic" stability result for the Prékopa-Leindler inequality, in terms of the transportation distance. Using rather standard arguments, it follows that non-isomorphic stability results in terms of the transportation distance yield stability in terms of the L_1 distance. For example, combining Proposition 2.9 in Bubeck, Eldan, Lehec [44] and Proposition 10 in Eldan, Klartag [66] leads to such an implication. However, due to its isomorphic nature, the current result in in [65] fails to obtain a meaningful bound in terms of the L_1 distance.

A local variant of the Prekopa-Leindler inequality for log-concave functions was proved by Brascamp and Lieb [42] (Theorem 4.2), that is equivalent to the commonly known Poincare-type Brascamp-Lieb inequality (Theorem 4.1 also in [42]). Furthermore, a stability version of this Brascamp-Lieb inequality is presented in Livshyts [112].

Recently, there has been notable breakthrough in obtaining stability results for geometric functional inequalities. Fusco, Maggi, Pratelli [79] considered the isoperimetric inequality and obtained an optimal stability version with respect to the symmetric difference metric, and this result was further extended to the case of the Brunn-Minkowski inequality by Figalli, Maggi, Pratelli [69, 70]. Barthe, Böröczky, Fradelizi [21] provided stronger versions of the functional Blaschke-Santaló inequality. Ghilli, Salani [81], Rossi, Salani [134, 135] and Balogh, Kristály [14] (later even on Riemannian manifolds) proved stronger versions of the Borell-Brascamp-Lieb inequality. Figalli, Zhang [71] (extending Bianchi, Egnell [26] and Figalli, Neumayer [74]), Nguyen [127] and Wang [150] did the same for the Sobolev inequality. Stability results for the log-Sobolev inequality was considered by Gozlan [82], and some related inequalities by Caglar, Werner [46], Cordero-Erausquin [57], Kolesnikov, Kosov [104].

To prove Theorem 2.3.2, we first prove a stability version with $\lambda = \tau = \frac{1}{2}$. We may assume $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1$ and $\sup f = 1$. We make use of the following stability version by Ball, Böröczky for $n = 1$.

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THEOREM 2.3.3. *If f, g are log-concave densities on $\mathbb{R}$, and h is log-concave satisfying with $h(\frac{r+s}{2}) \geq \sqrt{f(r)g(s)}$ $r, s \in \mathbb{R}$, and*

$$\int_{\mathbb{R}} h \leq 1+\varepsilon,$$

for $\varepsilon \in (0,1)$, then there exists $w \in \mathbb{R}$ such that

$$\begin{aligned}\int_{\mathbb{R}} |f(t)-h(t+w)|\,dt &\leq c\cdot\sqrt[3]{\varepsilon}|\ln\varepsilon|^{\frac{4}{3}}\\ \int_{\mathbb{R}} |g(t)-h(t-w)|\,dt &\leq c\cdot\sqrt[3]{\varepsilon}|\ln\varepsilon|^{\frac{4}{3}}.\end{aligned}$$

Next we consider and compare the level sets of f, g, h in Theorem 2.3.2. Let

$$\begin{aligned}\Phi_t &= \{x\in\mathbb{R}^n : f(x)\geq t\} \text{ and } F(t)=|\Phi_t|\\ \Psi_t &= \{x\in\mathbb{R}^n : g(x)\geq t\} \text{ and } G(t)=|\Psi_t|\\ \Omega_t &= \{x\in\mathbb{R}^n : h(x)\geq t\} \text{ and } H(t)=|\Omega_t|.\end{aligned}$$

The condition on f, g, h yields that if $\Phi_r, \Psi_s \neq \emptyset$ for $r, s > 0$, then

$$\tfrac{1}{2}(\Phi_r+\Psi_s)\subset\Omega_{\sqrt{rs}}.$$

Therefore the Brunn-Minkowski inequality implies that

$$H(\sqrt{rs}) \geq \left(\frac{F(r)^{\frac{1}{n}}+G(s)^{\frac{1}{n}}}{2}\right)^n \geq \sqrt{F(r)\cdot G(s)}$$

for all $r, s > 0$. In particular, we have

$$\sqrt{\int_0^\infty F(t)\,dt\cdot\int_0^\infty G(t)\,dt} \leq \int_0^\infty H(t) \leq (1+\varepsilon)\sqrt{\int_0^\infty F(t)\,dt\cdot\int_0^\infty G(t)\,dt}$$

We use the following stability version of the product form of the Brunn-Minkowski inequality on $\mathbb{R}^n$. Since

$$\begin{aligned}\frac{1}{2}\left(|K|^{\frac{1}{n}}+|C|^{\frac{1}{n}}\right) &= |K|^{\frac{1}{2n}}|C|^{\frac{1}{2n}}\left[1+\frac{1}{2}\left(\sigma(K,C)^{\frac{1}{4n}}-\sigma(K,C)^{\frac{-1}{4n}}\right)^2\right]\\ &\geq |K|^{\frac{1}{2n}}|C|^{\frac{1}{2n}}\left[1+\frac{(\sigma(K,C)-1)^2}{32n^2\sigma(K,C)^{\frac{4n-1}{2n}}}\right],\end{aligned}$$

from Figalli, Maggi, Pratelli's [70] stability version of the Brunn-Minkowski inequality Theorem 2.2.1, and denoting $\sigma=\sigma(K,C)=\max\{\frac{|C|}{|K|},\frac{|K|}{|C|}\}$ we deduce that

$$\left|\tfrac{1}{2}(K+C)\right| \geq \sqrt{|K|\cdot|C|}\left[1+\frac{(\sigma-1)^2}{32n\sigma^2}+\frac{n\gamma^*(n)}{\sigma^{\frac{1}{n}}}\cdot A(K,C)^2\right]. \tag{2.7}$$

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Combining the stability result for $n = 1$ and the above product form of Brunn-Minkowski stability, we arrive at the conclusion that Φ_t and Ψ_t are "almost translates". More precisely consider the following convex bodies in $\mathbb{R}^{n+1}$:

$$\begin{aligned} K = K_{\xi,f} &= \{x + u_0 \ln t : x \in \Phi_\xi \text{ and } \xi \le t \le f(x)\} \\ C = C_{\xi,g} &= \{x + u_0 \ln t : x \in \Psi_\xi \text{ and } \xi \le t \le g(x)\} \\ L = L_{\xi,h} &= \{x + u_0 \ln t : x \in \Omega_\xi \text{ and } \xi \le t \le h(x)\}. \end{aligned}$$

where $\xi \approx \varepsilon^{\frac{1}{18}}$. $h\left(\frac{1}{2}x + \frac{1}{2}y\right) \ge f(x)^{\frac{1}{2}} g(y)^{\frac{1}{2}}$ implies

$$\frac{1}{2}K + \frac{1}{2}C \subset L$$

Thus we convert a problem involving log-concave functions to a problem concerning convex bodies. Using certain estimates for the volume of the level sets, we deduce that $|V(K) - V(C)|$ and $|V(C) - V(L)|$ are "small", and using the product form of Brunn-Minkowski stability (2.7), we deduce that K, C, L are essentially translates and conclude the stability version of the Prékopa-Leindler inequality for $\lambda = \frac{1}{2}$. See Chapter 3 for a complete presentation of the results.

We note here that Böröczky, Figalli, Ramos [36] (Theorem 1.6) have provided the first quantitative stability version of the Prékopa-Leindler inequality for arbitrary measurable functions as follows.

THEOREM 2.3.4 (Böröczky, Figalli, Ramos). *For $\tau \in \left(0, \frac{1}{2}\right]$ and $\lambda \in [\tau, 1-\tau]$, if $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ be measurable functions such that $h((1-\lambda)x + \lambda y) \ge f(x)^{1-\lambda} g(y)^\lambda$ for all $x, y \in \mathbb{R}^n$, and*

$$\int_{\mathbb{R}^n} h < (1+\varepsilon) \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda} \quad \text{for some } \varepsilon > 0,$$

the there are a computable dimensional constant Θ_n and computable constants $Q_n(\tau)$ and $M_n(\tau)$ depending only on n and τ such that the following holds: If $0 < \varepsilon < e^{-M_n(\tau)}$, then there exist $\tilde{h}$ log-concave and $w \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |h - \tilde{h}| + \int_{\mathbb{R}^n} \left|a^\lambda f - \tilde{h}(\cdot + \lambda w)\right| + \int_{\mathbb{R}^n} \left|a^{\lambda-1} g - \tilde{h}(\cdot + (\lambda - 1)w)\right| < \frac{\varepsilon^{Q_n(\tau)}}{\tau^{\Theta_n}} \int_{\mathbb{R}^n} h,$$

where $a = \int_{\mathbb{R}^n} g / \int_{\mathbb{R}^n} f$.

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2.4 Equivalent forms of Brunn-Minkowski inequality

Minkowski showed that for $\alpha, \beta \geq 0$ the volume of the linear combination $\alpha K + \beta C$ of convex bodies K and C can be expressed as a polynomial in α and β as follows

$$V(\alpha\, K + \beta\, C) = \sum_{i=0}^{n} \binom{n}{i} V(K, C; i)\alpha^{n-i}\beta^{i}. \tag{2.8}$$

The coefficients $V(K, C; i)$ called the mixed volumes (see Schneider [141]). For a fixed i, $V(K, C; i)$ is positive and continuous in both variables. Some of the properties of the mixed volumes are given below:

- $V(\alpha K, \beta C; i) = \alpha^{n-i}\beta^{i} V(K, C; i)$ for $\alpha, \beta > 0$
- $V(K, C; i) = V(C, K; n-i)$
- $V(\Phi K + x, \Phi C + y; i) = V(K, C; i)$ for $x, y \in \mathbb{R}^n$ and $\Phi \in \mathrm{SL}(n)$

Several of the mixed volumes have geometric significance, for example, $V(K, K; i) = V(K, C; 0) = V(K)$ and $\frac{1}{n} V(K, B^n; 1) = \mathcal{H}(\partial K)$ is the surface area of K.

In fact, Minkowski defined mixed volumes for n convex bodies as follows

$$V\left(\lambda_1 K_1 + \ldots + \lambda_m K_m\right) = \sum_{i_1, \ldots, i_n = 1}^{m} V(K_{i_1}, \ldots, K_{i_n}) \cdot \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} \tag{2.9}$$

$V(C_1, \ldots, C_n)$ is non-negative, symmetric and continuous in its arguments wrt the Hausdorff metric. $V(K, C; i)$ then denotes the mixed volume of $n - i$ copies of K and i copies of C.

Considering the first derivative $f'(\lambda)$ of the concave function $f(\lambda) = V((1-\lambda)K + \lambda C)^{\frac{1}{n}}$ on $[0, 1]$, it can be shown that the Brunn-Minkowski inequality (2.2) is equivalent to the following, known as Minkowski's inequality:

$$V(K, C; 1)^n \geq V(K)^{n-1} V(C), \tag{2.10}$$

where equality holds if and only if K, C are homothetic. Considering the second derivative $f''(\lambda)$ yields Minkowski's second inequality which is also equivalent to the Brunn-Minkowski inequality

$$V(K, C; 1)^2 \geq V(K) V(K, C; 2), \tag{2.11}$$

Recent results by van Handel, Shenfeld [144] have dealt with the equality conditions of Minkowski's second inequality.

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Here, we note that that since the surface area measure (see section 2.5) is the first variation of the volume, that is, for a convex body C in $\mathbb{R}^n$, we have

$$nV(K,C;1) = \lim_{\varepsilon\to 0^+} \frac{V(K+\varepsilon C) - V(K)}{\varepsilon} = \int_{S^{n-1}} h_C\, dS_K, \tag{2.12}$$

the Minkowski inequality (2.10) for the case $V(K) = V(C)$ can be alternatively expressed as

$$\int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_K\, dS_K \tag{2.13}$$

where equality holds if and only if K and C are translates.

As such we have the following equivalent formulations of the Brunn-Minkowski inequality for convex bodies K, C in $\mathbb{R}^n$:

- $V(\alpha K + \beta C)^{\frac{1}{n}} \geq \alpha V(K)^{\frac{1}{n}} + \beta V(C)^{\frac{1}{n}}$
- $V((1-\lambda)K + \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda}$
- $f(\lambda) = V((1-\lambda)K + \lambda C)^{\frac{1}{n}}$ is concave on $[0,1]$
- Minkowski inequality: $V(K,C;1)^n \geq V(K)^{n-1}V(C)$
- Minkowski's second inequality: $V(K,C;1)^2 \geq V(K)V(K,C;2)$.

Here, we mention the important Alexandrov-Fenchel inequality (see Alexandrov [3, 5], Schneider [141]) given by

$$V(K_1,K_2,K_3,\ldots,K_n)^2 \geq V(K_1,K_1,K_3,\ldots,K_n)V(K_2,K_2,K_3,\ldots,K_n)$$

which generalizes both Minkowski's first and second inequalities. For the equality conditions in this case, we refer to van Handel, Shenfeld [143, 144].

2.5 Surface area measure and Minkowski problem

Surface area measure

For a convex body K, $u \in S^{n-1}$ is said to be an exterior unit normal of $x \in \partial K$ if $\langle x, u\rangle = h_K(u)$. We call a boundary point $x \in \partial K$ smooth if x has a unique exterior unit normal. $\partial' K$ denotes the set of all smooth boundary points of ∂K. It is a well known fact that $\mathcal{H}(\partial K \backslash \partial' K) = 0$ (see Schneider [141]) and $\partial' K$ is a Borel set.

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The spherical Gauss map $\nu_K : \partial' K \to S^{n-1}$ assigns to each $x \in \partial' K$ its unique exterior normal. ν_K is continuous on $\partial' K$. For a Borel set $\omega \subset S^{n-1}$, $\nu_K^{-1}(\omega)$ is the set of all points of $\partial' K$ that have exterior unit normal in ω.

For $K \in \mathcal{K}_o^n$, the surface area measure of K is a Borel measure on S^{n-1} given by

$$S_K(\omega) = \mathcal{H}(\nu_K^{-1}(\omega))$$

for any Borel set $\omega \subset S^{n-1}$. We have the basic volume formula

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) d\, S_K(u).$$

In the case of a polytope P with facets $F_1, \ldots, F_k$, with corresponding exterior unit normals $u_1, \ldots, u_k$, the surface area measure is concentrated on $\{u_1, \ldots, u_k\}$. And for $i = 1, \ldots, k$,

$$S_P(u_i) = \mathcal{H}(F_i)$$

which is the surface area of the facet F_i. So, the surface area measure in this case is a discrete measure, that can be written as

$$S_K = a_1 \delta_{u_1} + \ldots + a_k \delta_{u_k}$$

where $a_i = \mathcal{H}(F_i)$ and δ_{u_i} is the delta measure concentrated on u_i.

We say ∂K is C_+^2 if it is C^2 and has positive Gaussian curvature. We write $\kappa(u)$ to denote the Gaussian curvature at $x \in \partial K$ with exterior unit normal $u \in S^{n-1}$. Then

$$d\, S_K = \kappa^{-1} d\,\mathcal{H} = \det(\nabla^2 h + hId) d\,\mathcal{H}$$

on S^{n-1}, where $h = h_K|_{S^{n-1}}$, and ∇h and $\nabla^2 h$ denote the gradient and hessian of h with respect to a moving orthonormal frame. Here $\det(\nabla^2 h + hId) > 0$, and S_K is absolute continuous.

Minkowski problem

Given a Borel measure μ on S^{n-1}, the Minkowski problem looks for the necessary and sufficient conditions such that

$$\mu = S_K \tag{2.14}$$

The Minkowski problem has the following associated Monge-Ampère equation on the sphere S^{n-1}:

$$\det(\nabla^2 h + h\,\mathrm{Id}) = f \tag{2.15}$$

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where f is a given non-negative function with positive integral. In the case when μ is not absolute continuous wrt the Lebesgue measure, we call $h = h_K|_{S^{n-1}}$ an Alexandrov solution of (2.15) if the surface area measure of K satisfies $S_k = \mu$.

Minkowski [122, 123] provided the solution along with its uniqueness in the case whe μ is discrete or absolutely continuous. For the general case when μ is any general measure Alexandrov [2, 4, 5] (see also Fenchel, Jensen [68], Lewy [109]) showed that there exists a convex body K with $\mu = S_K$ if and only if for any linear $(n-1)$-dimensional subspace $L \subset \mathbb{R}^n$

$$\mu(L \cap S^{n-1}) < \mu(S^{n-1}) \tag{2.16}$$

$$\int_{S^{n-1}} u \, dS_K(u) = o; \tag{2.17}$$

and the solution is unique upto translation, that is, for convex bodies K and C, $S_K = S_C$ if and only if K and C are translates.

2.6 L_p-Minkowski linear combinations

Recall that for any sub-additive, positive homogeneous function h on $\mathbb{R}^n$, there exists a unique compact convex set K in $\mathbb{R}^n$ such that h is the support function of K. For two convex bodies K and C in $\mathbb{R}^n$ and $\alpha, \beta \geq 0$ the Minkowski linear combination is given by

$$\alpha K + \beta C = \{\alpha x + \beta y : x \in K, y \in C\}$$

If h_K and h_C are the support functions of K and C respectively, then it readily follows that the support function of the Minkowski linear combination $\alpha K + \beta C$ is given by $h_{\alpha K + \beta C} = \alpha h_K + \beta h_C$.

In view of this, we can equivalently define the Minkowski linear combination as the unique convex body whose support function is $\alpha h_K + \beta h_C$. For $p > 1$, Firey similarly defined the L_p linear combination by

$$h_{\alpha K +_p \beta C} = (\alpha h_K^p + \beta h_C^p)^{\frac{1}{p}}$$

Minkowski's inequality implies that $h_{\alpha K +_p \beta C} = (\alpha h_K^p + \beta h_C^p)^{\frac{1}{p}}$ is subadditive for $p \geq 1$. However, for $p < 1$ this definition fails. To get around this, following Böröczky, Lutwak, Yang, Zhang [39], we define the L_p linear combination in terms of the Alexandrov body.

Alexandrov body/ Wulff shape

A continuous function $f : S^{n-1} \to (0, \infty)$ defines a family of hyperplanes given by

$$H_{\mathrm{u}} = \{x \in \mathbb{R}^n : \langle x, u \rangle = f(u)\}.$$

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for each $u \in S^{n-1}$. If we take the intersection of the half-spaces bounded by these hyperplanes, we get the convex body

$$\begin{aligned} W(f) &= \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : \langle x, u \rangle \le f(u)\} \\ &= \left\{x \in \mathbb{R}^n : \langle x, u \rangle \le f(u) \forall\, u \in S^{n-1}\right\} \end{aligned}$$

known as the Alexandrov body or the Wulff shape of f. Note that $W(f)$ is a convex body containing the origin in its interior since f is strictly positive and continuous. Equivalently we can define $W(f)$ as the unique maximal element, with respect to set inclusion of

$$\left\{K \in \mathcal{K}^n_{(o)} : h_K \le f\right\}.$$

It readily follows that $h_{W(f)} \le f$, where $h_{W(f)}$ is the support function of $W(f)$. And in fact, we have $h_{W(f)} = f$ almost everywhere with respect to the surface area measure $S_{W(f)}$. $h_{W(f)}(u) = f(u)$ for any u in the support of $S_{W(f)}$ and if $W(f)$ has a smooth boundary, $h_{W(f)} = f$. For a convex body K with support function h_K, K itself is the Alexandrov body of h_K, that is, $W(h_K) = K$.

Then for $p \in (0, 1)$, we define the L_p linear combination for $\lambda \in (0, 1)$ by

$$\begin{aligned} (1-\lambda)K +_p \lambda C &= W\left(\left((1-\lambda)\, h_K(u)^p + \lambda\, h_C(u)^p\right)^{\frac{1}{p}}\right) \\ &= \{x \in \mathbb{R}^n : \langle x, u \rangle \le \left((1-\lambda)\, h_K(u)^p + \lambda\, h_C(u)^p\right)^{\frac{1}{p}} \ \ \forall u \in S^{n-1}\} \end{aligned}$$

where as for the case $p = 0$, the L_0 or logarithmic linear combination is

$$\begin{aligned} (1-\lambda)K +_0 \lambda C &= W(h_K(u)^{1-\lambda} h_C(u)^{\lambda}) \\ &= \{x \in \mathbb{R}^n : \langle x, u \rangle \le h_K(u)^{1-\lambda} h_C(u)^{\lambda}\ \forall u \in S^{n-1}\} \end{aligned}$$

Next we note some properties of the L_0 and L_p linear combinations.

The L_0 linear combination is linear invariant in the sense that if $\Phi \in \mathrm{GL}(n)$, then $\Phi\left((1-\lambda)K +_0 \lambda C\right) = (1-\lambda)\Phi(K) +_0 \lambda\Phi(C)$. It follows that if K and C are invariant under Φ, then so also is $(1-\lambda)K +_0 \lambda C$. If $\{h_K = 0\} = \{h_C = 0\}$, the L_0 linear combination of K and C is a convex body. If K and C are polytopes, so is $(1-\lambda)K +_0 \lambda C$. However, even if K, C have C^2_+ boundaries, $(1-\lambda)K +_0 \lambda C$ may contain segments and may not be C^2_+. Crasta, Fragalà [58] provide a functional analogue of the L_0 sum.

For $p > 0$, we also have that if $\Phi \in \mathrm{GL}(n)$, then $\Phi\left((1-\lambda)K +_p \lambda C\right) = (1-\lambda)\Phi(K) +_p \lambda\Phi(C)$. And again, L_p combination preserves polytopes, but may not preserve smooth boundaries.

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2.7 Cone volume measure, log-Minkowski problem, log-Brunn-Minkowski conjecture

Cone volume measure

For a convex body K in $\mathbb{R}^n$ containing the origin, the cone volume measure, V_K on S^{n-1} is defined by

$$dV_K = \frac{1}{n} h_K \, dS_K$$

where S_K is the surface area measure of K, and h_K is its support function. So, the total measure of S^{n-1} is $V_K(S^{n-1}) = V(K)$. The origin of the name is best illustrated if we consider the case of polytopes. If P is a polytope containing the origin, with facets F_i and corresponding exterior unit normal u_i for $i = 1, \ldots, k$, then the cone volume measure is concentrated on $\{u_1, \ldots, u_k\}$ and we have for $i = 1, \ldots, k$

$$V_P(u_i) = \frac{1}{n} \cdot h_K \cdot \mathcal{H}(F_i)$$

which is the volume of the cone $\mathrm{conv}\{o, F_i\}$. So, the cone volume measure in this case is a discrete measure, that can be written as

$$V_k = v_1 \delta_{u_1} + \ldots + v_k \delta_{u_k}$$

where $v_i = V(\mathrm{conv}\{o, F_i\})$ and δ_{u_i} is the delta measure concentrated on u_i. For $K \in \mathcal{K}^n_{(o)}$, we can write

$$V_K(\omega) = \frac{1}{n} \int_{x \in \nu_K^{-1}(\omega)} \langle x, \nu_K(x) \rangle d\,\mathcal{H}(x)$$

for a Borel set $\omega \subset S^{n-1}$ where $\nu_K : \partial' K \to S^{n-1}$ is the spherical Gauss map that assigns to $x \in \partial' K$ its unique exterior unit normal.

log-Minkowski problem

Given a non-negative measurable function f on S^{n-1} with $0 < \int_{S^{n-1}} f \, d\mathcal{H} < \infty$, the Monge-Ampère equation on the sphere S^{n-1} corresponding to the logarithmic Minkowski problem is given by

$$h \det(\nabla^2 h + h \, \mathrm{Id}) = nf \tag{2.18}$$

Given μ, a finite non-trivial Borel measure on S^{n-1} we call a non negative function h on S^{n-1} which is the restriction of the support function h_K of a convex body K to S^{n-1}, an Alexandrov solution to the problem 2.18, if

$$d\mu = dV_K = \tfrac{1}{n} h_K \, dS_K. \tag{2.19}$$

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The logarithmic Minkowski problem then asks for the necessary and sufficient conditions for μ to be the cone volume measure of a convex body K in $\mathbb{R}^n$ containing the origin.

The concept of cone volume measure was first introduced by Firey in his paper [76] and has since become a widely employed concept following Gromov, Milman [85], for example, in Guédon, Mendelson, Naor [22], Naor [124], Paouris, Werner [128]. Firey [76] posed the log-Minkowski problem in 1974 and verified that in the case when f is a positive constant function, (2.18) has a unique even solution provided by a centered ball. The general case of uniqueness of solution for a positive constant function f without the evenness condition was established by Andrews [6] if $n = 2, 3$, and Brendle, Choi, Daskalopoulos [43] if $n \geq 4$. Chen, Li, Zhu [50] showed that uniqueness may not hold if f is not a constant function. The log-Minkowski problem remains open.

log-Minkowski conjectures

Lutwak [116]'s famous logarithmic Minkowski conjecture from 1993 states that given an even positive function f, (2.18) has a unique even solution. A more restricted version of this is the following Conjecture 2.7.1.

Conjecture 2.7.1 (log-Minkowski Conjecture 1). *Given a positive, even, C^∞ function f,*

$$h\det(\nabla^2 h + h\,\mathrm{Id}) = nf \tag{2.20}$$

has a unique even solution.

The above conjecture is very closely related to the following logarithmic analogue of Minkowski's inequality (2.13) (see Böröczky, Lutwak, Yang, Zhang [39] for origin symmetric bodies and Böröczky, Kalantzopoulos [38] for centered convex bodies). In fact, it follows from Böröczky, Lutwak, Yang, Zhang [40], that Conjecture 2.7.1 is equivalent to Conjecture 2.7.2 for origin symmetric convex bodies with C^∞_+ boundaries.

Conjecture 2.7.2 (log-Minkowski Conjecture 2). *If K and C are convex bodies in $\mathbb{R}^n$ whose centroid is the origin, then*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K \geq \frac{V(K)}{n} \log \frac{V(C)}{V(K)} \tag{2.21}$$

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for compact convex sets K_i, C_i of dimension at least one such that K_i, C_i are dilates for $i = 1, \ldots, m$ and $\sum_{i=1}^m \dim K_i = n$.

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If $V(K) = V(C)$ in 2.21, then it can be equivalently written as

$$\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K \tag{2.22}$$

with the same equality case as above.

More precisely, the log-Minkowski Conjecture 2.7.1 can be expressed as follows: if $V_K = V_C$ for convex bodies K and C in $\mathbb{R}^n$ with their centroids at the origin,then the equality conditions in Conjecture 2.7.2 hold.

Nayar and Tkocz [125] provide examples that show that the choice of the right translates of K and C are important in Conjecture 2.7.2. It's important to note that Conjecture 2.7.2 is invariant under applying the same non-singular linear transformation to both K and C.

log-Minkowski conjecture: some known cases

- **origin symmetric convex bodies in $\mathbb{R}^2$:** Böröczky, Lutwak, Yang, Zhang [39] proved the log-Minkowski conjecture 2.7.2 in this case. However, the general planar case is still open.

- **complex bodies (in $\mathbb{C}^n$):** Rotem [136] verified the conjecture for complex bodies.

- **K and C are invariant under n-hyperplane symmetry:** When K and C are invariant under reflections through n indpendent hyperplanes, Böröczky, Kalantzopoulos [38] verified the conjecture together with the equality characterization. Further, Böröczky and De [33] verified a stability version in this case (see Chapter 4).

- **unconditional convex bodies:** The case of unconditional bodies is a particular case of the above. Saroglou [137] earlier verified the conjecture in this case along with complete equality characterization.

- **C is origin symmetric, K is a zonoid:** van Handel [148] verified the conjecture in this case with equality characterization only for K with C^2_+ boundary.

- **C is a centered convex body, K is a centered ellipsoid:** Guan, Ni [86] verifies the conjecture in this case via the Jensen and Blaschke-Santaló inequality.

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- **local versions:** The local estimates by Kolesnikov, Milman [106], and the use of the continuity method in PDE by Chen, Huang, Li, Liu [48] show that the log-Minkowski conjecture 2.7.2 holds for origin symmetric convex bodies K, C when K is close to being a centered ellipsoid (with the equality characterization when K has C^2_+ boundary). Putterman [133] also provides a recent proof of the same. Here K is said to be close to an origin symmetric ellipsoid E if there exists $c_n > 0$ depending only on n, such that

  $$E \subset K \subset (1 + c_n)E.$$

  For $q > 2$, and the dimension n high enough, Kolesnikov, Milman [106] show that similar results hold for linear images of Hausdorff neighborhoods of l_q balls.

  Colesanti, Livshyts, Marsiglietti [55], Kolesnikov, Livshyts [105] and Hosle, Kolesnikov, Livshyts [95] also consider local versions of the log-Minkowski conjecture 2.7.2.

- **$n = 2$, convex bodies K and C are in dilated position:** In $\mathbb{R}^2$, if the convex bodies K and C are in the so called "dilation position" as described by Xi, Leng [151], the log-Minkowski inequality 2.7.2 holds together with the equality characterization. In fact, Xi, Leng [151] also showed that Dar's conjecture (2.4) holds in the plane for convex bodies in dilated position.

Existence of solution to the log-Minkowski problem: some known cases

Building on previous partial results and related research by Andrews [6], Chou, Wang [53], He, Leng, Li [90], Henk, Schürman,Wills [92], Stancu [145], and Xiong [152], Böröczky, Lutwak, Yang, and Zhang [40] gave the following characterization of even cone volume measures via the "subspace concentration condition"s:

THEOREM 2.7.3. *Given a non-trival finite even Borel measure μ on S^{n-1}, there exists an origin symmetric convex body $K \in \mathcal{K}^n_e$ with $\mu = V_K$ if and only if*

(i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *for any proper linear subspace $L \subset \mathbb{R}^n$;*

(ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *equality in (i) is equivalent to the existence of a complementary linear subspace $L' \subset \mathbb{R}^n$ with* $\operatorname{supp} \mu \subset L \cup L'$.

The cone volume measure V_K satisfies (ii) if and only if $K = C + C'$ for compact convex sets $C \subset L^\perp$ and $C' \subset L'^\perp$. A finite Borel measure μ on S^{n-1} is said to satisfy the

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strict subspace concentration condition if $\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1})$ for any proper linear subspace $L \subset \mathbb{R}^n$.

If a finite non-trivial even measure μ on S^{n-1} is invariant under linear reflections $A_1, \dots A_n$ through n independent hyperplanes, then Böröczky, Kalantzopoulos [38] showed that there exists a convex body K invariant under the same reflections such that $\mu = V_K$ if and only if K satisfies the subspace concentration condition (i) and (ii) for any proper subspace also invariant under the same reflections.

Böröczky, Henk [31] (see also Henk, Linke [91]) showed that the cone volume measure of a centered convex body satisfies the subspace concentration condition whereas in [32], the same authors show that if $K \in \mathcal{K}^n$ satisfies $V_K(L \cap S^{n-1}) \geq (1-\varepsilon) \cdot \frac{\dim L}{n} \cdot V(K)$ for a proper linear subspace $L \subset \mathbb{R}^n$ and a small $\varepsilon > 0$, then K is close to the sum of two lower dimensional compact convex sets lying in complementary subspaces.

Freyer, Henk, Kipp [78] provide certain so-called Affine Subspace Concentration Conditions for the cone volume measure of centered polytopes.

In the case of possibly non-even measures, the log Minkowski problem (2.19) remains wide open. Chen, Li, Zhu [50] offer the best sufficient condition that a non-trival finite Borel measure μ on S^{n-1} is a cone volume measure if it satisfies the subspace concentration condition (solving for example the case of absolutely continuous measures). Some obstruction is provided by Böröczky, Hegedűs [37] where they characterized the restriction of a cone volume measure to a pair of antipodal points.

log-Brunn-Minkowski conjecture

Recall that the logarithmic or L_0 linear combination for convex bodies $K, C \in \mathcal{K}_o^n$ and for $\lambda \in (0,1)$ is given by

$$(1-\lambda)K +_0 \lambda C = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K(u)^{1-\lambda} h_C(u)^{\lambda} \ \forall u \in S^{n-1}\}.$$

The following log-Brunn-Minkowski conjecture was proposed by Böröczky, Lutwak, Yang, Zhang [39] for origin symmetric convex bodies and by Martin Henk for centered convex bodies.

Conjecture 2.7.4 (log-Brunn-Minkowski Conjecture). *If $\lambda \in (0,1)$ and K and C are centered convex bodies in $\mathbb{R}^n$, then*

$$V((1-\lambda)K +_0 \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda} \tag{2.23}$$

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with equality if and only if $K = K_1 + \ldots + K_m$ *and* $C = C_1 + \ldots + C_m$ *for compact convex sets* K_i, C_i *of dimension at least one such that* K_i, C_i *are dilates for* $i = 1, \ldots, m$ *and* $\sum_{i=1}^m \dim K_i = n$*.*

Note that for convex bodies K and C containing the origin in their interior, we have $(1-\lambda)K +_0 \lambda C \subset (1-\lambda)K + \lambda C$. However, $(1-\lambda)K +_0 \lambda C$ can be much smaller than $(1-\lambda)K + \lambda C$. For example consider $K = [\frac{-1}{t}, \frac{1}{t}] \times [-t, t]$ and $C = [-t, t] \times [\frac{-1}{t}, \frac{1}{t}]$ in $\mathbb{R}^2$ for $t > 0$. then

$$\begin{aligned} \tfrac{1}{2} K +_0 \tfrac{1}{2} C &= [-1, 1]^2 \\ \tfrac{1}{2} K + \tfrac{1}{2} C &= \left[-\tfrac{1}{2}(t + \tfrac{1}{t}), \tfrac{1}{2}(t + \tfrac{1}{t})\right]^2. \end{aligned} \tag{2.24}$$

So as we increase t, $\frac{1}{2} K + \frac{1}{2} C$ becomes much larger than $\frac{1}{2} K +_0 \frac{1}{2} C$. And as such the log-Brunn-Minkowski inequality is significantly stronger than the Brunn-Minkowski inequality for centered convex bodies.

log-Brunn-Minkowski conjecture: some equivalent and related formulations

Böröczky, Lutwak, Yang, Zhang [39] show that if the log-Brunn-Minkowski (2.23) holds for $K, C \in \mathcal{K}_o^n$ and for all $\lambda \in (0,1)$, it implies the log-Minkowski inequality (2.21). In fact, according to [39] for any family $\mathcal{F}$ of convex bodies closed under L_0 linear combination, the log-Minkowski inequality (2.21) for all $K, C \in \mathcal{F}$ is equivalent to the log-Brunn-Minkowski inequality (2.23) for all $K, C \in \mathcal{F}$ and $\lambda \in (0,1)$.

Kolesnikov, Milman [106] and Putterman [133] have derived the following conjectured inequality for for origin symmetric convex bodies K and C in $\mathbb{R}^n$

$$\frac{V(K,C;1)^2}{V(K)} \geq \frac{n-1}{n} V(K,C;2) + \frac{1}{n} \int_{S^{n-1}} \frac{h_C^2}{h_K^2}\, dV_K \tag{2.25}$$

which is stronger than Minkowski's second inequality (2.11), and is equivalent to the log-Brunn-Minkowski conjecture without the characterization of equality.

And thus for origin symmetric convex bodies K and C in $\mathbb{R}^n$, we have the following three equivalent forms of the log-Brunn-Minkowski conjecture, without the characterization of equality in the third case:

- $V((1-\lambda)K +_0 \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda}$
- $\int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K \geq \frac{V(K)}{n} \log \frac{V(C)}{V(K)}$

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- $\frac{V(K,C;1)^2}{V(K)} \geq \frac{n-1}{n} V(K,C;2) + \frac{1}{n} \int_{S^{n-1}} \frac{h_C^2}{h_K^2} \, dV_K$

Kolesnikov, Milman [106] derives another equivalent formulation in terms of the so-called Hilbert-Brunn-Minkowski operator.

Saroglou [137] shows that for origin symmetric convex bodies the log-Brunn-Minkowski conjecture is equivalent to the so-called B-property which says that for an origin symmetric convex body K in $\mathbb{R}^n$ and a positive definite diagonal matrix in $\mathrm{GL}(n, \mathbb{R})$, the function $s \mapsto V([-1,1]^n \cap \Phi^s K)$ is log-concave for $s \in \mathbb{R}$.

Nayar, Tkocz [126] provides another equivalent form of the log-Brunn-Minkowsi conjecture for origin symmetric convex bodies in terms of the "strong B-property": if L is an n-dimensional subspace of $\mathbb{R}^N$ ($N > n$), then $\mathrm{V}(L \cap \prod_{i=1}^N [-e^{t_i}, e^{t_i}])$ is a log-concave function of $(t_1, \dots, t_N) \in \mathbb{R}^N$.

According to Saroglou [138] if the log-Brunn-Minkowski Conjecture (2.23) holds for any origin symmetric convex bodies K and C and $\lambda \in (0,1)$, then it holds for any even log-concave measure μ on $\mathbb{R}^n$:

$$\mu((1-\lambda)K +_0 \lambda C) \geq \mu(K)^{1-\lambda} \mu(C)^{\lambda}. \tag{2.26}$$

Also in Saroglou [138], it is shown that if (2.26) holds for the Gaussian measure $\mu = \gamma_n$, it implies the log-Brunn-Minkowski Conjecture (2.23) for origin symmetric convex bodies.

log-Brunn-Minkowski conjecture: some known cases

The Log-Brunn-Minkowski Conjecture 2.7.4 remains open. Here we list some cases in which it has been verified.

Since the log-Minkowski and log-Brunn-Minkowski conjectures are equivalent on a family of convex bodies containing the origin and closed under L_0 linear combination (especially in the case of origin symmetric bodies), all the verified cases of the log-Minkowski conjecture in these cases also hold for the log-Brunn-Minkowski conjecture.

Böröczky, Lutwak, Yang, Zhang [39] verified it in $\mathbb{R}^2$ for origin symmetric convex bodies, although the general planar case remains open. Rotem [136] proved it for complex bodies. Saroglou [137] treated the unconditional case along with the equality characterization. The case when the convex bodies are invariant under n-hyperplane symmetries was verified by Böröczky, Kalantzopoulos [38] and a stability version was also provided by Böröczky and De [33] (see section 2.8) and Chapter 4 for more details).

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When K and C are origin symmetric convex bodies in the neighborhood of a fixed centered ellipsoid E, that is, when $E \subset K, C \subset (1+c_n)E$ for some $c_n > 0$ depending only on n, Chen, Huang, Li, Liu [48] extended the local estimate of Kolesnikov, Milman [106] to establish the log-Brunn-Minkowski conjecture in this case, where as [106] and the method of [48] provide an analogous result for linear images of l_q balls for $q > 2$ if the dimension n is high enough. Chen, Feng, Liu [49] established some partial results in $\mathbb{R}^3$. Earlier, Colesanti, Livshyts, Marsiglietti [55] had handled the case when K and C are in a C^2 neighbourhood of E.

Stability of solution to the log-Minkowski problem

Stability of the solution to the log-Minkowski problem has been established in certain cases where we are aware of the solution's uniqueness. For convex bodies invariant under reflections through n independent hyperplanes (which includes the case of unconditional convex bodies), stability of the solution was established by Böröczky, De [35] (see section 2.9 and Chapter 5). Ivaki [101] verified a stability version in the case when the given function f in (2.18) is positive constant in which case Firey had shown the only origin symmetric solution is the centered ball. In $\mathbb{R}^3$, and for a possibly non-even f that is C^α and close to a constant function, Chen, Feng, Liu [49] proved the uniqueness results while Andrews [6] and Brendle, Choi, Daskalopoulos [43] provided uniqueness results if $n \geq 4$. Establishing a stability version in this case remains open.

2.8 Stability of log-Minkowski and log-Brunn-Minkowski inequalities under n-hyperplane symmetries

We say that $A \in \mathrm{GL}(n)$ is a linear reflection associated to the linear $(n-1)$-dimensional space $H \subset \mathbb{R}^n$ if A fixes the points of H and $\det A = -1$. In this case, there exists $u \in \mathbb{R}^n \backslash H$ such that $Au = -u$ where the invariant subspace $\mathbb{R}u$ is uniquely determined (see Davis [60], Humphreys [98], Vinberg [149]). It follows that a linear reflection A is a classical "orthogonal" reflection if and only if $A \in O(n)$, that is, when $H = u^\perp$.

Here, we consider the case of the log-Brunn-Minkowski (or equivalently the log-Minkowski) conjecture where the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1, \ldots, A_n$ through n independent hyperplanes. In that case, we say K and C are invariant under n-hyperplane symmetry. Note that the case of unconditional convex bodies (symmetric with respect to reflections through the co-ordinate hyperplanes) is a particular case.

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Saroglou [137] verified the conjectures in the case of unconditional bodies along with the characterization of equality cases. Based on ideas from Barthe, Fradelizi [19] and Barthe, Cordero-Erausquin [18] where they verified the classical Mahler conjecture and Slicing conjecture, and following the result by Saroglou [137], Böröczky, Kalantzopoulos [38] verified the logarithmic Brunn-Minkowski and Minkowski conjectures under hyperplane symmetry assumption.

THEOREM 2.8.1 (Böröczky, Kalantzopoulos)**.** *If the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1,\ldots,A_n$ through n hyperplanes $H_1,\ldots,H_n$ with $H_1\cap\ldots\cap H_n=\{o\}$, then*

$$V((1-\lambda)\cdot K+_0\lambda\cdot C) \;\geq\; V(K)^{1-\lambda}V(C)^{\lambda} \tag{2.27}$$

$$\int_{S^{n-1}}\log\frac{h_C}{h_K}\,dV_K \;\geq\; \frac{V(K)}{n}\log\frac{V(C)}{V(K)}, \tag{2.28}$$

with equality in either inequality if and only if $K=K_1+\ldots+K_m$ and $C=C_1+\ldots+C_m$ for compact convex sets $K_1,\ldots,K_m,C_1,\ldots,C_m$ of dimension at least one and invariant under $A_1,\ldots,A_n$ where K_i and C_i are dilates, $i=1,\ldots,m$, and $\sum_{i=1}^m\dim K_i=n$.

Our main results here are the following stability versions of the log-Brunn-Minkowski and log-Minkowski inequalities.

THEOREM 2.8.2. *If $\lambda\in[\tau,1-\tau]$ for $\tau\in(0,\frac{1}{2}]$, the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1,\ldots,A_n$ through n hyperplanes $H_1,\ldots,H_n$ with $H_1\cap\ldots\cap H_n=\{o\}$, and*

$$V((1-\lambda)\cdot K+_0\lambda\cdot C)\leq(1+\varepsilon)V(K)^{1-\lambda}V(C)^{\lambda}$$

for $\varepsilon>0$, then for some $m\geq 1$, there exist compact convex sets $K_1,C_1,\ldots,K_m,C_m$ of dimension at least one and invariant under $A_1,\ldots,A_n$ where K_i and C_i are dilates, $i=1,\ldots,m$, and $\sum_{i=1}^m\dim K_i=n$ such that

$$K_1+\ldots+K_m\subset\;K\;\subset\left(1+c^n\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(K_1+\ldots+K_m)$$

$$C_1+\ldots+C_m\subset\;C\;\subset\left(1+c^n\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(C_1+\ldots+C_m)$$

where $c>1$ is an absolute constant.

From Theorem 2.8.2, in turn, we deduce a stability version of the log-Minkowski inequality.

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THEOREM 2.8.3. *If the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1, \ldots, A_n$ through n hyperplanes $H_1, \ldots, H_n$ with $H_1 \cap \ldots \cap H_n = \{o\}$, and*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \frac{dV_K}{V(K)} \leq \frac{1}{n} \cdot \log \frac{V(C)}{V(K)} + \varepsilon$$

for $\varepsilon > 0$, then for some $m \geq 1$, there exist compact convex sets $K_1, C_1, \ldots, K_m, C_m$ of dimension at least one and invariant under $A_1, \ldots, A_n$ where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$ such that

$$\begin{aligned} K_1 + \ldots + K_m \subset \quad K \quad &\subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(K_1 + \ldots + K_m) \\ C_1 + \ldots + C_m \subset \quad C \quad &\subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(C_1 + \ldots + C_m) \end{aligned}$$

where $c > 1$ is an absolute constant.

Ivaki [99], Theorem 2.1 provides an improved version of Theorem 2.8.3 when K is a ball centered at the origin (and hence $m = 1$) and C does not need to satisfy any symmetry assumption (only translated in a suitable way) with an error term of the order of $\varepsilon^{\frac{1}{n+1}}$ instead of $\varepsilon^{\frac{1}{95n}}$.

The main ideas behind establishing Theorems 2.8.2 and 2.8.2 are as follows. First we derive some stability versions for unconditional convex bodies.

If K and C are unconditional convex bodies in $\mathbb{R}^n$ and $\lambda \in (0, 1)$, then the co-ordinate wise product is defined as follows

$$\begin{aligned} K^{1-\lambda} \cdot C^{\lambda} \quad = \quad &\{(\pm|x_1|^{1-\lambda}|y_1|^{\lambda}, \ldots, \pm|x_n|^{1-\lambda}|y_n|^{\lambda}) \in \mathbb{R}^n : \\ &(x_1, \ldots, x_n) \in K \quad \text{and} \quad (y_1, \ldots, y_n) \in C\}. \end{aligned}$$

$K^{1-\lambda} \cdot C^{\lambda}$ is known to be an unconditional body. Hölder's inequality implies that (see Saroglou [137])

$$K^{1-\lambda} \cdot C^{\lambda} \subset (1 - \lambda) \cdot K +_0 \lambda \cdot C.$$

Therefore, we have the following inequality.

THEOREM 2.8.4 (Bollobas & Leader, Uhrin, Saroglou)**.** *If K and C are unconditional bodies and $\lambda \in (0, 1)$, then*

$$V((1 - \lambda) \cdot K +_0 \lambda \cdot C) \geq V(K^{1-\lambda} \cdot C^{\lambda}) \geq V(K)^{1-\lambda} V(C)^{\lambda}. \tag{2.29}$$

Saroglou [137] also characterized the equality case as follows

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(i) $V(K^{1-\lambda}\cdot C^{\lambda}) = V(K)^{1-\lambda}V(C)^{\lambda}$ if and only if $C = \Phi K$ for a positive definite diagonal matrix Φ.

(ii) $V((1-\lambda)\cdot K +_0 \lambda\cdot C) = V(K)^{1-\lambda}V(C)^{\lambda}$ if and only if $K = K_1\oplus\ldots\oplus K_m$ and $L = L_1\oplus\ldots\oplus L_m$ for unconditional compact convex sets $K_1,\ldots,K_m,L_1,\ldots,L_m$ of dimension at least one where K_i and L_i are dilates, $i=1,\ldots,m$.

Using the stability version of the Prékopa-Leindler inequality for log-concave functions (Theorem 2.3.2) and noting that $f(x_1,\ldots,x_n) = \mathbf{1}_{K_+}(e^{x_1},\ldots,e^{x_n})e^{x_1+\ldots+x_n}$, $g(x_1,\ldots,x_n) = \mathbf{1}_{C_+}(e^{x_1},\ldots,e^{x_n})e^{x_1+\ldots+x_n}$ and $h(x_1,\ldots,x_n) = \mathbf{1}_{(K^{1-\lambda}\cdot C^{\lambda})_+}(e^{x_1},\ldots,e^{x_n})$ $e^{x_1+\ldots+x_n}$ are log-concave, we derive the following stability version of the Bollobas-Leader inequality (the second inequality in Theorem 2.8.4).

THEOREM 2.8.5. *If $\lambda\in[\tau,1-\tau]$ for $\tau\in(0,\frac{1}{2}]$, and the unconditional convex bodies K and C in $\mathbb{R}^n$ satisfy*

$$V(K^{1-\lambda}\cdot C^{\lambda}) \le (1+\varepsilon)V(K)^{1-\lambda}V(C)^{\lambda}$$

for $\varepsilon>0$, then there exists positive definite diagonal matrix Φ such that

$$V(K\Delta(\Phi C)) < c^n n^n\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{19}}V(K) \quad \text{and} \quad V((\Phi^{-1}K)\Delta C) < c^n n^n\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{19}}V(C)$$

where $c>1$ is an absolute constant.

Combining, Theorem 2.8.5 and a stability version for the first inequality in Theorem 2.8.4 when $C=\Phi K$ is a linear image of K under a positive definite diagonal matrix Φ, leads to the following stability version of the log-Brunn-Minkowski inequality in the case of unconditional convex bodies.

THEOREM 2.8.6. *If $\lambda\in[\tau,1-\tau]$ for $\tau\in(0,\frac{1}{2}]$, and the unconditional convex bodies K and C in $\mathbb{R}^n$ satisfy*

$$V((1-\lambda)\cdot K +_0 \lambda\cdot C) \le (1+\varepsilon)V(K)^{1-\lambda}V(C)^{\lambda}$$

for $\varepsilon>0$, then for some $m\ge 1$, there exist $\theta_1,\ldots,\theta_m>0$ and unconditional compact convex sets $K_1,\ldots,K_m$ such that $\operatorname{lin} K_i$, $i=1,\ldots,m$, are complementary coordinate subspaces, and

$$K_1\oplus\ldots\oplus K_m \subset \quad K \quad \subset \left(1+c^n\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(K_1\oplus\ldots\oplus K_m)$$

$$\theta_1K_1\oplus\ldots\oplus\theta_mK_m \subset \quad C \quad \subset \left(1+c^n\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(\theta_1K_1\oplus\ldots\oplus\theta_mK_m)$$

where $c>1$ is an absolute constant.

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When a convex body K is symmetric under reflections through n independent hyperplanes, it is invariant under a Coxeter group G of rank n. Using ideas from Barthe, Fradelizi [19] and Barthe, Cordero-Erausquin [18], we see that G has a simplicial cone C as fundamental domain, and reflections through the walls of C generate G. We can then map C into a "co-ordinate corner" ($\mathbb{R}^n_{\geq 0}$) by a linear transform. Then the already established results for unconditional convex bodies apply, and we deduce Theorem 2.8.2 and subsequently Theorem 2.8.3. For a complete presentation of the results, see Chapter 4.

2.9 Stability of log-Minkowski problem under n-hyperplane symmetries

Under n-hyperplane symmetry assumption, Böröczky, Kalantzopoulos [38] provided the following characterization of cone-volume measures. Here, we note that for any group $G \subset O(n)$ acting on $\mathbb{R}^n$ without non-zero fixed points, there exist only finitely many G invariant linear subspaces of $\mathbb{R}^n$ where G is a Coxeter group if it is generated by reflections through n independent hyperplanes.

THEOREM 2.9.1 (Böröczky, Kalantzopoulos)**.** *Let $G \subset O(n)$ be a Coxeter group acting on $\mathbb{R}^n$ without non-zero fixed points. For a finite non-trivial Borel measure μ on S^{n-1} invariant under G, there exists a G invariant Alexandrov solution of the logarithmic Minkowski problem* (2.19) *if and only if*

(i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *for any G-invariant proper linear subspace L;*

(ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *in (i) for an invariant proper linear subspace L is equivalent to* $\operatorname{supp} \mu \subset L \cup L^{\perp}$.

In addition, if strict inequality holds in (i) for each G-invariant proper linear subspace L, then the G invariant solution is unique.

We note that the measure in Theorem 2.9.1 may not be even; for example, possibly $\mu = V_K$ for a regular simplex K whose centroid is the origin.

Böröczky, Kalantzopoulos [38] showed that $V_K(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot V(K)$ holds in Theorem 2.9.1 (i) for a proper invariant subspace L if and only if $K = (K \cap L) \oplus (K \cap L^{\perp})$. Further, according to [38], $V_K = V_C$ holds for convex bodies K and C in $\mathbb{R}^n$ invariant under a Coxeter group $G \subset O(n)$ acting on $\mathbb{R}^n$ without non-zero fixed points if and

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only if $V(K) = V(C)$, and $K = K_1 \oplus \ldots \oplus K_m$ and $C = C_1 \oplus \ldots \oplus C_m$ for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one and invariant under G where K_i and C_i are dilates for $i = 1, \ldots, m$. Naturally, if $m = 1$, then $K = C$.

[38] also verified the log-Minkowski inequality for convex bodies with n hyperplane symmetry as follows.

THEOREM 2.9.2 (Böröczky, Kalantzopoulos). *If the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1, \ldots, A_n$ through n independent linear $(n-1)$-planes $H_1, \ldots, H_n$, then*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K \geq \frac{V(K)}{n} \log \frac{V(C)}{V(K)},$$

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one and invariant under $A_1, \ldots, A_n$ where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$.

Further, Boroczky, De [33] proved the stability version of the logarithmic-Minkowski inequality Theorem 2.8.3 (see Chapter 4) for convex bodies with many hyperplane symmetries.

Our main goal here is to establish a stability version of of Theorem 2.9.1 under the same condition of n hyperplane symmetry. In order to prepare for the stability version Theorem 2.9.3 of Theorem 2.9.1, for any compact $X \subset S^{n-1}$ and $\varrho \in [0, 2]$, we consider the tube

$$\Psi(X, \varrho) = \{u \in S^{n-1} : \exists x \in X,\ \|x - u\| \leq \varrho\}.$$

The cone volume measure V_K of a convex body K readily satisfies $dV_{tK} = t^n dV_K$ for $t > 0$. Therefore, when comparing the cone volume measures of convex bodies K and C, we may asssume that $V(K) = V(C) = 1$, and hence V_K and V_C are probability measures on S^{n-1}.

One natural distance to consider between two probability measures μ and ν on S^{n-1} is the l_1 Wasserstein distance. The family of Lipschitz functions on S^{n-1}, for for $\theta > 0$ is given by

$$\mathrm{Lip}_\theta = \Big\{f : S^{n-1} \to \mathbb{R} : \forall a, b \in S^{n-1},\ |f(a) - f(b)| \leq \theta \|a - b\|\Big\}. \tag{2.30}$$

Now the Wasserstein distance of the Borel probability measures μ and ν on S^{n-1} is

$$d_W(\mu, \nu) = \sup \left\{ \int_{S^{n-1}} f\, d\mu - \int_{S^{n-1}} f\, d\nu : f \in \mathrm{Lip}_1 \right\}.$$

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It is known that convergence of a sequence of probability measures with respect to the Wasserstein distance is equivalent with weak convergence.

First we establish some estimates bounding the diameter of a G-invariant convex body K in terms of V_K, and a condition yielding that a convex body with hyperplane symmetries is not close to be the direct sum of lower dimensional invariant compact convex sets; then using Theorem 2.9.2 and Theorem 2.8.3, we establish the following stability version of Theorem 2.9.1 (see Chapter 5 for details).

THEOREM 2.9.3. *Let $G \subset O(n)$ be a Coxeter group acting on $\mathbb{R}^n$ without non-zero fixed points. If μ_1 and μ_2 are G-invariant Borel probability measures on S^{n-1}, and*

$$\begin{aligned} \mu_1\Big(\Psi(L\cap S^{n-1},\delta)\Big) &\leq (1-\tau)\cdot \tfrac{\dim L}{n}, \\ \mu_2\Big(\Psi(L\cap S^{n-1},\delta)\Big) &\leq (1-\tau)\cdot \tfrac{\dim L}{n} \end{aligned} \tag{2.31}$$

for $\delta,\tau \in (0,\frac{1}{2})$ and for any G-invariant proper subspace L, then the unique G invariant Alexandrov solution h_i of the logarithmic Minkowski problem (2.19) for $\mu=\mu_i$, $i=1,2$, satisfies

$$\|h_1-h_2\|_\infty \leq \gamma_0\cdot d_W(\mu_1,\mu_2)^{\frac{1}{95n}} \tag{2.32}$$

$$r_0 \leq h_1, h_2 \leq R_0 \tag{2.33}$$

where for some absolute constant $c>1$, we have

- *$R_0=n$, $r_0=\frac{1}{e}$, $\gamma_0=c^n$ and the condition (2.31) is irrelevant provided the action of G is irreducible;*
- *$R_0=\left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$, $r_0=\frac{n^{\frac{n}{2}}}{6^n}\left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}$ and $\gamma_0=\frac{c^n}{\tau}\cdot\delta^{\frac{-3n}{\tau}}n^{\frac{12n}{\tau}}$ provided the action of G is reducible.*

Next, we show that Theorem 2.9.3 can actually be extended to the case when $\mu_1(S^{n-1})\neq\mu_2(S^{n-1})$. In this case, instead of the Wasserstein distance we use the bounded Lipschitz distance $d_{\rm bL}(\mu,\nu)$ of two Borel measures μ and ν on S^{n-1} (see Dudley [64]) given by

$$d_{\rm bL}(\mu,\nu)=\sup\left\{\int_{S^{n-1}} f\,d\mu-\int_{S^{n-1}} f\,d\nu : \ f\in {\rm Lip}_1 \text{ and } \|f\|_\infty\leq 1\right\}.$$

We have the following stability version for the case $\mu_1(S^{n-1})\neq\mu_2(S^{n-1})$.

COROLLARY 2.9.4. *Let $G\subset O(n)$ be a Coxeter group acting on $\mathbb{R}^n$ without non-zero fixed points. If μ_1 and μ_2 are G-invariant finite Borel measures on S^{n-1} satisfying*

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$d_{\mathrm{bL}}(\mu_1,\mu_2) \le M = \min\{\mu_1(S^{n-1}),\mu_2(S^{n-1})\} > 0$ *and*

$$\begin{aligned} \mu_1\Big(\Psi(L\cap S^{n-1},\delta)\Big) &\le (1-\tau)\cdot\tfrac{\dim L}{n}, \\ \mu_2\Big(\Psi(L\cap S^{n-1},\delta)\Big) &\le (1-\tau)\cdot\tfrac{\dim L}{n} \end{aligned} \tag{2.34}$$

for $\delta,\tau\in(0,\frac{1}{2})$ and for any G-invariant proper subspace L, then the unique G invariant Alexandrov solution h_i of the logarithmic Minkowski problem (2.19) for $\mu=\mu_i$, $i=1,2$, satisfies

$$\|h_1-h_2\|_\infty \le \gamma_0 M^{\frac{1}{n}}\cdot d_{\mathrm{bL}}(\mu_1,\mu_2)^{\frac{1}{95n}} \tag{2.35}$$

$$r_0 M^{\frac{1}{n}} \le h_1,h_2 \le R_0 M^{\frac{1}{n}} \tag{2.36}$$

where for some absolute constant $c>1$, we have

- $R_0=2n$, $r_0=\frac{1}{e}$, $\gamma_0=c^n$ *and the condition (2.34) is irrelevant provided the action of G is irreducible;*
- $R_0=2\left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$, $r_0=\frac{n^{\frac{n}{2}}}{5^n}\left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}$ *and* $\gamma_0=\frac{c^n}{\tau}\cdot\delta^{\frac{-3n}{\tau}}n^{\frac{12n}{\tau}}$ *provided the action of G is reducible.*

We note here that the error term in Theorem 2.9.3 in terms of ε is not far from being optimal (see Chapter 5).

Next, we consider two partial converses of Theorem 2.9.3 to show that concerning Theorem 2.9.3, both the conditions involved and the conclusion are of the right kind. The first result does not require any symmetry assumption.

THEOREM 2.9.5. *Let μ_1 and μ_2 be finite Borel measures on S^{n-1} such that there exists Alexandrov solution h_i of the logarithmic Minkowski problem (2.19) for $\mu=\mu_i$ and $i=1,2$. If $h_1,h_2<R$ for $R>0$, then*

$$d_{\mathrm{bL}}(\mu_1,\mu_2)\le\gamma(R,n)\cdot\sqrt{\|h_1-h_2\|_\infty}$$

where $\gamma(R,n)>0$ depends on R and n.

The proof of (2.9.5) is based on the argument of Hug, Schneider [97] where they prove that if $R>0$ and K and C are convex bodies in $\mathbb{R}^n$ satisfying $K,C\subset RB^n$, then

$$d_{\mathrm{bL}}(S_K,S_C)\le\tilde\gamma(R,n)\cdot\sqrt{d_\infty(K,C)} \tag{2.37}$$

where $\tilde\gamma(R,n)>0$ depends on R and n.

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Secondly, we show that if we have almost equality in Theorem 2.9.1 (ii) for measures μ_1 and μ_2 and a proper linear subspace L invariant under reflections through independent hyperplanes $H_1, \dots, H_n$, then even if μ_1 and μ_2 are close, it is possible that the solutions h_1 and h_2 of (2.19) are arbitrarily far away.

To show this, we use ideas from Böröczky, Henk [32] where the authors characterized convex bodies with centroid at the origin and satisfying almost equality in Theorem 2.9.1 (ii) as follows: if $\varepsilon \in (0, \tilde{\varepsilon}_0)$ and the convex body $K \subset \mathbb{R}^n$ has its centroid at the origin, and satisfies

$$V_K(L \cap S^{n-1}) \geq (1-\varepsilon) \cdot \frac{d}{n} \cdot V(K)$$

for a linear d-space L with $1 \leq d < n$, then

$$(1 - \tilde{\gamma} \cdot \varepsilon^{\frac{1}{5n}})(C + M) \subset K \subset C + M \tag{2.38}$$

for some compact convex set $C \subset L^{\perp}$, and complementary d-dimensional compact convex set M where $\tilde{\varepsilon}_0, \tilde{\gamma} > 0$ depend on the dimension n.

Further, using some estimates for the symmetric difference metric also found in [32] together with the above, we establish the following.

THEOREM 2.9.6. *Let $G \subset O(n)$ be a group acting without non-zero fixed points on $\mathbb{R}^n$, and let h be a positive G-invariant Alexandrov solution of (2.19) for a probability measure μ on S^{n-1} with $h < R$ for $R > \sqrt{n}$ such that*

$$\mu\Big(\Psi(L \cap S^{n-1}, \delta)\Big) \geq (1-\varepsilon) \cdot \frac{\dim L}{n}$$

for $\varepsilon \in (0, \frac{\varepsilon_0}{R^n})$, $\delta \in (0, \varepsilon]$ and a G-invariant proper subspace L where $\varepsilon_0 > 0$ depends on n. Then for any $t > 1$, there exists a positive G-invariant Alexandrov solution h_t of (2.19) for a probability measure μ_t on S^{n-1} such that

$$\begin{aligned} \|h - h_t\|_\infty &\geq t \\ d_W(\mu, \mu_t) &\leq \gamma(R,n)\varepsilon^{\frac{1}{10n}} \end{aligned}$$

where $\gamma(R,n) > 0$ depends on R and n.

For a complete presentation of the results, see Chapter 5.

2.10 L_p-Brunn-Minkowski Theory

Initiated by Lutwak [115, 116, 117], L_p-Brunn-Minkowski theory has underwent rapid development to become a main research area in modern convex geometry and geometric analysis.

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L_p-surface area measure

For $p \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$, the L_p-surface area measure $S_{K,p}$ on S^{n-1} is defined by

$$dS_{K,p} = h_K^{1-p}\, dS_K \tag{2.39}$$

For a convex body $K \in \mathcal{K}_{(o)}^n$ and any Borel set $\omega \subset S^{n-1}$, we can write

$$S_{K,p}(\omega) = \int_{x \in \nu_K^{-1}(\omega)} \langle x, \nu_K(x)\rangle^{1-p} d\mathcal{H}(x)$$

where ν_K is the spherical Gauss map. The cases $p = 1$ and $p = 0$ correspond to the surface area measure and the cone volume measure respectively.

L_p-Minkowski problem

For $p \in \mathbb{R}$, the Monge-Ampére equation on S^{n-1} corresponding to the L_p-Minkowski problem is

$$\begin{aligned} \det(\nabla^2 h + h\,\mathrm{Id}) &= h^{p-1} f \quad \text{if} \quad p > 1 \\ h^{1-p} \det(\nabla^2 h + h\,\mathrm{Id}) &= f \quad \text{if} \quad p \leq 1 \end{aligned} \tag{2.40}$$

where $f \in L_1(S^{n-1})$ is non-negative with $\int_{S^{n-1}} f d\mathcal{H} > 0$.

For a finite non-trivial Borel measure μ on S^{n-1}, $h = h_K|_{S^{n-1}}$ for a convex body $K \in \mathcal{K}_o^n$ is an Alexandrov solution of the L_p-Minkowski problem if

$$\begin{aligned} dS_K &= h_K^{p-1}\, d\mu \quad \text{if} \quad p > 1 \\ h_K^{1-p} dS_K &= d\mu \quad \text{if} \quad p \leq 1. \end{aligned} \tag{2.41}$$

Some known existence and uniqueness results

- $p > 1, p \neq n$: According to Hug, Lutwak, Yang, Zhang [96] (improving on Chou, Wang [53]), the L_p Minkowski problem (2.41) has a unique Alexandrov solution if and only if μ is not concentrated on any closed hemisphere (that is $L \cap S^{n-1}$ for a subspace L of codimension 1).

- $p = n$: According to Hug, Lutwak, Yang, Zhang [96], if μ not concentrated onto any closed hemisphere, there exists a convex body $K \in \mathcal{K}_o^n$ and $c > 0$ such that $\mu = c \cdot S_{K,n}$.

- $p = 1, 0$: Corresponds to classical and log-Minkowski problems.

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- $p \in (0,1)$: Chen, Li, Zhu [51] shows if μ is not concentrated onto any great subsphere, an Alexandrov solution exists, that is, there exists $K \in \mathcal{K}_o^n$ such that $S_{K,p} = \mu$. In $\mathbb{R}^2$, Böröczky, Trinh [41] provide complete characterization of L_p surface area measures. For $n \geq 3$, Bianchi, Böröczky, Colesanti, Yang [25] show the following: if L is the linear hull of $\operatorname{supp} \mu$ in $\mathbb{R}^n$ and $1 \leq \dim L \leq n-1$ where L, and if $\operatorname{supp} \mu$ is contained in a closed hemisphere centered at a point of $L \cap S^{n-1}$, then μ is an L_p surface area measure. Saroglou [139] provides some restriction. If $\mu(\omega) = V(\omega \cap L)$ for any Borel $\omega \subset S^{n-1}$, then μ is not a L_p surface area measure.

- $p \in (-n, 0)$: According to Bianchi, Böröczky, Colesanti, Yang [25], if μ has a density f with respect to $\mathcal{H}$ such that $f \in L_{\frac{n}{n+p}}(S^{n-1})$, then (2.40) has a solution. For $p < 0$ and discrete μ that is not concentrated on any closed hemisphere, if any n unit vectors in the support of μ are independent Zhu [154] provides a solution to the L_p-Minkowski problem.

- $p = -n$: Jian, Lu, Zhu [103] show the existence of a solution when f in (2.40) is unconditional and satisfies some other conditions. Li, Guang, Wang [88], Chou, Wang [53], Du [62] also provide some related results for $p = -n$.

- $p < -n$: According to a recent result by Li, Guang, Wang [87] there exists a C^4 solution of (2.40) for any positive C^2 f. Du [62] provides an example of a non-negative C^α function f that is positive everywhere but a fixed pair of antipodal points for which no solution exists.

For $p < 1$, the solution of the L_p-Minkowski problem (2.40) may not be unique even if f is positive and continuous. Examples are found in Chen, Li, Zhu [50, 51], Milman [121], Jian, Lu, Wang [102], Li, Liu, Lu [111], Li [110].

If f is a constant function in the L_p-Minkowski problem, combining Lutwak [116], Andrews [6], Andrews, Guan, Ni [7] and Brendle, Choi, Daskalopoulos [43], we have if $p > -n$, only solutions are centered balls; if $p = -n$, centered ellipsoids, where as there are multiple solutions if $p < -n$. See also Crasta, Fragalá [58], Ivaki, Milman [100] and Saroglou [140]. Ivaki [101] has provided stability versions in these cases. However, no stability results are known for possibly non-even solutions.

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L_p-Minkowski and L_p-Brunn-Minkowski conjectures

Analogous to Lutwak's log-Minkowski conjecture for the case $p = 0$, for $p \in (0, 1)$, it is also conjectured that the L_p-Minkowski problem (2.39) has a unique even solution for any positive, C^∞ and even f. The following conjecture is more general (for origin symmetric convex bodies see Böröczky, Lutwak, Yang, Zhang [39])

Conjecture 2.10.1 (**L_p-Minkowski Conjecture 1**). *For $p \in (0, 1)$ and centered convex bodies K and C in $\mathbb{R}^n$, if $S_{K,p} = S_{C,p}$, then $K = C$.*

Recall that for $p > 0$, $\alpha, \beta > 0$ and convex bodies K, C containing the origin,the L_p linear combination is given by

$$\alpha K +_p \beta C = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq [\alpha\, h_K(u)^p + \beta\, h_C(u)^p]^{\frac{1}{p}}\ \forall u \in S^{n-1}\}.$$

Note that if $p \geq 1$, $h_{\alpha K +_p \beta C} = [\alpha\, h_K^p + \beta\, h_C^p]^{\frac{1}{p}}$ by Minkowski's inequality. Firey [75] showed that if $p > 1$, $K, C \in \mathcal{K}_o^n$, then the Brunn-Minkowski inequality implies the L_p-Brunn-Minkowski inequality

$$V(\alpha\, K +_p \beta\, C)^{\frac{p}{n}} \geq \alpha\, V(K)^{\frac{p}{n}} + \beta\, V(C)^{\frac{p}{n}} \tag{2.42}$$

for any $\alpha, \beta > 0$ with equality if and only if K and C are dilates; which can be equivalently written in the following form

$$V((1-\lambda)K +_p \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda} \tag{2.43}$$

for $\lambda \in (0, 1)$ with equality if and only if $K = C$.

Analogous to the classical mixed volumes, Lutwak [115] defined the L_p mixed volumes as follows

$$V_p(K, C) = \frac{p}{n} \lim_{t\to 0^+} \frac{V(K +_p t\, C) - V(K)}{t} = \frac{1}{n} \int_{S^{n-1}} h_C^p\, dS_{K,p} = \int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K,$$

Note here that $V_1(K, C) = V(K, C; 1)$. Taking the first derivative of $\lambda \mapsto V((1-\lambda)K +_p \lambda C)^{\frac{p}{n}}$ leads to the L_p-Minkowski inequality

$$V_p(K, C) \geq V(K)^{\frac{n-p}{n}} V(C)^{\frac{p}{n}} \tag{2.44}$$

for $p > 1$ and $K, C \in \mathcal{K}_{(o)}^n$ where equality holds if and only if K and C are dilates.

In the case where $p > 1$, and $K, C \in \mathcal{K}_{(o)}^n$ with $V(K) = V(C)$, we have the following equivalent form

$$\int_{S^{n-1}} h_C^p\, dS_{K,p} \geq \int_{S^{n-1}} h_K^p\, dS_{K,p} \tag{2.45}$$

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where equality holds if and only if $K = C$.

For $p > 1$, Zhang [153], Ludwig, Xiao, Zhang [114] and Lutwak, Yang, Zhang [118] have extended the L_p-Brunn-Minkowski inequality to certain families of non-convex sets.

For $p \in (0,1)$, it is known that the L_p-Minkowski and the L_p-Brunn-Minkowski inequalities do not hold for general convex bodies $K, C \in \mathcal{K}^n_{(o)}$. But it is a conjecture of Böröczky, Lutwak, Yang, Zhang [39] that they hold at least for origin symmetric convex bodies (Böröczky, Kalantzopoulos [38] for centered convex bodies).

Conjecture 2.10.2 (**L_p-Minkowski Conjecture 2**). *If $p \in (0,1)$, and K, C are centered convex bodies in $\mathbb{R}^n$, then*

$$V_p(K,C) \geq V(K)^{\frac{n-p}{n}} V(C)^{\frac{p}{n}} \tag{2.46}$$

with equality if and only if K and C are dilates;

or equivalently,

$$\int_{S^{n-1}} h_C^p \, dS_{K,p} \geq \int_{S^{n-1}} h_K^p \, dS_{K,p} \tag{2.47}$$

when $V(K) = V(C)$ and equality holds if and only if $K = C$.

Conjecture 2.10.3 (**L_p-Brunn-Minkowski Conjecture**). *If $p \in (0,1)$, and K, C are centered convex bodies in $\mathbb{R}^n$, then*

$$V(\alpha\, K +_p \beta\, C)^{\frac{p}{n}} \geq \alpha\, V(K)^{\frac{p}{n}} + \beta\, V(C)^{\frac{p}{n}} \tag{2.48}$$

for any $\alpha, \beta > 0$ with equality if and only if K and C are dilates;

or equivalently,

$$V((1-\lambda)K +_p \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda} \tag{2.49}$$

for $\lambda \in (0,1)$ with equality if and only if $K = C$.

For $0 \leq q < p$, from Jensen's inequality we have $(1-\lambda)K +_q \lambda \subset (1-\lambda)K +_p \lambda C$. Then it follows from (2.43) that the L_q-Brunn-Minkowski conjecture implies the L_p-Brunn-Minkowski conjecture . Consequently we see that the log-Brunn-Minkowski conjecture would yield the L_p-Brunn-Minkowski conjecture for $p \in (0,1)$, which in turn would lead

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to the Brunn-Minkowski inequality. Moreover, according to Kolesnikov-Milman [106] the validity of the L_p-Minkowski conjecture for some $p \in (0,1)$ would also lead to the characterization of the equality case for the L_q-Minkowski inequality when $q \in (p,1)$.

According to Böröczky, Lutwak, Yang, Zhang [39], the L_p-Brunn-Minkowski inequality implies the L_p-Minkowski inequality, and for a family $\mathcal{F}$ of convex bodies closed under L_p linear combination, the L_p-Minkowski inequality (2.44) for all $K, C \in \mathcal{F}$ is equivalent to the L_p Brunn-Minkowski inequality (2.42) for all $K, C \in \mathcal{F}$ and $\alpha, \beta > 0$. Particularly it holds for the family of origin symmetric convex bodies.

Kolesnikov, Milman [106] and Putterman [133] derive the following conjectured strengthening of Minkowski's second inequality for origin symmetric convex bodies $K, C \subset \mathbb{R}^n$:

$$\frac{V(K,C;1)^2}{V(K)} \geq \frac{n-1}{n-p} V(K,C;2) + \frac{1-p}{n-p} \int_{S^{n-1}} \frac{h_C^2}{h_K^2}\, dV_K \tag{2.50}$$

which is equivalent to the L_p-Brunn-Minkowski conjecture without the characterization of equality.

So for $p \in (0,1)$ and origin symmetric convex bodies K and C in $\mathbb{R}^n$ we have the following three equivalent forms of the L_p-Brunn-Minkowski conjecture (without the characterization of equality in the case of the third formulation):

- $V((1-\lambda)K +_p \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda}$ for $\lambda \in (0,1)$;
- $V_p(K,C) \geq V(K)^{\frac{n-p}{n}} V(C)^{\frac{p}{n}}$;
- $\frac{V(K,C;1)^2}{V(K)} \geq \frac{n-1}{n-p} V(K,C;2) + \frac{1-p}{n-p} \int_{S^{n-1}} \frac{h_C^2}{h_K^2}\, dV_K$.

Some known cases of conjectures 2.10.1, 2.10.2 and 2.10.3

Böröczky, Lutwak, Yang, Zhang [39] have verified the conjectures in the planar case $n = 2$. Combining the work of Kolesnikov, Milman [106] and Cheng, Huang, Li, Liu [48] (see also Putterman [133]), we have that the L_p-Minkowski and the L_p-Brunn-Minkowski conjectures hold for $p \in (0,1)$ close to 1. More precisely, we have the following theorem

THEOREM 2.10.4. *If K, C are origin-symmetric convex bodies in $\mathbb{R}^n$ $(n \geq 3)$, and $p \in (p_n, 1)$ for $0 < p_n < 1 - \frac{c}{n(\log n)^{10}}$ for an absolute constant $c > 0$, then the L_p-Brunn-Minkowski and L_p-Minkowski conjectures* (2.42), (2.43), (2.44) *and* (2.45) *hold, together with the characterization of the equality cases.*

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Further, all the known cases of the log-Minkowski and log-Brunn-Minkowski conjectures imply the validity of the L_p-Minkowski and L_p-Brunn-Minkowksi conjectures ($p \in (0,1)$) in those cases. For a comprehensive survey of the state of the art, we refer to Böröczky [30].

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Chapter 3

Stability of the Prékopa-Leindler inequality for log-concave functions

3.1 Introduction

In this chapter, our main goal is to eastablish a stability version of Prékopa-Leindler inequality, a generalization of the Brunn-Minkowski inequality. The following multiplicative version from [10] is often more useful and is more convenient for geometric applications.

THEOREM 3.1.1 (Prékopa-Leindler)**.** *If $\lambda \in (0,1)$ and h, f, g are non-negative integrable functions on $\mathbb{R}^n$ satisfying $h((1-\lambda)x+\lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \cdot \left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{3.1}$$

It follows from Theorem 3.1.1 that the Prékopa-Leindler inequality has the following multifunctional form which resembles Barthe's Reverse Brascamp-Lieb inequality [20]. If $\lambda_1, \ldots, \lambda_m > 0$ satisfy $\sum_{i=1}^m \lambda_i = 1$ and $f_1, \ldots, f_m$ are non-negative integrable functions on $\mathbb{R}^n$, then

$$\int^*_{\mathbb{R}^n} \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f_i(x_i)^{\lambda_i}\, dz \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i\right)^{\lambda_i} \tag{3.2}$$

where $*$ denotes the outer integral in case the integrand is not measurable.

A function $f : \mathbb{R}^n \to [0, \infty)$ is said to have *positive integral* if f is measurable and $0 < \int_{\mathbb{R}^n} f < \infty$. If $\Gamma \subset \mathbb{R}^n$ is a convex set, a function $f : \Gamma \to [0, \infty)$ is called *log-concave*

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if for any $x, y \in \Gamma$ and $\alpha, \beta \in [0,1]$ with $\alpha+\beta=1$, we have $f(\alpha x+\beta y) \geq f(x)^\alpha g(y)^\beta$. Dubuc [63] has characterized the equality case in Theorem 3.1.1 as follows.

THEOREM 3.1.2 (Dubuc). *If $\lambda \in (0,1)$ and $h, f, g : \mathbb{R}^n \to [0,\infty)$ have positive integral, satisfy $h((1-\lambda)x+\lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for $x, y \in \mathbb{R}^n$ and equality holds in (3.1), then f, g, h are log-concave up to a set of measure zero, and there exist $a > 0$ and $z \in \mathbb{R}^n$ such that*

$$\begin{aligned} f(x) &= a^\lambda h(x-\lambda z) \\ g(x) &= a^{-(1-\lambda)} h(x+(1-\lambda)z) \end{aligned}$$

for almost all x.

Our goal in this chapter is to prove a stability version of the Prékopa-Leindler inequality Theorem 3.1.1 at least for log-concave functions.

THEOREM 3.1.3. *For some absolute constant $c > 1$, if $\tau \in (0, \frac{1}{2}]$, $\tau \leq \lambda \leq 1-\tau$, $h, f, g : \mathbb{R}^n \to [0,\infty)$ are integrable such that $h((1-\lambda)x+\lambda\, y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for $x, y \in \mathbb{R}^n$, h is log-concave and*

$$\int_{\mathbb{R}^n} h \leq (1+\varepsilon) \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^\lambda$$

for $\varepsilon \in (0,1]$, then there exists $w \in \mathbb{R}^n$ such that setting $a = \int_{\mathbb{R}^n} f / \int_{\mathbb{R}^n} g$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x) - a^\lambda h(x-\lambda\, w)|\, dx &\leq c^n n^n \sqrt[19]{\frac{\varepsilon}{\tau}} \cdot \int_{\mathbb{R}^n} f \\ \int_{\mathbb{R}^n} |g(x) - a^{-(1-\lambda)} h(x+(1-\lambda)w)|\, dx &\leq c^n n^n \sqrt[19]{\frac{\varepsilon}{\tau}} \cdot \int_{\mathbb{R}^n} g. \end{aligned}$$

Remark According to Lemma 3.7.3 (i), if f and g are log-concave, then

$$h(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda$$

is log-concave, as well, and hence Theorem 3.1.3 applies.

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Ball, Böröczky [13] have proved a statement similar to Theorem 3.1.3 for even log-concave functions and $\tau = \frac{1}{2}$, with an error term of the order of $\varepsilon^{\frac{1}{6}} |\log \varepsilon|^{\frac{2}{3}}$. A stability version of the Prékopa-Leindler inequality for log-concave functions was also given by Bucur, Fragalà [45] in terms of the weaker notion of bounding the (translative) distance of all one dimensional projections instead of the "translative" L_1 distance considered here.

Here we present a version of Theorem 3.1.3 analogous to Theorem 3.3.2. If f, g are non-negative functions on $\mathbb{R}^n$ with $0 < \int_{\mathbb{R}^n} f < \infty$ and $0 < \int_{\mathbb{R}^n} g < \infty$, then for the probability densities

$$\tilde{f} = \frac{f}{\int_{\mathbb{R}^n} f} \quad \text{and} \quad \tilde{g} = \frac{g}{\int_{\mathbb{R}^n} g},$$

we define

$$\widetilde{L}_1(f,g) = \inf_{v \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\tilde{f}(x-v) - \tilde{g}(x)| \, dx. \tag{3.3}$$

COROLLARY 3.1.4. *If $\tau \in (0, \frac{1}{2}]$, $\lambda \in [\tau, 1-\tau]$ and f, g are log-concave functions with positive integral on $\mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda \, dz \geq \left(1 + \gamma \cdot \tau \cdot \widetilde{L}_1(f,g)^{19}\right) \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\cdot \int_{\mathbb{R}^n} g\right)^\lambda$$

where $\gamma = c^n / n^{19n}$ for some absolutute constant $c \in (0,1)$.

We also deduce the following stability version of (3.2) from Theorem 3.1.3.

THEOREM 3.1.5. *For some absolute constant $c > 1$, if $\tau \in (0, \frac{1}{m}]$, $m \geq 2$, $\lambda_1, \ldots, \lambda_m \in [\tau, 1-\tau]$ satisfy $\sum_{i=1}^m \lambda_i = 1$ and $f_1, \ldots, f_m$ are log-concave functions with positive integral on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f_i(x_i)^{\lambda_i} \, dz \leq (1+\varepsilon) \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i\right)^{\lambda_i}$$

for $\varepsilon \in (0,1]$, then for the log-concave $h(z) = \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f(x_i)^{\lambda_i}$, there exist $a_1, \ldots, a_m > 0$ and $w_1, \ldots, w_m \in \mathbb{R}^n$ such that $\sum_{i=1}^m \lambda_i w_i = o$ and for $i = 1, \ldots, m$, we have

$$\int_{\mathbb{R}^n} |f_i(x) - a_i h(x + w_i)| \, dx \leq c^n n^n m^5 \sqrt[19]{\frac{\varepsilon}{m\tau}} \cdot \int_{\mathbb{R}^n} f_i.$$

Remark $a_i = \dfrac{\left(\int_{\mathbb{R}^n} f_i\right)^{1-\lambda_i}}{\prod_{j \neq i} \left(\int_{\mathbb{R}^n} f_j\right)^{\lambda_j}}$ for $i = 1, \ldots, m$ in Theorem 3.9.4.

For the log-concavity of the h in Theorem 3.9.4 see Corollary 3.9.1. In the next Section 3.2 we review some known stability versions of the Prekopa-Leindler inequality for functions on $\mathbb{R}$, and in Section 3.3 we outline the idea for the proofs of Theorem 3.1.3, Corollary 3.1.4 and Theorem 3.1.5.

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3.2 Stability versions of the one dimensional Prékopa-Leindler inequality

Ball, Böröczky [12] showed the following stability version of the Prekopa-Leindler inequality Theorem 3.1.1 for log-concave functions on $\mathbb{R}$ ($n = 1$).

THEOREM 3.2.1. *There exists a positive absolute constant c with the following property: If h, f, g are non-negative integrable functions with positive integrals on $\mathbb{R}$ such that h is log-concave, $h(\frac{r+s}{2}) \geq \sqrt{f(r)g(s)}$ for $r, s \in \mathbb{R}$, and*

$$\int_{\mathbb{R}} h \leq (1+\varepsilon)\sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g},$$

for $\varepsilon \in (0,1)$, then there exists $b \in \mathbb{R}$ such that for $a = \sqrt{\int_{\mathbb{R}} g / \int_{\mathbb{R}} f}$, we have

$$\begin{aligned}\int_{\mathbb{R}} |f(t) - a\,h(t+b)|\,dt &\leq c \cdot \sqrt[3]{\varepsilon}|\ln\varepsilon|^{\frac{4}{3}} \cdot \int_{\mathbb{R}} f(t)\,dt \\ \int_{\mathbb{R}} |g(t) - a^{-1}h(t-b)|\,dt &\leq c \cdot \sqrt[3]{\varepsilon}|\ln\varepsilon|^{\frac{4}{3}} \cdot \int_{\mathbb{R}} g(t)\,dt.\end{aligned}$$

Remark If f and g are log-concave probability distributions, then $a = 1$, and if in addition f and g have the same expectation, then even $b = 0$ can be assumed.

Combining Theorem 3.2.1, Lemma 3.7.3 (ii) and Lemma 3.7.4 gives us the following more precise stability version of the one-dimensional Prekopa-Leindler inequality.

COROLLARY 3.2.2. *For some absolute constant $c > 1$, if $\tau \in (0, \frac{1}{2}]$, $\tau \leq \lambda \leq 1-\tau$, $h, f, g : \mathbb{R} \to [0,\infty)$ are integrable such that $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ for $x, y \in \mathbb{R}$, h is log-concave and*

$$\int_{\mathbb{R}} h \leq (1+\varepsilon)\left(\int_{\mathbb{R}} f\right)^{1-\lambda}\left(\int_{\mathbb{R}} g\right)^{\lambda}$$

for $\varepsilon \in (0,1]$, then there exists $w \in \mathbb{R}$ such that for $a = \int_{\mathbb{R}} g / \int_{\mathbb{R}} f$, we have

$$\begin{aligned}\int_{\mathbb{R}} |f(x) - a^{-\lambda}h(x - \lambda w)|\,dx &\leq c\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{3}} |\log\varepsilon|^{\frac{4}{3}} \cdot \int_{\mathbb{R}} f \\ \int_{\mathbb{R}} |g(x) - a^{1-\lambda}h(x + (1-\lambda)w)|\,dx &\leq c\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{3}} |\log\varepsilon|^{\frac{4}{3}} \cdot \int_{\mathbb{R}} g.\end{aligned}$$

We note here, as in C. Borell [29], and K.M. Ball [10], that assigning to any function $H : [0,\infty] \to [0,\infty]$ the function $h : \mathbb{R} \to [0,\infty]$ defined by $h(x) = H(e^x)e^x$, we have the version Theorem 3.2.3 of the Prékopa-Leindler inequality. If H is log-concave and decreasing, then h is log-concave.

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THEOREM 3.2.3. *If $H, F, G : [0,\infty] \to [0,\infty]$ are integrable functions satisfying $H(\sqrt{rs}) \geq \sqrt{F(r)G(s)}$ for $r, s \geq 0$, then*

$$\int_0^\infty H \geq \sqrt{\int_0^\infty F \cdot \int_0^\infty G}.$$

Then Theorem 3.2.1 gives us the follwing corollary:

COROLLARY 3.2.4. *There exists a positive absolute constant $c_0 > 1$ with the following property: If $H, F, G : [0,\infty] \to [0,\infty]$ are integrable functions with positive integrals such that H is log-concave and decreasing, $H(\sqrt{rs}) \geq \sqrt{F(r)G(s)}$ for $r, s \in [0,\infty]$, and*

$$\int_0^\infty H \leq (1+\varepsilon)\sqrt{\int_0^\infty F \cdot \int_0^\infty G}$$

for $\varepsilon \in [0, c_0^{-1})$, then there exist $a, b > 0$, such that

$$\begin{aligned} \int_0^\infty |F(t) - a\,H(b\,t)|\,dt &\leq c \cdot \sqrt[3]{\varepsilon} |\ln\varepsilon|^{\frac{4}{3}} \cdot \int_0^\infty F(t)\,dt \\ \int_0^\infty |G(t) - a^{-1}H(b^{-1}t)|\,dt &\leq c \cdot \sqrt[3]{\varepsilon} |\ln\varepsilon|^{\frac{4}{3}} \cdot \int_0^\infty G(t)\,dt. \end{aligned}$$

Remark If in adddition, F and G are decreasing log-concave probability distributions then $a = b$ can be assumed. The condition that H is log-concave and decreasing can be replaced by the one that $H(e^t)$ is log-concave.

For more general measurable functions, the stability of at least the one-dimensional Brunn-Minkowski inequality has been clarified by Christ [54] (see also Theorem 1.1 in Figalli, Jerison [73]).

THEOREM 3.2.5. *If $X, Y \subset \mathbb{R}$ are measurable with $|X|, |Y| > 0$, and $|X+Y| \leq |X| + |Y| + \delta$ for some $\delta \leq \min\{|X|, |Y|\}$, then there exist intervals $I, J \subset \mathbb{R}$ such that $X \subset I$, $Y \subset J$, $|I \backslash X| \leq \delta$ and $|J \backslash Y| \leq \delta$.*

3.3 Ideas to verify Theorem 3.1.3 and its consequences

In order to prove Theorem 3.1.3, we first consider Theorem 3.3.1 which is essentially the case $\lambda = \frac{1}{2}$ of Theorem 3.1.3 for log-concave functions and for small ε. We treat the general case later in Sections 3.7 and 3.8.

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THEOREM 3.3.1. *If $h, f, g : \mathbb{R}^n \to [0, \infty)$ are log-concave, f, g are probability distributions, $h(\frac{x+y}{2}) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$, and*

$$\int_{\mathbb{R}^n} h \leq 1 + \varepsilon$$

where $0 < \varepsilon < (cn)^{-n}$, then there exists $w \in \mathbb{R}^n$ such that

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x) - h(x - w)|\, dx &\leq \tilde{c}n^8 \cdot \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n \\ \int_{\mathbb{R}^n} |g(x) - h(x + w)|\, dx &\leq \tilde{c}n^8 \cdot \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n\end{aligned}$$

where $c, \tilde{c} > 1$ are absolute constants.

Our proof of the stability version Theorem 3.3.1 of the Prékopa-Leindler inequality stems from the follwing argument of Ball (*cf* [10] and Borell [29]) for proving the Prékopa-Leindler inequality based on the Brunn-Minkowski inequality.

Let $f, g, h : \mathbb{R}^n \to [0, \infty]$ have positive integrals and satisfy that $h(\frac{x+y}{2}) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$, and for $t > 0$, consider the level sets

$$\begin{aligned}\Phi_t &= \{x \in \mathbb{R}^n : f(x) \geq t\} \text{ and } F(t) = |\Phi_t| \\ \Psi_t &= \{x \in \mathbb{R}^n : g(x) \geq t\} \text{ and } G(t) = |\Psi_t| \\ \Omega_t &= \{x \in \mathbb{R}^n : h(x) \geq t\} \text{ and } H(t) = |\Omega_t|.\end{aligned}$$

As it was observed by Ball [10] and and Borell [29], it follows from the condition on f, g, h that if $\Phi_r, \Psi_s \neq \emptyset$ for $r, s > 0$, then

$$\tfrac{1}{2}(\Phi_r + \Psi_s) \subset \Omega_{\sqrt{rs}}.$$

Then the Brunn-Minkowski inequality gives us that

$$H(\sqrt{rs}) \geq \left(\frac{F(r)^{\frac{1}{n}} + G(s)^{\frac{1}{n}}}{2}\right)^n \geq \sqrt{F(r) \cdot G(s)}$$

for all $r, s > 0$. And the Prékopa-Leindler inequality would then follow from Theorem 3.2.3 as follows:

$$\int_{\mathbb{R}^n} h = \int_0^\infty H(t)\, dt \geq \sqrt{\int_0^\infty F(t)\, dt \cdot \int_0^\infty G(t)\, dt} = \sqrt{\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g}.$$

For the proof of Theorem 3.3.1, we will need the stability version Corollary 3.2.4 of the Prekopa-Leindler inequality for functions on $\mathbb{R}$, and a stability version of the product form of the Brunn-Minkowski inequality on $\mathbb{R}^n$.

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Recall the stability version of the Brunn-Minkowski inequality by Figalli, Maggi, Pratelli [70].

THEOREM 3.3.2 (Figalli, Maggi, Pratelli). *For* $\gamma^*(n) = (\frac{(2-2^{\frac{n-1}{n}})^{\frac{3}{2}}}{122n^7})^2$*, and for any convex bodies* K *and* C *in* $\mathbb{R}^n$*,*

$$|K+C|^{\frac{1}{n}} \geq (|K|^{\frac{1}{n}} + |C|^{\frac{1}{n}}) \left[1 + \frac{\gamma^*}{\sigma(K,C)^{\frac{1}{n}}} \cdot A(K,C)^2\right].$$

Since

$$\begin{aligned} \frac{1}{2}\left(|K|^{\frac{1}{n}} + |C|^{\frac{1}{n}}\right) &= |K|^{\frac{1}{2n}}|C|^{\frac{1}{2n}}\left[1 + \frac{1}{2}\left(\sigma(K,C)^{\frac{1}{4n}} - \sigma(K,C)^{\frac{-1}{4n}}\right)^2\right] \\ &\geq |K|^{\frac{1}{2n}}|C|^{\frac{1}{2n}}\left[1 + \frac{(\sigma(K,C)-1)^2}{32n^2\sigma(K,C)^{\frac{4n-1}{2n}}}\right], \end{aligned}$$

using the notation $\sigma = \sigma(K,C) = \max\{\frac{|C|}{|K|}, \frac{|K|}{|C|}\}$, we conclude from the stability version Theorem 3.3.2 that

$$\left|\tfrac{1}{2}(K+C)\right| \geq \sqrt{|K|\cdot|C|}\left[1 + \frac{(\sigma-1)^2}{32n\sigma^2} + \frac{n\gamma^*(n)}{\sigma^{\frac{1}{n}}} \cdot A(K,C)^2\right]. \tag{3.4}$$

Note that the volume of the symmetric difference $|K\Delta C|$ of convex bodies K and C is a metric on convex bodies in $\mathbb{R}^n$. We use this fact in the following consequence of Theorem 3.3.2:

LEMMA 3.3.3. *If* $\eta \in (0, \frac{1}{122n^7})$ *and* $K, C.L$ *are convex bodies in* $\mathbb{R}^n$ *such that* $|C| = |K|$*,* $|L| \leq (1+\eta)|K|$ *and* $\frac{1}{2}K + \frac{1}{2}C \subset L$*, then there exists* $w \in \mathbb{R}^n$ *such that*

$$|K\Delta(L-w)| \leq 245n^7\sqrt{\eta}\,|K| \quad \textit{and} \quad |C\Delta(L+w)| \leq 245n^7\sqrt{\eta}\,|K|.$$

Proof: We may assume that $|C| = |K| = 1$. Theorem 3.3.2 implies that there exists $z \in \mathbb{R}^n$, such that

$$|K \cap (C-z)| \geq 1 - \sqrt{\frac{\eta}{\gamma^*(n)}} > 1 - 122n^7\sqrt{\eta}.$$

From $z + [K\cap(C-z)] \subset C$, we have that $M = \frac{1}{2}z + [K\cap(C-z)] \subset L$. Then it follows from $|L| \leq 1+\eta$ that $|L\Delta M| < \eta + 122n^7\sqrt{\eta} < 123n^7\sqrt{\eta}$. Denoting $w = \frac{1}{2}z$, we have

$$|K\Delta(L-w)| \leq |K\Delta(M-w)| + |(M-w)\Delta(L-w)| < 245n^7\sqrt{\eta}.$$

We also get $|C\Delta(L+w)| < 245n^7\sqrt{\eta}$ by a similar argument. $\square$

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In Section 3.4 we discuss some fundamental estimates for log-concave functions. We compare the level sets of f, g and h in Theorem 3.3.1 in Section 3.5, and finally complete the argument for Theorem 3.3.1 in Section 3.6. Theorem 3.1.3 for small ε is verified in Section 3.7. Section 3.8 deals with the proofs of Theorem 3.1.3 and Corollary 3.1.4. And finally in Section 3.9, we prove Theorem 3.1.5.

3.4 Some properties of log-concave functions

First, we characterize a log-concave function φ on $\mathbb{R}^n$, $n \geq 2$ with positive integral; namely, if $0 < \int_{\mathbb{R}^n} \varphi < \infty$. For any measurable function φ on $\mathbb{R}^n$, we denote

$$M_\varphi = \sup \varphi.$$

LEMMA 3.4.1. *Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be log-concave. Then φ has positive integral if and only if φ is bounded, $M_\varphi > 0$, and for any $t \in (0, M_\varphi)$, the level set $\{\varphi > t\}$ is bounded and has non-empty interior.*

Proof: If φ has positive integral, then $M_\varphi > 0$, and there exists some $t_0 \in (0, M_\varphi)$ such that the n-dimensional measure of $\{\varphi > t_0\}$ is positive. As $\{\varphi > t_0\}$ is convex, it has non-empty interior. It follows from the log-concavity of φ that the level set $\{\varphi > t\}$ has non-empty interior for any $t \in (0, M_\varphi)$. In turn, we deduce that φ is bounded from the log-concavity of φ and $\int_{\mathbb{R}^n} \varphi < \infty$.

Next we suppose that that there exists $t \in (0, M_\varphi)$ such that the level set $\{\varphi > t\}$ is unbounded and seek a contradiction. As $\{\varphi > t\}$ is convex, there exists a $u \in S^{n-1}$ such that $x + su \in \mathrm{int}\{\varphi > t\}$ for any $x \in \mathrm{int}\{\varphi > t\}$ and $s \geq 0$. We conclude that $\int_{\mathbb{R}^n} \varphi = \infty$, contradicting the assumption $\int_{\mathbb{R}^n} \varphi < \infty$. Therefore the level set $\{\varphi > t\}$ is bounded for any $t \in (0, M_\varphi)$.

Assuming that the conditions of Lemma 3.4.1 hold, we readily have $\int_{\mathbb{R}^n} \varphi > 0$. To show $\int_{\mathbb{R}^n} \varphi < \infty$, we choose $x_0 \in \mathbb{R}^n$ such that $\varphi(x_0) > 0$, and let B be an n-dimensional ball centered at x_0 and radius $\varrho > 0$ containing $\{\varphi > \frac{1}{e}\,\varphi(x_0)\}$. Let us consider

$$\psi(x) = \varphi(x_0) e^{-\frac{\|x - x_0\|}{\varrho}}.$$

It follows from the log-concavity of φ that $\varphi(x) \leq \psi(x)$ if $\|x - x_0\| \geq \varrho$, and hence

$$\int_{\mathbb{R}^n} \varphi \leq \int_B \varphi + \int_{\mathbb{R}^n \setminus B} \psi < \infty,$$

verifying Lemma 3.4.1. $\square$

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If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a bounded measurable function, and $t \in \mathbb{R}$, we denote the level set

$$\Xi_{\varphi,t} = \{x \in \mathbb{R}^n : \varphi(x) \geq t\}.$$

For a log-concave function φ with positive integral, it's symmetric decreasing rearrangement $\varphi^* : \mathbb{R}^n \to \mathbb{R}$ is characterized by the following properties. For any $t > 0$, we have

$$|\Xi_{\varphi^*,t}| = |\Xi_{\varphi,t}|$$

and whenever $|\Xi_{\varphi,t}| > 0$, $\Xi_{\varphi^*,t}$ is a Euclidean ball centered at the origin o. Moreover,

$$M_\varphi = \max_{x\in\mathbb{R}^n} \varphi(x) = \max_{x\in\mathbb{R}^n} \varphi^*(x) = \varphi^*(o).$$

Here, Lemma 3.4.1 implies that φ^* is well-defined. It follows that φ^* is log-concave as well and

$$\int_{\mathbb{R}^n} \varphi = \int_0^\infty |\Xi_{\varphi,t}|\, dt = \int_0^\infty |\Xi_{\varphi^*,t}|\, dt = \int_{\mathbb{R}^n} \varphi^*.$$

Denote by B^n the Euclidean ball centered at the origin with $|B^n| = \kappa_n$, and so, the surface area of S^{n-1} is $n\kappa_n$. The symmetric decreasing rearrangement of log-concave functions satisfies the following useful property that if $\Xi_{\varphi^*,s\,M_\varphi} = \varrho\, B^n$, and if we write $s = e^{-\gamma\,\varrho}$ for $\gamma, \varrho > 0$, then

$$\begin{array}{rcll} \varphi^*(x) & \geq & M_\varphi e^{-\gamma\|x\|} & \text{if } \|x\| \leq \varrho \\ \varphi^*(x) & \leq & M_\varphi e^{-\gamma\|x\|} & \text{if } \|x\| \geq \varrho. \end{array} \tag{3.5}$$

Note that when $s = e^{-\gamma\,\varrho}$, we have

$$|\Xi_{\varphi,sM_\varphi}| = |\Xi_{\varphi^*,sM_\varphi}| = \kappa_n \varrho^n. \tag{3.6}$$

We will make use of the following related integral which follows from induction on n

$$\int_0^\infty e^{-\gamma r} r^{n-1}\, dr = (n-1)! \cdot \gamma^{-n}. \tag{3.7}$$

LEMMA 3.4.2. *If φ is a log-concave probability density on $\mathbb{R}^n$, then for $\tau \in (0,1)$, we have*

$$|\Xi_{\varphi,(1-\tau)M_\varphi}| \geq \frac{1}{n!+1} \cdot \frac{\tau^n}{M_\varphi}. \tag{3.8}$$

Proof: Here, we are gonna make use of the property (3.5). We choose $\gamma, \varrho > 0$ such that $\Xi_{\varphi^*,(1-\tau)\,M_\varphi} = \varrho\, B^n$ with $1 - \tau = e^{-\gamma\,\varrho}$. From $e^{-\gamma\,\varrho} > 1 - \gamma\,\varrho$, we have $\gamma\,\varrho \geq \tau$.

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Then using (3.6) and (3.7), we have

$$
\begin{aligned}
1 &= \int_{\mathbb{R}^n} \varphi^*(x)\,dx \le |\varrho\, B^n| \cdot M_\varphi + \int_{\mathbb{R}^n \setminus \varrho\, B^n} \varphi^*(x)\,dx \\
&\le |\Xi_{\varphi,(1-\tau)M_\varphi}| \cdot M_\varphi + \int_{\mathbb{R}^n} M_\varphi e^{-\gamma\,\|x\|}\,dx \\
&= M_\varphi \cdot |\Xi_{\varphi,(1-\tau)M_\varphi}| + M_\varphi n\kappa_n \int_0^\infty e^{-\gamma r} r^{n-1}\,dr \\
&= M_\varphi \cdot |\Xi_{\varphi,(1-\tau)M_\varphi}| + M_\varphi n!\kappa_n \cdot \gamma^{-n} \\
&\le M_\varphi \cdot |\Xi_{\varphi,(1-\tau)M_\varphi}| + M_\varphi n!\kappa_n \cdot \frac{\varrho^n}{\tau^n} \\
&= M_\varphi \cdot |\Xi_{\varphi,(1-\tau)M_\varphi}| \left(1 + \frac{n!}{\tau^n}\right),
\end{aligned}
$$

and that concludes the proof of (3.8). □

To see that the estimate (3.8) is close to being optimal, let's consider the probability density $\varphi(x) = M_\varphi e^{-\gamma\|x\|}$ for an appropriately chosen $\gamma > 0$. Then for $\tau \in (0, \frac{1}{n})$, we have $|\Xi_{\varphi,(1-\tau)M_\varphi}| = \frac{|\ln(1-\tau)|^n}{n!M_\varphi} < \frac{e\tau^n}{n!M_\varphi}$.

If μ_φ is the probability measure corresponding to the log-concave probability density φ (that is, $d\mu_\varphi(x) = \varphi(x)\,dx$), then it follows from Lovász, Vempala [113] Lemma 5.16 that for $s \in (0, e^{-4(n-1)})$,

$$
\mu_\varphi(\varphi < sM_\varphi) \le \frac{e^{n-1}}{(n-1)^{n-1}} \cdot s \cdot |\ln s|^{n-1} \le s \cdot |\ln s|^n. \tag{3.9}
$$

Next we derive the following necessary estimate.

LEMMA 3.4.3. *If $s \in (0, e^{-4(n-1)})$ and φ is a log-concave probability density on $\mathbb{R}^n$, then*

$$
\begin{aligned}
|\Xi_{\varphi,sM_\varphi}| &< \frac{2|\ln s|^n}{n!M_\varphi}, && (3.10)\\
\int_0^{sM_\varphi} |\Xi_{\varphi,t}|\,dt &< \left(1 + \frac{1}{M_\varphi}\right) \cdot s \cdot |\ln s|^n. && (3.11)
\end{aligned}
$$

Proof: Once again, we will make use of the property (3.5). Choose $\gamma, \varrho > 0$ satisfying $\Xi_{\varphi^*,s\,M_\varphi} = \varrho\, B^n$ with $s = e^{-\gamma\,\varrho}$. From $s \in (0, e^{-4(n-1)})$, it follows that

$$
\gamma\varrho > 4(n-1). \tag{3.12}
$$

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Using (3.7) and integrating by parts, we have

$$\int_{\varrho}^{\infty} e^{-\gamma r} r^{n-1}\, dr = e^{-\gamma\varrho} \int_0^{\infty} e^{-\gamma r} r^{n-1}\, dr \cdot \sum_{k=0}^{n-1} \frac{(\gamma\varrho)^k}{k!}. \tag{3.13}$$

Now, using the estimate $(n-1)! > \frac{(n-1)^{n-1}}{e^{n-1}}$, we have for $k = 1, \dots, n-1$,

$$(n-1) \cdot \ldots \cdot (n-k) > \frac{(n-1)^k}{e^k}. \tag{3.14}$$

Next, noting that $s \geq 4$ implies $1 + e \cdot s < e^{\frac{3}{4}s}$ and using (3.14), we have that if $t > 4(n-1)$, then

$$\sum_{k=0}^{n-1} \frac{t^k}{k!} < \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{et}{n-1}\right)^k = \left(1 + \frac{et}{n-1}\right)^{n-1} < e^{\frac{3}{4}t}. \tag{3.15}$$

Then, combining (3.12), (3.13) and (3.15), it follows that

$$\begin{aligned} \int_{\varrho}^{\infty} e^{-\gamma r} r^{n-1}\, dr \;&<\; e^{-\gamma\varrho} \int_0^{\infty} e^{-\gamma r} r^{n-1}\, dr \cdot e^{\frac{3}{4}\gamma\varrho} = e^{-\frac{1}{4}\gamma\varrho} \int_0^{\infty} e^{-\gamma r} r^{n-1}\, dr \\ &<\; \frac{1}{2} \int_0^{\infty} e^{-\gamma r} r^{n-1}\, dr. \end{aligned} \tag{3.16}$$

It follow from (3.5), (3.16) and (3.6) that

$$\begin{aligned} 1 \;&\geq\; \int_{\varrho B^n} \varphi^*(x)\, dx \geq \int_{\varrho B^n} M_\varphi e^{-\gamma\|x\|}\, dx \\ &=\; n\kappa_n M_\varphi \int_0^{\varrho} e^{-\gamma r} r^{n-1}\, dr \\ &\geq\; n\kappa_n M_\varphi \cdot \frac{1}{2} \int_0^{\infty} e^{-\gamma r} r^{n-1}\, dr = \frac{n! M_\varphi \kappa_n \varrho^n}{2\gamma^n \varrho^n} \\ &=\; \frac{n! M_\varphi}{2} \cdot \frac{|\Xi_{\varphi,s}|}{|\ln s|^n}. \end{aligned}$$

And hence, for $s \in (0, e^{-4(n-1)})$ we have that

$$|\Xi_{\varphi,s}| \leq \frac{2|\ln s|^n}{n! M_\varphi}.$$

This together with (3.9) gives us that if $s \in (0, e^{-4(n-1)})$, then

$$\int_0^{sM_\varphi} |\Xi_{\varphi,t}|\, dt = |\Xi_{\varphi,s}| \cdot s + \mu_\varphi(\varphi < sM_\varphi) < e\left(1 + \frac{1}{M_\varphi}\right) s \cdot |\ln s|^n,$$

which concludes the proof of (3.11). $\square$

Once again if we consider the probability density $\varphi(x) = M_\varphi e^{-\gamma\|x\|}$ for appropriately chosen $\gamma > 0$, then $|\Xi_{\varphi,sM_\varphi}| = \frac{|\ln s|^n}{n! M_\varphi}$. And hence, the estimate (3.10) is close to being optimal.

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3.5 The measure of the level sets in Theorem 3.3.1

Here, we consider the functions f, g, h as in Theorem 3.3.1.We may assume that

$$f(o) = \max\{f(x) : x \in \mathbb{R}^n\} \text{ and } g(o) = \max\{g(x) : x \in \mathbb{R}^n\}. \tag{3.17}$$

Next let's consider the following bounded convex sets (as per Lemma 3.4.1): for $t > 0$, let

$$\begin{aligned} \Phi_t &= \Xi_{f,t} = \{x \in \mathbb{R}^n : f(x) \geq t\} \text{ and } F(t) = |\Phi_t| \\ \Psi_t &= \Xi_{g,t} = \{x \in \mathbb{R}^n : g(x) \geq t\} \text{ and } G(t) = |\Psi_t| \\ \Omega_t &= \Xi_{h,t} = \{x \in \mathbb{R}^n : h(x) \geq t\} \text{ and } H(t) = |\Omega_t| \end{aligned}$$

Note then that (3.17) implies

$$o \in \Phi_t \cap \Psi_t. \tag{3.18}$$

Further, we have

$$\int_0^\infty F = \int_0^{M_f} F = \int_{\mathbb{R}^n} f = 1 \text{ and } \int_0^\infty G = \int_0^{M_g} G = \int_{\mathbb{R}^n} g = 1. \tag{3.19}$$

We note here as in K.M. Ball [10], that the condition satisfied by f, g, h means that if $\Phi_r, \Psi_s \neq \emptyset$ for $r, s > 0$, then

$$\tfrac{1}{2}(\Phi_r + \Psi_s) \subset \Omega_{\sqrt{rs}}. \tag{3.20}$$

Consequently, it readily follows from the Brunn-Minkowski inequality that for all $r, s > 0$, we have

$$H(\sqrt{rs}) \geq \left(\frac{F(r)^{\frac{1}{n}} + G(s)^{\frac{1}{n}}}{2}\right)^n \geq \sqrt{F(r) \cdot G(s)}. \tag{3.21}$$

Consider the absolute constant $c_0 > 1$ of Corollary 3.2.4 and for $0 < \varepsilon < 1/c_0$, denote by

$$\omega(\varepsilon) = c_0 \cdot \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}} \tag{3.22}$$

the error estimate in Corollary 3.2.4.

The main result of this section is the following estimate.

LEMMA 3.5.1. *If $0 < \varepsilon < \frac{1}{cn^4}$ for a suitable absolute constant $c > 1$, then*

$$\begin{aligned} \textstyle\int_0^\infty \big| |\Phi_t| - |\Omega_t| \big| \, dt &\leq 97\sqrt{n}\sqrt{\omega(\varepsilon)} \\ \textstyle\int_0^\infty \big| |\Psi_t| - |\Omega_t| \big| \, dt &\leq 97\sqrt{n}\sqrt{\omega(\varepsilon)}. \end{aligned} \tag{3.23}$$

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Proof: We choose the absolute constant $c > 1$ in the condition $0 < \varepsilon < \frac{1}{cn^4}$ such that (*cf* (3.22))

$$\omega(\varepsilon) < \frac{1}{4 \cdot 24^2 n}. \tag{3.24}$$

This is equivalent to (3.34) below.

Note here that as we had defined before Φ_t, Ψ_t, Ω_t are convex bodies, and the corresponding functions $F(t), G(t), H(t)$ are decreasing and log-concave. It follows from (3.19) that F, G are probability distributions on $[0, \infty)$.

Note that $\int_0^\infty H = \int_{\mathbb{R}^n} h \leq 1 + \varepsilon$. Then Corollary 3.2.4 yields that there exists $b > 0$ such that

$$\begin{array}{rcl} \int_0^\infty |bF(bt) - H(t)|\, dt & \leq & \omega(\varepsilon) \\ \int_0^\infty |b^{-1}G(b^{-1}t) - H(t)|\, dt & \leq & \omega(\varepsilon). \end{array} \tag{3.25}$$

WLOG, we can assume $b \geq 1$.

Let's denote, for $t > 0$,

$$\begin{array}{rcl} \widetilde{\Phi}_t & = & b^{\frac{1}{n}} \Phi_{bt} \quad \text{if } \widetilde{\Phi}_t \neq \emptyset \\ \widetilde{\Psi}_t & = & b^{\frac{-1}{n}} \Psi_{b^{-1}t} \quad \text{if } \widetilde{\Psi}_t \neq \emptyset. \end{array}$$

Then we have that $|\widetilde{\Phi}_t| = bF(bt)$, $|\widetilde{\Psi}_t| = b^{-1}G(b^{-1}t)$, and (3.25) gives us

$$\int_0^\infty \big|\,|\widetilde{\Phi}_t| - H(t)\big|\, dt \;\leq\; \omega(\varepsilon) \tag{3.26}$$

$$\int_0^\infty \big|\,|\widetilde{\Psi}_t| - H(t)\big|\, dt \;\leq\; \omega(\varepsilon). \tag{3.27}$$

From (3.20), we have that if $\widetilde{\Phi}_t \neq \emptyset$ and $\widetilde{\Psi}_t \neq \emptyset$, then

$$\tfrac{1}{2}(b^{\frac{-1}{n}} \widetilde{\Phi}_t + b^{\frac{1}{n}} \widetilde{\Psi}_t) \subset \Omega_t. \tag{3.28}$$

Next, we dissect $[0, \infty)$ into I and J in such a way that $t \in I$, if $\frac{3}{4} H(t) < |\widetilde{\Phi}_t| < \frac{5}{4} H(t)$ and $\frac{3}{4} H(t) < |\widetilde{\Psi}_t| < \frac{5}{4} H(t)$, and $t \in J$ otherwise. Since $\varepsilon < \frac{1}{cn^4}$ and we choose $c > 1$ in a way such that (3.34) holds, it follows from (3.26) and (3.27) that

$$\int_J H(t)\, dt \leq 4 \int_J \Big(\big|\,|\widetilde{\Phi}_t| - H(t)\big| + \big|\,|\widetilde{\Psi}_t| - H(t)\big|\Big)\, dt \leq 8\omega(\varepsilon) < \frac{1}{2}. \tag{3.29}$$

For I, Prékopa-Leindler inequality and (3.29) gives us

$$\int_I H(t)\, dt \geq 1 - \int_J H(t)\, dt > \frac{1}{2}. \tag{3.30}$$

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Let's denote for $t \in I$, $\alpha(t) = |\widetilde{\Phi}_t|/H(t)$ and $\beta(t) = |\widetilde{\Psi}_t|/H(t)$. Then $\frac{3}{4} < \alpha(t), \beta(t) < \frac{5}{4}$. And it follows from (3.26) and (3.27) that

$$\int_0^\infty H(t) \cdot (|\alpha(t) - 1| + |\beta(t) - 1|)\, dt \leq 2\omega(\varepsilon). \tag{3.31}$$

Using γ^* and $A(\cdot,\cdot)$ as in Theorem 3.3.2 and (3.4), we define

$$\begin{aligned}
\sigma(t) &= \sigma\left(b^{\frac{-1}{n}}\widetilde{\Phi}_t, b^{\frac{1}{n}}\widetilde{\Psi}_t\right) = \max\left\{\frac{b^2\beta(t)}{\alpha(t)}, \frac{\alpha(t)}{b^2\beta(t)}\right\}\\
\eta(t) &= \frac{(\sigma(t)-1)^2}{32n\sigma(t)^2} + \frac{n\gamma^*}{\sigma(t)^{\frac{1}{n}}} \cdot A(\widetilde{\Phi}_t, \widetilde{\Psi}_t)^2,
\end{aligned}$$

For $t \in I$, it follows from $\alpha(t), \beta(t) > \frac{3}{4}$ that $\sqrt{\alpha(t)\cdot\beta(t)} \geq 1 - \max\{0, 1-\alpha(t)\} - \max\{0, 1-\beta(t)\} > \frac{1}{2}$. Then (3.4) and (3.28) give us

$$\begin{aligned}
H(t) &\geq \sqrt{\left|b^{\frac{-1}{n}}\widetilde{\Phi}_t\right| \cdot \left|b^{\frac{1}{n}}\widetilde{\Psi}_t\right|}(1+\eta(t)) = H(t)\cdot\sqrt{\alpha(t)\cdot\beta(t)}(1+\eta(t))\\
&\geq H(t)\cdot(1 - \max\{0, 1-\alpha(t)\} - \max\{0, 1-\beta(t)\})\,(1+\eta(t))\\
&\geq H(t)\cdot(1 - |\alpha(t)-1| - |\beta(t)-1| + \tfrac{1}{2}\,\eta(t)),
\end{aligned}$$

which together with (3.31) yield

$$\int_I H(t)\cdot\eta(t)\, dt \leq 4\omega(\varepsilon). \tag{3.32}$$

In order to estimate $b \geq 1$ (see (3.35)), we claim that for $t \in I$,

$$|\alpha(t)-1| + |\beta(t)-1| + \eta(t) \geq \frac{(b-1)^2}{32nb^2}. \tag{3.33}$$

Indeed note first that if $\alpha(t) < b\beta(t)$, then $\sigma(t) > b$, which gives us

$$\eta(t) > \frac{(b-1)^2}{32nb^2}.$$

Next, we consider the case when $\alpha(t) \geq b\beta(t)$ in (3.33). If $\beta(t) \geq 1$, then we have

$$|\alpha(t)-1| + |\beta(t)-1| \geq |\alpha(t)-1| \geq b-1 \geq \frac{b-1}{b}.$$

On the other hand, if $1/b \leq \beta(t) \leq 1$, then

$$|\alpha(t)-1| + |\beta(t)-1| \geq b\beta(t) - 1 + 1 - \beta(t) \geq \frac{b-1}{b}.$$

And finally if $\beta(t) \leq 1/b$, then

$$|\alpha(t)-1| + |\beta(t)-1| \geq |\beta(t)-1| \geq 1 - \frac{1}{b} = \frac{b-1}{b}.$$

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Thus we have that the claim in (3.33) indeed holds.

Combining (3.30), (3.31), (3.32) and (3.33), it follows that

$$\begin{aligned}\frac{(b-1)^2}{64nb^2} &\leq \int_I H(t)\cdot\frac{(b-1)^2}{32nb^2}\,dt \\ &\leq \int_0^\infty H(t)\cdot(\eta(t)+|\alpha(t)-1|+|\beta(t)-1|)\,dt \leq 6\,\omega(\varepsilon),\end{aligned}$$

which gives us

$$\frac{b-1}{b} \leq 24\sqrt{n}\sqrt{\omega(\varepsilon)}.$$

Recall that in the condition $\varepsilon < \frac{1}{cn^4}$, we had chosen $c > 1$ large enough (*cf.* (3.24)) that

$$24\sqrt{n}\sqrt{\omega(\varepsilon)} < \frac{1}{2}. \tag{3.34}$$

If $b > 2$, $\frac{b-1}{b} > \frac{1}{2}$ would contradict (3.34). So we must have that $b < 1$ which gives us from (3.34) that

$$b-1 \leq 48\sqrt{n}\,\sqrt{\omega(\varepsilon)}. \tag{3.35}$$

We then claim that

$$\begin{aligned}\textstyle\int_0^\infty \Big||\Phi_t| - |\widetilde{\Phi}_t|\Big|\,dt &\leq 96\sqrt{n}\,\sqrt{\omega(\varepsilon)} \\ \textstyle\int_0^\infty \Big||\Psi_t| - |\widetilde{\Psi}_t|\Big|\,dt &\leq 96\sqrt{n}\,\sqrt{\omega(\varepsilon)}.\end{aligned} \tag{3.36}$$

It follows from $|\Phi_{bt}| \leq |\Phi_t|$ that,

$$\begin{aligned}\int_0^\infty \Big||\Phi_t| - |\widetilde{\Phi}_t|\Big|\,dt &= \int_0^\infty ||\Phi_t| - b|\Phi_{bt}||\,dt \\ &\leq \int_0^\infty ||\Phi_t| - b|\Phi_t||\,dt + b\int_0^\infty ||\Phi_t| - |\Phi_{bt}||\,dt \\ &= (b-1) + b\int_0^\infty |\Phi_t| - |\Phi_{bt}|\,dt \\ &= 2(b-1) \leq 96\sqrt{n}\,\sqrt{\omega(\varepsilon)}.\end{aligned}$$

Similarly, $|\Psi_t| \leq |\Psi_{b^{-1}t}|$ gives us

$$\begin{aligned}\int_0^\infty \Big||\Psi_t| - |\widetilde{\Psi}_t|\Big|\,dt &= \int_0^\infty \Big||\Psi_t| - b^{-1}|\Psi_{b^{-1}t}|\Big|\,dt \\ &\leq \int_0^\infty \Big||\Psi_t| - b^{-1}|\Psi_t|\Big|\,dt + b^{-1}\int_0^\infty ||\Psi_t| - |\Psi_{b^{-1}t}||\,dt \\ &= (1-b^{-1}) + b^{-1}\int_0^\infty |\Psi_{b^{-1}t}| - |\Psi_t|\,dt \\ &= 2(1-b^{-1}) \leq 96\sqrt{n}\,\sqrt{\omega(\varepsilon)},\end{aligned}$$

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and hence (3.36) indeed holds.

Finally using (3.26), (3.27) and (3.36) gives us that (3.23) holds, concluding the proof. $\square$

Next we derive the following corollary of Lemma 3.23

COROLLARY 3.5.2. *There exists an absolute constant $c > 1$ such that if $0 < \varepsilon < (cn)^{-n}$, then $\frac{1}{2} < M_f/M_g < 2$ and $\frac{1}{2} < M_f/M_h < 2$.*

Proof: From (3.23), we have

$$\int_0^\infty \big|\, |\Phi_t| - |\Psi_t| \,\big| \, dt \le 194\sqrt{\omega(\varepsilon)}. \tag{3.37}$$

We may assume that $1 = M_f \ge M_g$. Note that $|\Psi_t| = 0$ if $t > M_g$. Then using (3.37), (3.8) and $k! < (\frac{k}{e})^k \sqrt{2\pi(k+1)}$, we have

$$\begin{aligned} 194\sqrt{\omega(\varepsilon)} &\ge \int_{M_g}^1 |\Phi_t| \, dt \ge \frac{1}{2 \cdot n!} \int_{M_g}^1 (1-t)^n \, dt \\ &= \frac{1}{2 \cdot n!} \frac{(1-M_g)^{n+1}}{n+1} > \frac{e^{n+1}}{2(n+1)^{n+1}\sqrt{2\pi(n+2)}} \cdot (1-M_g)^{n+1}, \end{aligned}$$

which gives us

$$1 - M_g < c_1 n \omega(\varepsilon)^{\frac{1}{2(n+1)}}$$

for an absolute constant $c_1 > 0$. Then (3.22) yields that for some absolute constant $c > 1$, if $0 < \varepsilon < (cn)^{-n}$, then $M_g > \frac{1}{2}$ which gives us $\frac{1}{2} < M_f/M_g < 2$.

By a similar argument, $\frac{1}{2} < M_f/M_h < 2$ analogously follows from (3.23). $\square$

3.6 Proof of Theorem 3.3.1

using all the same notation as in Section 3.5, we assume here that $f(o) = 1$, that is,

$$f(o) = M_f = 1 \text{ and } g(o) = M_g. \tag{3.38}$$

First we assume that for a large enough absolute constant $c > 1$,

$$\varepsilon < c^{-n} n^{-n}. \tag{3.39}$$

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Then (3.38), (3.39) and Corollary 3.5.2 give us

$$\begin{array}{rcl} \frac{1}{2} < & g(o) = M_g & < 2 \\ \frac{1}{2} < & M_h & < 2. \end{array} \tag{3.40}$$

here we assume that $\mathbb{R}^n$ is a linear subspace of $\mathbb{R}^{n+1}$, and denote by u_0 the $(n+1)$th basis vector in $\mathbb{R}^{n+1}$ orthogonal to $\mathbb{R}^n$. Let

$$\xi = \frac{\sqrt[6]{\omega(\varepsilon)}}{|\ln \omega(\varepsilon)|^{\frac{1}{2}}}. \tag{3.41}$$

It follows from (3.39) that

$$\xi < \frac{e^{-4(n-1)}}{2} \quad \text{and} \quad 6e\xi \cdot |\ln \xi|^n < \frac{1}{2}. \tag{3.42}$$

Using $M_f = 1$ (see (3.38)), $\frac{1}{2} < M_g, M_h < 2$ (see (3.40)), (3.11)) and (3.42) with the substitution $s = \ln t$, we have

$$\int_\xi^1 |\Phi_t|\,dt = \int_\xi^{M_f} |\Phi_t|\,dt \;>\; 1 - 2e \cdot \xi \cdot |\ln \xi|^n > \frac{1}{2} \tag{3.43}$$

$$\begin{aligned} \int_\xi^2 |\Psi_t|\,dt = \int_\xi^{M_g} |\Psi_t|\,dt \;&>\; 1 - 3e \cdot \frac{\xi}{M_g} \cdot \left|\ln \frac{\xi}{M_g}\right|^n \\ &>\; 1 - 6e\xi \cdot |\ln \xi|^n > \frac{1}{2} \end{aligned} \tag{3.44}$$

$$\int_\xi^2 |\Omega_t|\,dt = \int_\xi^{M_h} |\Omega_t|\,dt \;>\; 1 - 6e\xi \cdot |\ln \xi|^n > \frac{1}{2}. \tag{3.45}$$

Next we define the following convex bodies in $\mathbb{R}^{n+1}$:

$$K = K_{\xi,f} \;=\; \{x + u_0 \ln t : x \in \Phi_\xi \text{ and } \xi \leq t \leq f(x)\} \tag{3.46}$$

$$C = C_{\xi,g} \;=\; \{x + u_0 \ln t : x \in \Psi_\xi \text{ and } \xi \leq t \leq g(x)\} \tag{3.47}$$

$$L = L_{\xi,h} \;=\; \{x + u_0 \ln t : x \in \Omega_\xi \text{ and } \xi \leq t \leq h(x)\}. \tag{3.48}$$

Denote by $V(\cdot)$ the volume ($(n+1)$-dimensional Lebesgue measure) in $\mathbb{R}^{n+1}$. Then (3.43) and (3.44) gives us

$$\begin{array}{rcl} V(K) & = & \int_{\ln \xi}^0 |\Phi_{e^s}|\,ds = \int_\xi^1 |\Phi_t| \cdot \frac{1}{t}\,dt \geq \int_\xi^1 |\Phi_t|\,dt > \frac{1}{2}, \\ V(C) & = & \int_{\ln \xi}^{\ln 2} |\Psi_{e^s}|\,ds = \int_\xi^2 |\Psi_t| \cdot \frac{1}{t}\,dt \geq \int_\xi^2 |\Psi_t| \cdot \frac{1}{2}\,dt > \frac{1}{4}. \end{array} \tag{3.49}$$

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Note here that K is contained in a right cylinder whose base is a translate of Φ_ξ and height is $|\ln \xi|$, and C is contained in a right cylinder whose base is a copy of Ψ_ξ and height is $|\ln \xi| + \ln 2 < 2|\ln \xi|$. Then it follows from (3.10) that

$$\begin{aligned} V(K) &\leq \tfrac{2}{n!} \cdot |\ln \xi|^{n+1}, \\ V(C) &\leq \tfrac{4}{n!} \cdot |\ln \xi|^{n+1}. \end{aligned} \tag{3.50}$$

Using (3.23), $f(o) = 1$, (3.40) and the substitution $s = \ln t$ gives us

$$\begin{aligned} |V(K) - V(L)| &= \left| \int_{\ln \xi}^{\ln 2} (|\Phi_{e^s}| - |\Omega_{e^s}|)\, ds \right| = \left| \int_{\xi}^{2} (|\Phi_t| - |\Omega_t|) \cdot \frac{1}{t}\, dt \right| \\ &\leq \frac{1}{\xi} \int_{\xi}^{2} ||\Phi_t| - |\Omega_t||\, dt \leq 97\sqrt{n}\, \frac{\sqrt{\omega(\varepsilon)}}{\xi}. \end{aligned} \tag{3.51}$$

Similarly, we have

$$|V(C) - V(L)| \leq 97\sqrt{n}\, \frac{\sqrt{\omega(\varepsilon)}}{\xi}. \tag{3.52}$$

Then from (3.51) and (3.52), it follows that

$$|V(C) - V(K)| \leq 194\sqrt{n}\, \frac{\sqrt{\omega(\varepsilon)}}{\xi}. \tag{3.53}$$

We note here that $h(\frac{x+y}{2}) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$ in Theorem 3.3.1 gives us

$$\tfrac{1}{2} K + \tfrac{1}{2} C \subset L. \tag{3.54}$$

LEMMA 3.6.1. *If $\varepsilon < c^{-n} n^{-n}$ as in (3.39), there exist $w \in \mathbb{R}^n$ and absolute constant $\gamma > 1$ such that*

$$\begin{aligned} V(K\Delta(L - w)) &\leq \gamma n^8 \cdot \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi}} \cdot |\ln \xi|^{\frac{n+1}{2}} \\ V(C\Delta(L + w)) &\leq \gamma n^8 \cdot \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi}} \cdot |\ln \xi|^{\frac{n+1}{2}}. \end{aligned}$$

Proof: We start by showing a slightly weaker statement, where we allow the translation vectors to be chosen from $\mathbb{R}^{n+1}$, and not only from $\mathbb{R}^n$. We claim that there exist $\tilde{w} \in \mathbb{R}^{n+1}$ and absolute constant $\gamma > 1$ such that

$$V(K\Delta(L - \tilde{w})) \leq \frac{\gamma n^8}{3} \cdot \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi}} \cdot |\ln \xi|^{\frac{n+1}{2}} \tag{3.55}$$

$$V(C\Delta(L + \tilde{w})) \leq \frac{\gamma n^8}{3} \cdot \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi}} \cdot |\ln \xi|^{\frac{n+1}{2}}. \tag{3.56}$$

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In order to show (3.55) and (3.56), we consider a homothetic copy $K_0 \subset K$ of K and homothetic copy $C_0 \subset C$ of C such that $V(K_0) = V(C_0) = \min\{V(K), V(C)\}$. We have either $K = K_0$ or $C = C_0$, and

$$\tfrac{1}{2}\,K_0 + \tfrac{1}{2}\,C_0 \subset L. \tag{3.57}$$

Then using (3.51) and (3.52), and noting that either $K = K_0$ or $C = C_0$, we have

$$V(L) - V(K_0) \leq 97\sqrt{n} \cdot \frac{\sqrt{\omega(\varepsilon)}}{\xi\, V(K_0)} \cdot V(K_0). \tag{3.58}$$

Recall that $\xi = \sqrt[6]{\omega(\varepsilon)}/|\ln \omega(\varepsilon)|^{\frac{1}{2}}$. Then (3.39) (provided $\tilde{c}$ is large enough) and (3.49) give us that $\frac{\sqrt{\omega(\varepsilon)}}{\xi\, V(K_0)} < 2\omega(\varepsilon)^{\frac{1}{3}}|\ln \omega(\varepsilon)|^{\frac{1}{2}}$ is small enough for Lemma 3.3.3 to be applied to K_0, C_0 and L. And hence using (3.57), (3.58) and Lemma 3.3.3, it follows that there exist $\tilde{w} \in \mathbb{R}^{n+1}$ and an absolute constant $\gamma_0 > 1$ such that

$$\begin{aligned} V(K_0\Delta(L-\tilde{w})) &\leq \gamma_0 n^8 \cdot \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi\, V(K_0)}} \cdot V(K_0) = \gamma_0 n^8 \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi}} \cdot \sqrt{V(K_0)} \\ V(C_0\Delta(L+\tilde{w})) &\leq \gamma_0 n^8 \cdot \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi}} \cdot \sqrt{V(K_0)}. \end{aligned}$$

Then (3.49), (3.53) and the properties of K_0 and C_0 yield that

$$\begin{aligned} V(K\Delta(L-\tilde{w})) &\leq (\gamma_0+1) n^8 \cdot \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi}} \cdot \sqrt{V(K)} \\ V(C\Delta(L+\tilde{w})) &\leq (\gamma_0+1) n^8 \cdot \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi}} \cdot \sqrt{V(K)}. \end{aligned}$$

Then, (3.50) implies (3.55) and (3.56).

We next show that if $w \in \mathbb{R}^n$ is such that $\tilde{w} = w + pu_0$ for $p \in \mathbb{R}$, then

$$V(K\Delta(L-w)) \leq 3V(K\Delta(L-\tilde{w})) \tag{3.59}$$
$$V(C\Delta(L+w)) \leq 3V(C\Delta(L+\tilde{w})). \tag{3.60}$$

Here, we may assume $p \neq 0$ and we consider the cases $p < 0$ and $p \geq 0$.

For $p < 0$, let us consider

$$K_{(p)} = \{x \in K : \ln \xi \leq \langle x, u_0\rangle < |p| + \ln \xi\} \subset K \backslash (L - \tilde{w}).$$

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Note that Φ_t is decreasing as a set as $t>0$ increases. Then using Fubini's theorem, it follows

$$V(K\Delta(K+|p|u_0)) = 2V(K_{(p)}) \le 2V(K\Delta(L-\tilde{w})).$$

And the using the triangle inequality for the symmetric difference metric, we get

$$\begin{aligned} V(K\Delta(L-w)) &= V((K+|p|\,u_0)\Delta(L-\tilde{w})) \\ &\le V((K+|p|\,u_0)\Delta K) + V(K\Delta(L-\tilde{w})) \\ &\le 3V(K\Delta(L-\tilde{w})). \end{aligned}$$

Similarly, for $p>0$, let

$$L_{(p)} = \{x \in L :\ \ln\xi \le \langle x, u_0\rangle < p + \ln\xi\}$$

with

$$L_{(p)} + \tilde{w} \subset (L+\tilde{w})\backslash K.$$

Note that Ω_t is decreasing as a set as $t>0$ increases and it follows

$$V((L+\tilde{w})\Delta(L+w)) = 2V(L_{(p)}) \le 2V(K\Delta(L+\tilde{w}));$$

Then using the triangle inequality for the symmetric difference metric, we have

$$V(K\Delta(L+w)) \le V(K\Delta(L+\tilde{w})) + V((L+\tilde{w})\Delta(L+w)) \le 3V(K\Delta(L+\tilde{w})),$$

which concludes the proof of (3.59).

Finally, (3.55), (3.56), (3.59) and (3.60) together give us Lemma 3.6.1. □

Proof of Theorem 3.3.1 Here we can assume that f and g are log-concave probability distributions with

$$f(o) = M_f = 1 \ \text{ and } \ g(o) = M_g.$$

From (3.40), we have $\frac{1}{2} < M_g, M_h < 2$. Let $K, C, L \subset \mathbb{R}^{n+1}$ be the convex bodies as defined in (3.46), (3.47) and (3.48) and let $w \in \mathbb{R}^n$ be the translating vector from Lemma 3.6.1. Note here that $\frac{1}{2} < M_f, M_h < 2$, and hence $\frac{\xi}{M_f}, \frac{\xi}{M_g} < 2\xi$ and both $1+\frac{1}{M_f}$ and $1+\frac{1}{M_h}$ are at most 3. As before we denote $\xi = \sqrt[6]{\omega(\varepsilon)}/|\ln\omega(\varepsilon)|^{\frac{1}{2}}$ and let γ be the absolute constant in Lemma 3.6.1. Then for the functions f, g, h, $n \ge 2$, (3.11)

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(compare the condition (3.42)) and Lemma 3.6.1 give us

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x) - h(x-w)|\,dx &= \int_0^2 |\Phi_t \Delta(\Omega_t - w)|\,dt \\
&\leq \int_\xi^2 |\Phi_t \Delta(\Omega_t - w)|\,dt + \int_0^\xi |\Phi_t|\,dt + \int_0^\xi |\Omega_t|\,dt \\
&\leq 2\int_\xi^2 |\Phi_t \Delta(\Omega_t - w)| \cdot \frac{1}{t}\,dt + \int_0^\xi |\Phi_t|\,dt + \int_0^\xi |\Omega_t|\,dt \\
&= 2V(K\Delta(L-w)) + \int_0^\xi |\Phi_t|\,dt + \int_0^\xi |\Omega_t|\,dt \\
&\leq \gamma n^8 \cdot \frac{\sqrt[4]{\omega(\varepsilon)}}{\sqrt{\xi}} \cdot |\ln \xi|^{\frac{n+1}{2}} + 2\cdot 3 \cdot (2\xi) \cdot |\ln(2\xi)|^n \\
&\leq 2\gamma n^8 \cdot \sqrt[6]{\omega(\varepsilon)} \cdot |\ln \omega(\varepsilon)|^{n-\frac{1}{4}}.
\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}^n} |g(x) - h(x+w)|\,dx \leq 2\gamma n^8 \cdot \sqrt[6]{\omega(\varepsilon)} \cdot |\ln \omega(\varepsilon)|^{n-\frac{1}{4}}.$$

Note that $\omega(\varepsilon) = c_0 \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}}$ for an absolute constant $c_0 > 1$ (*cf.* (3.22)). It follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x) - h(x-w)|\,dx &\leq \gamma_0 n^8 \cdot \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n \\
\int_{\mathbb{R}^n} |g(x) - h(x+w)|\,dx &\leq \gamma_0 n^8 \cdot \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n
\end{aligned}$$

where $\gamma_0 > 1$ is an absolute constant. This concludes the proof of Theorem 3.3.1. □

3.7 A version of Theorem 3.1.3 when ε is small

In this section we prove the following version of Theorem 3.1.3 for small ε.

THEOREM 3.7.1. *For some absolute constant $c > 1$, if $\tau \in (0, \frac{1}{2}]$, $\lambda \in [\tau, 1-\tau]$, $h, f, g : \mathbb{R}^n \to [0, \infty)$ are integrable such that $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for $x, y \in \mathbb{R}^n$, h is log-concave and*

$$\int_{\mathbb{R}^n} h \leq (1+\varepsilon) \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^\lambda$$

for $\varepsilon \in (0, \tau\varepsilon_0)$ for $\varepsilon_0 = c^{-n} n^{-n}$, then there exists $w \in \mathbb{R}^n$ such that setting $a = \int_{\mathbb{R}^n} f / \int_{\mathbb{R}^n} g$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x) - a^\lambda h(x - \lambda w)|\,dx &\leq cn^8 \sqrt[18]{\frac{\varepsilon}{\tau}} \cdot \left|\log \frac{\varepsilon}{\tau}\right|^n \int_{\mathbb{R}^n} f \\
\int_{\mathbb{R}^n} |g(x) - a^{-(1-\lambda)} h(x + (1-\lambda)w)|\,dx &\leq cn^8 \sqrt[18]{\frac{\varepsilon}{\tau}} \cdot \left|\log \frac{\varepsilon}{\tau}\right|^n \int_{\mathbb{R}^n} g.
\end{aligned}$$

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The log-concave hull $\tilde{f} : \mathbb{R}^n \to [0,\infty)$ of a bounded measurable function $f : \mathbb{R}^n \to [0,\infty)$ is given by

$$\tilde{f}(z) = \sup_{\substack{z=\sum_{i=1}^k \alpha_i x_i \\ \sum_{i=1}^k \alpha_i=1,\ \forall \alpha_i \geq 0}} \prod_{i=1}^k f(x_i)^{\alpha_i}.$$

Showing $\tilde{f}$ is log-concave, is equivalent to proving that if $\varepsilon, \alpha, \beta \in (0,1)$ and $x, y \in \mathbb{R}^n$, then

$$\tilde{f}(\alpha x + \beta y) \geq (1-\varepsilon)\tilde{f}(x)^\alpha \tilde{f}(y)^\beta. \tag{3.61}$$

Note that there exist $x_1, \ldots, x_k, y_1, \ldots, y_m \in \mathbb{R}^n$ and $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m \geq 0$ with $\sum_{i=1}^k \alpha_i = 1$, $\sum_{j=1}^m \beta_j = 1$, $x = \sum_{i=1}^k \alpha_i x_i$ and $y = \sum_{i=1}^k \beta_j x_j$ such that

$$(1-\varepsilon)\tilde{f}(x) \geq \prod_{i=1}^k f(x_i)^{\alpha_i} \quad \text{and} \quad (1-\varepsilon)\tilde{f}(y) \geq \prod_{j=1}^m f(y_j)^{\beta_j}.$$

We have

$$\alpha x + \beta y = \sum_{i=1}^k \sum_{j=1}^m (\alpha_i \beta_j \alpha x_i + \alpha_i \beta_j \beta y_j) \quad \text{where} \quad \sum_{i=1}^k \sum_{j=1}^m (\alpha_i \beta_j \alpha + \alpha_i \beta_j \beta) = 1. \tag{3.62}$$

It follows that

$$\begin{aligned} \tilde{f}(\alpha x + \beta y) &\geq \prod_{i=1}^k \prod_{j=1}^m f(x_i)^{\alpha_i \beta_j \alpha} f(y_j)^{\alpha_i \beta_j \beta} \\ &= \left(\prod_{i=1}^k f(x_i)^{\alpha_i} \right)^\alpha \left(\prod_{j=1}^m f(y_j)^{\beta_j} \right)^\beta \\ &\geq (1-\varepsilon)^\alpha \tilde{f}(x)^\alpha (1-\varepsilon)^\beta \tilde{f}(y)^\beta = (1-\varepsilon)\tilde{f}(x)^\alpha \tilde{f}(y)^\beta, \end{aligned}$$

which proves that $\tilde{f}$ is log-concave via (3.61).

We note that if $a_0 > 0$ and $z_0 \in \mathbb{R}^n$ and $f_0(z) = a_0 f(z - z_0)$, then

$$\widetilde{f_0}(z) = a_0 \tilde{f}(z - z_0). \tag{3.63}$$

In order to prove Theorem 3.7.1, we first derive the follwing three technical lemmas, Lemma 3.7.2, Lemma 3.7.3 and Lemma 3.7.4 about log-concave functions.

LEMMA 3.7.2. *If $\lambda \in (0,1)$, h is a log-concave function on $\mathbb{R}^n$ with positive integral, and $f, g : \mathbb{R}^n \to [0,\infty)$ are measurable satisfying $\int_{\mathbb{R}^n} f > 0$, $\int_{\mathbb{R}^n} g > 0$ and $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for $x, y \in \mathbb{R}^n$, then f and g are bounded, and their log-concave hulls $\tilde{f}$ and $\tilde{g}$ satisfy that $h((1-\lambda)x + \lambda y) \geq \tilde{f}(x)^{1-\lambda} \tilde{g}(y)^\lambda$ for $x, y \in \mathbb{R}^n$.*

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Remark The Prékopa-Leindler inequality implies $\int_{\mathbb{R}^n} \tilde{f} < \infty$ and $\int_{\mathbb{R}^n} \tilde{g} < \infty$.

Proof: Choose $y_0 \in \mathbb{R}^n$ with $g(y_0) > 0$. Then for any $x \in \mathbb{R}^n$, we have $h((1-\lambda)x+\lambda y_0) \geq f(x)^{1-\lambda} g(y_0)^{\lambda}$. It follows that

$$f(x) \leq \frac{h((1-\lambda)x+\lambda y_0)^{\frac{1}{1-\lambda}}}{g(y_0)^{\frac{\lambda}{1-\lambda}}} \leq \frac{M_h^{\frac{1}{1-\lambda}}}{g(y_0)^{\frac{\lambda}{1-\lambda}}},$$

that is, f is bounded. Similarly, we can show that g is bounded too.

It suffices to prove that $x, y \in \mathbb{R}^n$, and $\varepsilon \in (0,1)$

$$h((1-\lambda)x+\lambda y) \geq (1-\varepsilon)\tilde{f}(x)^{1-\lambda}\tilde{g}(y)^{\lambda}. \tag{3.64}$$

We choose $x_1, \ldots, x_k, y_1, \ldots, y_m \in \mathbb{R}^n$ and $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m \geq 0$ with $\sum_{i=1}^k \alpha_i = 1$, $\sum_{j=1}^m \beta_j = 1$, $x = \sum_{i=1}^k \alpha_i x_i$ and $y = \sum_{i=1}^k \beta_j x_j$ such that

$$(1-\varepsilon)\tilde{f}(x) \geq \prod_{i=1}^k f(x_i)^{\alpha_i} \quad \text{and} \quad (1-\varepsilon)\tilde{f}(y) \geq \prod_{j=1}^m f(y_j)^{\beta_j}.$$

Then (3.62) and the log-concavity of h give us

$$\begin{aligned} h((1-\lambda)x+\lambda y) &= h\left(\sum_{i=1}^k\sum_{j=1}^m \alpha_i\beta_j((1-\lambda)x_i+\lambda y_j)\right) \\ &\geq \prod_{i=1}^k\prod_{j=1}^m h((1-\lambda)x_i+\lambda y_j)^{\alpha_i\beta_j} \geq \prod_{i=1}^k\prod_{j=1}^m f(x_i)^{(1-\lambda)\alpha_i\beta_j} g(y_j)^{\lambda\alpha_i\beta_j} \\ &= \left(\prod_{i=1}^k f(x_i)^{\alpha_i}\right)^{1-\lambda}\left(\prod_{j=1}^m f(y_j)^{\beta_j}\right)^{\lambda} \\ &\geq (1-\varepsilon)^{1-\lambda}\tilde{f}(x)^{1-\lambda}(1-\varepsilon)^{\lambda}\tilde{g}(y)^{\lambda} = (1-\varepsilon)\tilde{f}(x)^{1-\lambda}\tilde{g}(y)^{\lambda}, \end{aligned}$$

which proves (3.64). $\square$

LEMMA 3.7.3. *Let $f, g : \mathbb{R}^n \to [0,\infty)$ be log-concave with positive integrals.*

(i) For $\lambda \in [0,1]$, the function $h_\lambda : \mathbb{R}^n \to [0,\infty)$ defined by

$$h_\lambda(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$$

is log-concave, has positive integral, and satisfies $h_0 = f$ and $h_1 = g$.

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(ii) The function $\lambda \mapsto \int_{\mathbb{R}^n} h_\lambda$ is log-concave for $\lambda \in [0,1]$.

Proof: From the definition of h_λ, it's immediately obvious that $h_0 = f$ and $h_1 = g$. Now we assume $\lambda \in (0,1)$. We show the log-concavity of h_λ by proving that if $z_1, z_2 \in \mathbb{R}^n$, $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and $\varepsilon \in (0,1)$, then

$$h_\lambda(\alpha z_1 + \beta z_2) \geq (1-\varepsilon) h_\lambda(z_1)^\alpha h_\lambda(z_2)^\beta. \tag{3.65}$$

Next, we choose $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ satisfying that $z_1 = (1-\lambda)x_1 + \lambda y_1$, $z_2 = (1-\lambda)x_2 + \lambda y_2$ and

$$f(x_1)^{1-\lambda} g(y_1)^\lambda \geq (1-\varepsilon) h_\lambda(z_1) \quad \text{and} \quad f(x_2)^{1-\lambda} g(y_2)^\lambda \geq (1-\varepsilon) h_\lambda(z_2).$$

Then we have that $\alpha z_1 + \beta z_2 = (1-\lambda)(\alpha x_1 + \beta x_2) + \lambda(\alpha y_1 + \beta y_2)$ and the log-concavity of f and g gives us

$$\begin{aligned}
h_\lambda(\alpha z_1 + \beta z_2) &= h_\lambda\Big((1-\lambda)(\alpha x_1 + \beta x_2) + \lambda(\alpha y_1 + \beta y_2)\Big) \\
&\geq f(\alpha x_1 + \beta x_2)^{1-\lambda} g(\alpha y_1 + \beta y_2)^\lambda \\
&\geq f(x_1)^{\alpha(1-\lambda)} f(x_2)^{\beta(1-\lambda)} g(y_1)^{\alpha(\lambda)} g(y_2)^{\beta(\lambda)} \\
&= \left(f(x_1)^{1-\lambda} g(y_1)^\lambda\right)^\alpha \left(f(x_2)^{1-\lambda} g(y_2)^\lambda\right)^\beta \\
&\geq (1-\varepsilon)^\alpha h_\lambda(z_1)^\alpha (1-\varepsilon)^\beta h_\lambda(z_2)^\beta = (1-\varepsilon) h_\lambda(z_1)^\alpha h_\lambda(z_2)^\beta.
\end{aligned}$$

This proves (3.65), and hence h_λ is log-concae.

Readily, $\int_{\mathbb{R}^n} h_\lambda > 0$. Lemma 3.4.1 gives us that $0 < M_f, M_g < \infty$ and hence

$$M = M_{h_\lambda} = M_f^{1-\lambda} M_g^\lambda.$$

If $h_\lambda(z) > t$, for some $t \in (0, M)$, then there exist $x, y \in \mathbb{R}^n$ such that

$$z = (1-\lambda)x + \lambda y \tag{3.66}$$

and $f(x)^{1-\lambda} g(y)^\lambda > t$. Then we have

$$f(x) > \left(\frac{t}{M_g^\lambda}\right)^{\frac{1}{1-\lambda}} \quad \text{and} \quad g(y) > \left(\frac{t}{M_f^{1-\lambda}}\right)^{\frac{1}{\lambda}}. \tag{3.67}$$

Then (3.66), (3.67) and Lemma 3.4.1 yield that $h_\lambda(z) > t$ is bounded, and hence, Lemma 3.4.1 implies h_λ has positive integral.

To show that $\lambda \mapsto \int_{\mathbb{R}^n} h_\lambda$ is log-concave for $\lambda \in [0,1]$ it is enough to prove that if $\lambda_1, \lambda_2 \in [0,1]$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$, then for $\lambda = \alpha\lambda_1 + \beta\lambda_2$, we have

$$\int_{\mathbb{R}^n} h_\lambda \geq \left(\int_{\mathbb{R}^n} h_{\lambda_1}\right)^\alpha \left(\int_{\mathbb{R}^n} h_{\lambda_2}\right)^\beta. \tag{3.68}$$

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Note here that due to the Prékopa-Leindler inequality Theorem 3.1.1, it suffices to show if $z = \alpha z_1 + \beta z_2$, $z_1, z_2 \in \mathbb{R}^n$, then

$$h_\lambda(z) \geq h_{\lambda_1}(z_1)^\alpha h_{\lambda_2}(z_2)^\beta.$$

And this would follow if we can show that if $z = \alpha z_1 + \beta z_2$ for $z_1, z_2 \in \mathbb{R}^n$ and $\varepsilon \in (0,1)$, then

$$h_\lambda(z) \geq (1-\varepsilon) h_{\lambda_1}(z_1)^\alpha h_{\lambda_2}(z_2)^\beta. \tag{3.69}$$

We have that for $i = 1, 2$, there exist $x_i, y_i \in \mathbb{R}^n$ such that

$$\begin{aligned} z_i &= (1-\lambda_i)x_i + \lambda_i y_i & (3.70)\\ f(x_i)^{1-\lambda_i} g(y_i)^{\lambda_i} &\geq (1-\varepsilon) h_{\lambda_i}(z_i). & (3.71) \end{aligned}$$

From $\lambda = \alpha\lambda_1 + \beta\lambda_2$ and $z = \alpha z_1 + \beta z_2$, it follows that $1 - \lambda = \alpha(1-\lambda_1) + \beta(1-\lambda_2)$ and

$$\begin{aligned} z &= \alpha z_1 + \beta z_2 = \alpha\Big[(1-\lambda_1)x_1 + \lambda_1 y_1\Big] + \beta\Big[(1-\lambda_2)x_2 + \lambda_2 y_2\Big] \\ & (1-\lambda) \cdot \left(\frac{\alpha(1-\lambda_1)}{1-\lambda} \cdot x_1 + \frac{\beta(1-\lambda_2)}{1-\lambda} \cdot x_2 \right) + \lambda \cdot \left(\frac{\alpha\lambda_1}{\lambda} \cdot y_1 + \frac{\beta\lambda_2}{\lambda} \cdot y_2 \right). \end{aligned}$$

The, the fact that f and g are log-concave, and (3.71) give us

$$\begin{aligned} h_\lambda(z) &\geq f\left(\frac{\alpha(1-\lambda_1)}{1-\lambda} \cdot x_1 + \frac{\beta(1-\lambda_2)}{1-\lambda} \cdot x_2 \right)^{1-\lambda} g\left(\frac{\alpha\lambda_1}{\lambda} \cdot y_1 + \frac{\beta\lambda_2}{\lambda} \cdot y_2 \right)^\lambda \\ &\geq f(x_1)^{\alpha(1-\lambda_1)} f(x_2)^{\beta(1-\lambda_2)} g(y_1)^{\alpha\lambda_1} g(y_2)^{\beta\lambda_2} \\ &= \left(f(x_1)^{1-\lambda_1} g(y_1)^{\lambda_1} \right)^\alpha \left(f(x_2)^{1-\lambda_2} g(y_2)^{\lambda_2} \right)^\beta \\ &\geq (1-\varepsilon)^\alpha h_{\lambda_1}(z_1)^\alpha (1-\varepsilon)^\beta h_{\lambda_2}(z_2)^\beta = (1-\varepsilon) h_{\lambda_1}(z_1)^\alpha h_{\lambda_2}(z_2)^\beta, \end{aligned}$$

which shows that (3.69), and hence so does (3.68). □

LEMMA 3.7.4. *For a fixed $\lambda \in (0,1)$, if $\eta \in (0, 2\cdot\min\{1-\lambda, \lambda\})$ and φ is a log-concave function on $[0,1]$ satisfying $\varphi(\lambda) \leq (1+\eta)\varphi(0)^{1-\lambda}\varphi(1)^\lambda$, then*

$$\varphi\left(\tfrac{1}{2}\right) \leq \left(1 + \frac{\eta}{\min\{1-\lambda, \lambda\}}\right) \sqrt{\varphi(0)\varphi(1)}$$

Proof: We may assume that $0 < \lambda < \frac{1}{2}$. Then we have $\lambda = (1-2\lambda)\cdot 0 + 2\lambda \cdot \frac{1}{2}$, $\varphi(\lambda) \leq (1+\eta)\varphi(0)^{1-\lambda}\varphi(1)^\lambda$. And sice φ is log-concave, it follows that

$$(1+\eta)\varphi(0)^{1-\lambda}\varphi(1)^\lambda \geq \varphi(\lambda) \geq \varphi(0)^{1-2\lambda}\varphi\left(\tfrac{1}{2}\right)^{2\lambda}.$$

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Then noting that $(1+\eta)^{\frac{1}{2\lambda}} \le e^{\frac{\eta}{2\lambda}} \le 1+\frac{\eta}{\lambda}$, it follows that

$$\varphi\left(\tfrac{1}{2}\right) \le (1+\eta)^{\frac{1}{2\lambda}}\sqrt{\varphi(0)\varphi(1)} \le \left(1+\frac{\eta}{\lambda}\right)\sqrt{\varphi(0)\varphi(1)}. \quad \square$$

Proof of Theorem 3.7.1: We may assume that the λ in Theorem 3.7.1 satisfies $0<\lambda\le\frac{1}{2}$. Then $\min\{1-\lambda,\lambda\}=\lambda$.

For suitable $d,e>0$ and $w\in\mathbb{R}^n$, we may replace $f(z)$ by $d\cdot f(z-w)$, $g(z)$ by $e\cdot g(z+w)$ and $h(z)$ by $d^{1-\lambda}e^{\lambda}h(z+(2\lambda-1)w)$ where e and d will be defined by (3.73) below, and w will be defined by (3.76) and (3.77).

Denote by $\tilde{f}$ and $\tilde{g}$ the log-concave hulls of f and g. Then from Lemma 3.7.2, we have

$$h((1-\lambda)x+\lambda y) \ge \tilde{f}(x)^{1-\lambda}\tilde{g}(y)^{\lambda} \quad \text{for } x,y\in\mathbb{R}^n. \tag{3.72}$$

By (3.63), we may assume

$$\int_{\mathbb{R}^n}\tilde{f} = \int_{\mathbb{R}^n}\tilde{g} = 1. \tag{3.73}$$

Let

$$h_t(z) = \sup_{z=(1-t)x+ty} \tilde{f}(x)^{1-t}\tilde{g}(y)^t$$

and

$$\varphi(t) = \int_{\mathbb{R}^n} h_t.$$

Then Lemma 3.7.3 implies that φ is log-concave on $[0,1]$ and that

$$\varphi(0)=\varphi(1)=1. \tag{3.74}$$

Using (3.72), (3.73), the Prékopa-Leindler inequality Theorem 3.1.1 and the conditions in Theorem 3.7.1 we have that

$$\begin{aligned} 1 &= \left(\int_{\mathbb{R}^n}\tilde{f}\right)^{1-\lambda}\left(\int_{\mathbb{R}^n}\tilde{g}\right)^{\lambda} \le \int_{\mathbb{R}^n}h_\lambda \le \int_{\mathbb{R}^n}h \le (1+\varepsilon)\left(\int_{\mathbb{R}^n}f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n}g\right)^{\lambda} \\ &\le (1+\varepsilon)\left(\int_{\mathbb{R}^n}\tilde{f}\right)^{1-\lambda}\left(\int_{\mathbb{R}^n}\tilde{g}\right)^{\lambda} = 1+\varepsilon. \end{aligned} \tag{3.75}$$

In view of (3.75) and (3.74), Lemma 3.7.4 gives us

$$\int_{\mathbb{R}^n} h_{1/2} = \varphi\left(\frac{1}{2}\right) \le \left(1+\frac{\varepsilon}{\lambda}\right)\sqrt{\varphi(0)\varphi(1)} = 1+\frac{\varepsilon}{\lambda}.$$

And then using Theorem 3.3.1 we have that there exists $w\in\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n}|\tilde{f}(z)-h_{1/2}(z+w)|\,dz \;\le\; \tilde{c}n^8\sqrt[18]{\frac{\varepsilon}{\lambda}}\cdot\left|\log\frac{\varepsilon}{\lambda}\right|^n, \tag{3.76}$$

$$\int_{\mathbb{R}^n}|\tilde{g}(z)-h_{1/2}(z-w)|\,dz \;\le\; \tilde{c}n^8\sqrt[18]{\frac{\varepsilon}{\lambda}}\cdot\left|\log\frac{\varepsilon}{\lambda}\right|^n. \tag{3.77}$$

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Note that the function $h_{1/2}$ does not change if we replace $f(z)$ by $f(z-w)$ and $g(z)$ by $g(z+w)$ (*cf.* (3.63)), and it follows that

$$\int_{\mathbb{R}^n} |\tilde{f} - h_{1/2}| \leq \tilde{c}n^8 \sqrt[18]{\frac{\varepsilon}{\lambda}} \cdot \left|\log \frac{\varepsilon}{\lambda}\right|^n, \tag{3.78}$$

$$\int_{\mathbb{R}^n} |\tilde{g} - h_{1/2}| \leq \tilde{c}n^8 \sqrt[18]{\frac{\varepsilon}{\lambda}} \cdot \left|\log \frac{\varepsilon}{\lambda}\right|^n. \tag{3.79}$$

In order to replace $h_{1/2}$ by h in (3.78) and (3.79), we claim that

$$\int_{\mathbb{R}^n} |h - h_{1/2}| \leq 5\tilde{c}n^8 \sqrt[18]{\frac{\varepsilon}{\lambda}} \cdot \left|\log \frac{\varepsilon}{\lambda}\right|^n. \tag{3.80}$$

Let

$$\begin{aligned} X_- &= \{x \in \mathbb{R}^n : h(x) \leq h_{1/2}(x)\} \\ X_+ &= \{x \in \mathbb{R}^n : h(x) > h_{1/2}(x)\}. \end{aligned}$$

Then (3.72) implies that for any $x \in X_-$, we have

$$h(x) \geq \tilde{f}(x)^{1-\lambda}\tilde{g}(x)^{\lambda} \geq \min\{\tilde{f}(x), \tilde{g}(x)\}.$$

And so, for $x \in X_-$, we have

$$0 \leq h_{1/2}(x) - h(x) \leq |h_{1/2}(x) - \tilde{f}(x)| + |h_{1/2}(x) - \tilde{g}(x)|.$$

Then it follows from (3.78) and (3.79) that

$$\begin{aligned} \int_{X_-} |h - h_{1/2}| &= \int_{X_-} (h_{1/2} - h) \leq \int_{X_-} (|h_{1/2} - \tilde{f}| + |h_{1/2} - \tilde{g}|) \\ &\leq 2\tilde{c}n^8 \sqrt[18]{\frac{\varepsilon}{\lambda}} \cdot \left|\log \frac{\varepsilon}{\lambda}\right|^n, \end{aligned} \tag{3.81}$$

Now, we have $\int_{\mathbb{R}^n} h < 1 + \varepsilon$ and $\int_{\mathbb{R}^n} h_{1/2} \geq 1$ by (3.75), and hence it follows from (3.81) that

$$\begin{aligned} \int_{X_+} |h - h_{1/2}| &= \int_{X_+} (h - h_{1/2}) = \int_{\mathbb{R}^n} h - \int_{\mathbb{R}^n} h_{1/2} + \int_{X_-} (h_{1/2} - h) \\ &\leq \varepsilon + \int_{X_-} (h_{1/2} - h) \leq 3\tilde{c}n^8 \sqrt[18]{\frac{\varepsilon}{\lambda}} \cdot \left|\log \frac{\varepsilon}{\lambda}\right|^n. \end{aligned} \tag{3.82}$$

Thus, by (3.81) and (3.82), our claim (3.80) holds.

Next in order to replace $\tilde{f}$ and $\tilde{g}$ by f and g in (3.78) and (3.79), , we claim that

$$\int_{\mathbb{R}^n} |f - \tilde{f}| \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^n} |g - \tilde{g}| \leq \varepsilon. \tag{3.83}$$

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We have $\tilde f \geq f$ and $\tilde g \geq g$, and $\int_{\mathbb{R}^n} g \leq \int_{\mathbb{R}^n} \tilde g = 1$. Then (3.75) gives us

$$\int_{\mathbb{R}^n} |f - \tilde f| = \int_{\mathbb{R}^n} \tilde f - \int_{\mathbb{R}^n} f \leq 1 - \frac{1}{1+\varepsilon} < \varepsilon.$$

Similarly, we can handle g and $\tilde g$, and thus, (3.83) holds.

Finally, using (3.78), (3.79), (3.80) and (3.83), we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f-h|\,dx &\leq 7\tilde c n^8 \sqrt[18]{\frac{\varepsilon}{\lambda}} \cdot \left|\log\frac{\varepsilon}{\lambda}\right|^n,\\ \int_{\mathbb{R}^n} |g-h|\,dx &\leq 7\tilde c n^8 \sqrt[18]{\frac{\varepsilon}{\lambda}} \cdot \left|\log\frac{\varepsilon}{\lambda}\right|^n,\end{aligned}$$

which, in turn, proves Theorem 3.7.1. □

3.8 Proof of Theorem 3.1.3 and Corollary 3.1.4

First we derive the following simple estimate.

LEMMA 3.8.1. *If $\varrho > 0$, $t > 1$ and $n \geq 2$, then*

$$(\log t)^n \leq \left(\frac{n\varrho}{e}\right)^n t^{\frac{1}{\varrho}}.$$

Proof: Denote $s = \log t$, and consider the function $\psi(s) = n\log s - \frac{s}{\varrho}$. Taking the derivative of $\psi(s)$, we have

$$\log\frac{(\log t)^n}{t^{\frac{1}{\varrho}}} = n\log s - \frac{s}{\varrho} \leq n(\log(n\varrho) - 1) = n\log\frac{n\varrho}{e},$$

which leads to Lemma 3.8.1. □

Proof of Theorem 3.1.3 and Corollary 3.1.4: We may assume that f and g are probability densities.

Theorem 3.7.1 and Lemma 3.8.1 then gives us that for some absolute constants $c_1, c_2 > 1$, if $\varepsilon < c_1^{-n} n^{-n} \cdot \tau$, then there exists $w \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x) - h(x - \lambda w)|\,dx \leq c_2^n n^n \sqrt[19]{\frac{\varepsilon}{\tau}} \tag{3.84}$$

$$\int_{\mathbb{R}^n} |g(x) - h(x + (1-\lambda)w)|\,dx \leq c_2^n n^n \sqrt[19]{\frac{\varepsilon}{\tau}}, \tag{3.85}$$

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which proves Theorem 3.1.3 in the case when $\varepsilon < c_1^{-n} n^{-n} \cdot \tau$.

Now, if $\varepsilon \geq c_1^{-n} n^{-n} \cdot \tau$, the left hand sides of (3.84) and (3.85) are at most $2+\varepsilon \leq 3$. Then for a suitable absolute constant $c_2 > 1$, both (3.84) and (3.85) hold. This completes the proof of Theorem 3.1.3.

For Corollary 3.1.4, the functions f and g are log-concave probability densities on $\mathbb{R}^n$. Let

$$h(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}.$$

Then Lemma 3.7.3 (i) implies that h is log-concave on $\mathbb{R}^n$. Using the same w as in (3.84) and (3.85), we have that

$$\widetilde{L}_1(f,g) \leq \int_{\mathbb{R}^n} |f(x+w) - g(x)| \, dx \leq 2c_2^n n^n \sqrt[19]{\frac{\varepsilon}{\tau}},$$

and that settles Corollary 3.1.4. □

3.9 Proof of Theorem 3.1.5

Lemma 3.7.3 (i) and induction on m gives us the following corollary.

COROLLARY 3.9.1. *If $\lambda_1, \ldots, \lambda_m > 0$ satisfy $\sum_{i=1}^m \lambda_i = 1$ and $f_1, \ldots, f_m$ are log-concave functions with positive integral on $\mathbb{R}^n$, then*

$$h(z) = \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f_i(x_i)^{\lambda_i}$$

is log-concave and has positive integral.

In order to prove Theorem 3.9.4, we first consider the case when each λ_i in Theorem 3.9.4 is $\frac{1}{m}$.

THEOREM 3.9.2. *Let $c > 1$ be the absolute constant in Theorem 3.7.1, let $\gamma_0 = cn^8$ and $\varepsilon_0 = c^{-n} n^{-n}$. If $f_1, \ldots, f_m$, $m \geq 2$ are log-concave probability densities on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \sup_{mz=\sum_{i=1}^m x_i} \prod_{i=1}^m f_i(x_i)^{\frac{1}{m}} \, dz \leq 1 + \varepsilon$$

for $0 < \varepsilon < \varepsilon_0 / m^4$, then for the log-concave $h(z) = \sup_{mz=\sum_{i=1}^m x_i} \prod_{i=1}^m f_i(x_i)^{\frac{1}{m}}$, there exist $w_1, \ldots, w_m \in \mathbb{R}^n$ such that $\sum_{i=1}^m w_i = o$ and

$$\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)| \, dx \leq m^4 \cdot \gamma_0 \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n.$$

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Proof: Here we prove that if $0 < \varepsilon < \varepsilon_0/4^{\lceil \log_2 m \rceil}$, then there exist $w_1, \ldots, w_m \in \mathbb{R}^n$ such that $\sum_{i=1}^m w_i = o$ and

$$\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)|\, dx \le 4^{\lceil \log_2 m \rceil} \cdot \gamma_0 \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n . \tag{3.86}$$

And then since $4^{\lceil \log_2 m \rceil} < 4^{2 \log_2 m} = m^4$, (3.86) implies Theorem 3.9.2.

To prove (3.86), we use induction on $\lceil \log_2 m \rceil \ge 1$ If $\lceil \log_2 m \rceil = 1$, and hence $m = 2$, then Lemma 3.7.3 (i) implies that

$$h(z) = \sup_{z = \lambda_1 x_1 + \lambda_2 x_2} f_1(x_1)^{\lambda_1} f_2(x_2)^{\lambda_2}$$

is log-concave, and in turn, Theorem 3.7.1 yields (3.86) in this case.

Next, we assume that $\lceil \log_2 m \rceil > 1$, and let $k = \lceil m/2 \rceil$, and hence $\lceil \log_2 (m-k) \rceil \le \lceil \log_2 k \rceil = \lceil \log_2 m \rceil - 1$. Let

$$\lambda = \frac{m-k}{m} \text{ satisfying } \frac{1}{3} \le \lambda \le \frac{1}{2},$$

and

$$\begin{aligned} h(z) &= \sup_{mz = \sum_{i=1}^m x_i} \prod_{i=1}^m f_i(x_i)^{\frac{1}{m}} & (3.87)\\ f(z) &= \sup_{kz = \sum_{i=1}^k x_i} \prod_{i=1}^k f_i(x_i)^{\frac{1}{k}} & (3.88)\\ g(z) &= \sup_{(m-k)z = \sum_{i=k+1}^m x_i} \prod_{i=k+1}^m f_i(x_i)^{\frac{1}{m-k}}. & (3.89) \end{aligned}$$

Corollary 3.9.1 then says that f, g, h are log-concave. Further, note that

$$h(z) = \sup_{z = \lambda x + (1-\lambda) y} f(x)^{1-\lambda} g(y)^{\lambda}.$$

Then Prékopa-Leindler inequality gives us that

$$\begin{aligned} \int_{\mathbb{R}^n} f &\ge 1 & (3.90)\\ \int_{\mathbb{R}^n} g &\ge 1 & (3.91)\\ \int_{\mathbb{R}^n} h &\ge \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda} \ge 1. & (3.92)\\ & & (3.93) \end{aligned}$$

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From $\int_{\mathbb{R}^n} h < 1+\varepsilon$, we get

$$\int_{\mathbb{R}^n} f \leq (1+\varepsilon)^{\frac{1}{1-\lambda}} \leq (1+\varepsilon)^3 \leq 1+4\varepsilon \tag{3.94}$$

$$\int_{\mathbb{R}^n} g \leq 1+4\varepsilon. \tag{3.95}$$

And it follows from Theorem 3.7.1 that for $a = \int_{\mathbb{R}^n} g / \int_{\mathbb{R}^n} f$, there exists $w \in \mathbb{R}^n$ such that

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x) - a^{\lambda} h(x - \lambda w)|\, dx &\leq \gamma_0 \sqrt[18]{\frac{\varepsilon}{1/3}} \cdot |\log \varepsilon|^n \int_{\mathbb{R}^n} f \\
&\leq 2\gamma_0 \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n
\end{aligned} \tag{3.96}$$

$$\begin{aligned}
\int_{\mathbb{R}^n} |g(x) - a^{-(1-\lambda)} h(x + (1-\lambda)w)|\, dx &\leq \gamma_0 \sqrt[18]{\frac{\varepsilon}{1/3}} \cdot |\log \varepsilon|^n \int_{\mathbb{R}^n} g \\
&\leq 2\gamma_0 \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n.
\end{aligned} \tag{3.97}$$

From (3.90), (3.91), (3.94), (3.95) we have that $1+4\varepsilon > a, a^{-1} > \frac{1}{1+4\varepsilon} > 1-4\varepsilon$. Then using $\frac{1}{3} \leq \lambda \leq \frac{2}{3}$, $\int_{\mathbb{R}^n} h < 1+\varepsilon$, (3.96) and (3.97), we have

$$\int_{\mathbb{R}^n} |f(x) - h(x - \lambda w)|\, dx \leq 4\gamma_0 \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n \tag{3.98}$$

$$\int_{\mathbb{R}^n} |g(x) - h(x + (1-\lambda)w)|\, dx \leq 4\gamma_0 \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n. \tag{3.99}$$

Since $\lceil \log_2(m-k) \rceil \leq \lceil \log_2 k \rceil = \lceil \log_2 m \rceil - 1$, using induction and (3.89), (3.88) and (3.87), we have that there exist $\tilde{w}_1, \dots, \tilde{w}_m \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{k} \tilde{w}_i = \sum_{j=k+1}^{m} \tilde{w}_j = o, \tag{3.100}$$

and if $i = 1, \dots, k$ and $j = k+1, \dots, m$, then

$$\int_{\mathbb{R}^n} |f_i(x) - f(x + \tilde{w}_i)|\, dx \leq 4^{\lceil \log_2 k \rceil} \gamma_0 \sqrt[18]{4\varepsilon} \cdot |\log \varepsilon|^n, \tag{3.101}$$

$$\int_{\mathbb{R}^n} |f_j(x) - g(x + \tilde{w}_j)|\, dx \leq 4^{\lceil \log_2 (m-k) \rceil} \gamma_0 \sqrt[18]{4\varepsilon} \cdot |\log \varepsilon|^n. \tag{3.102}$$

Using (3.98), (3.99), (3.101) and (3.102), we have that there exists $w_1, \dots, w_m \in \mathbb{R}^n$ such that for $i = 1, \dots, m$,

$$\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)|\, dx \leq 4 \cdot 4^{\lceil \log_2 m \rceil - 1} \gamma_0 \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n = 4^{\lceil \log_2 m \rceil} \gamma_0 \sqrt[18]{\varepsilon} \cdot |\log \varepsilon|^n;$$

where

$$\begin{aligned}
w_i &= -\lambda w - \tilde{w}_i \quad \text{for } i = 1, \dots, k, \\
w_j &= (1-\lambda) w - \tilde{w}_j \quad \text{for } j = k+1, \dots, m.
\end{aligned}$$

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From $\lambda = \frac{m-k}{m}$ and $1-\lambda = \frac{k}{m}$, we have

$$\sum_{i=1}^m w_i = -k\cdot\lambda\, w - \left(\sum_{i=1}^k \tilde{w}_i\right) + (m-k)(1-\lambda)w - \left(\sum_{j=k+1}^m \tilde{w}_j\right) = o,$$

which settles the proof of (3.86). $\square$

For $m \geq 2$, denote the $(m-1)$-simplex

$$\Delta^{m-1} = \{p = (p_1, \ldots, p_m) \in \mathbb{R}^m : p_1 + \ldots + p_m = 1\}.$$

The proof of Lemma 3.7.3 can be readily extended to show the following lemma.

LEMMA 3.9.3. *Let $f_1, \ldots, f_m$, $m \geq 2$ be log-concave probability densities on $\mathbb{R}^n$, $n \geq 2$. For $p = (p_1, \ldots, p_m) \in \Delta^{m-1}$, the function*

$$h_p(z) = \sup_{z=\sum_{i=1}^m p_i x_i} \prod_{i=1}^m f_i(x_i)^{p_i}$$

is log-concave on $\mathbb{R}^n$, and the function $p \mapsto \int_{\mathbb{R}^n} h_p$ is log-concave on Δ^{m-1}.

THEOREM 3.9.4. *For some absolute constant $\tilde{\gamma} > 1$, if $\tau \in (0, \frac{1}{m}]$, $m \geq 2$, $\lambda_1, \ldots, \lambda_m \in [\tau, 1-\tau]$ satisfy $\sum_{i=1}^m \lambda_i = 1$ and $f_1, \ldots, f_m$ are log-concave functions with positive integral on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f_i(x_i)^{\lambda_i}\, dz \leq (1+\varepsilon) \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i\right)^{\lambda_i}$$

for $0 < \varepsilon < \tau \cdot \tilde{\gamma}^{-n} n^{-n}/m^4$, then for the log-concave $h(z) = \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f(x_i)^{\lambda_i}$, there exist $a_1, \ldots, a_m > 0$ and $w_1, \ldots, w_m \in \mathbb{R}^n$ such that $\sum_{i=1}^m \lambda_i w_i = o$ and for $i = 1, \ldots, m$, we have

$$\int_{\mathbb{R}^n} |f_i(x) - a_i h(x + w_i)|\, dx \leq \tilde{\gamma} m^5 n^8 \sqrt[18]{\frac{\varepsilon}{m\tau}} \cdot \left|\log \frac{\varepsilon}{m\tau}\right|^n \int_{\mathbb{R}^n} f_i.$$

Remark $a_i = \dfrac{\left(\int_{\mathbb{R}^n} f_i\right)^{1-\lambda_i}}{\prod_{j\neq i}\left(\int_{\mathbb{R}^n} f_j\right)^{\lambda_j}}$ for $i = 1, \ldots, m$ in Theorem 3.9.4.

Proof: Let $\tau \in (0, \frac{1}{m}]$, let $\lambda_1, \ldots, \lambda_m \in [\tau, 1-\tau]$ with $\lambda_1 + \ldots + \lambda_m = 1$ and let $f_1, \ldots, f_m$ be log-concave with positive integralral as in Theorem 3.9.4. In particular,

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$\lambda = (\lambda_1, \dots, \lambda_m) \in \Delta^{m-1}$. We may assume that $f_1, \dots, f_m$ are probability densities, $\lambda_1 \geq \dots \geq \lambda_m$ and $\lambda_m < \frac{1}{m}$ (as if $\lambda_m = \frac{1}{m}$, then Theorem 3.9.2 implies Theorem 3.9.4).

Let $\tilde{p} = (\frac{1}{m}, \dots, \frac{1}{m})$, and for $i = 1, \dots, m$, let $v_{(i)} \in \mathbb{R}^m$ be the vector whose ith coordinate is 1, and the rest is 0, and hence $v_{(1)}, \dots, v_{(m)}$ are the vertices of Δ^{m-1}, and $\lambda = \sum_{i=1}^m \lambda_i v_{(i)}$. For $p = (p_1, \dots, p_m) \in \Delta^{m-1}$, we write

$$h_p(z) = \sup_{z=\sum_{i=1}^m p_i x_i} \prod_{i=1}^m f_i(x_i)^{p_i},$$

and hence $h_{v_{(i)}} = f_i$. According the conditions in Theorem 3.9.4,

$$\int_{\mathbb{R}^n} h_\lambda < 1 + \varepsilon. \tag{3.103}$$

Since $\lambda_i - \lambda_m \geq 0$ for $i = 1, \dots, m-1$ and $m\lambda_m < 1$, it follows that

$$q = \sum_{i=1}^{m-1} \frac{\lambda_i - \lambda_m}{1 - m\lambda_m} \cdot v_{(i)} \in \Delta^{m-1},$$

and Corollary 3.9.3 and $\int_{\mathbb{R}^n} h_{v_{(i)}} = \int_{\mathbb{R}^n} f_i = 1$ yield that $\int_{\mathbb{R}^n} h_q \geq 1$. Since

$$\lambda = m\lambda_m \tilde{p} + \sum_{i=1}^{m-1} (\lambda_i - \lambda_m) v_{(i)} = m\lambda_m \tilde{p} + (1 - m\lambda_m) q,$$

(3.103) and Corollary 3.9.3 imply that

$$1 + \varepsilon > \int_{\mathbb{R}^n} h_\lambda \geq \left(\int_{\mathbb{R}^n} h_q\right)^{1-m\lambda_m} \left(\int_{\mathbb{R}^n} h_{\tilde{p}}\right)^{m\lambda_m} \geq \left(\int_{\mathbb{R}^n} h_{\tilde{p}}\right)^{m\lambda_m};$$

therefore, $\varepsilon \leq m\tau \leq m\lambda_m$ yields

$$\int_{\mathbb{R}^n} h_{\tilde{p}} < (1+\varepsilon)^{\frac{1}{m\lambda_m}} < e^{\frac{\varepsilon}{m\lambda_m}} < 1 + \frac{2\varepsilon}{m\lambda_m}.$$

According to Theorem 3.9.2, there exist $w_1, \dots, w_m \in \mathbb{R}^n$ such that $\sum_{i=1}^m w_i = o$ and

$$\int_{\mathbb{R}^n} |f_i(x + w_i) - h_{\tilde{p}}(x)| \, dx \leq m^4 \cdot \gamma_0 \sqrt[18]{\frac{2\varepsilon}{m\lambda_m}} \cdot \left|\log \frac{2\varepsilon}{m\lambda_m}\right|^n$$

for $i = 1, \dots, m$. Replacing $f_i(x)$ by $f_i(x + w_i)$ for $i = 1, \dots, m$ does not change $h_{\tilde{p}}$ by the condition $\sum_{i=1}^m w_i = o$; therefore, we may assume that

$$\int_{\mathbb{R}^n} |f_i(x) - h_{\tilde{p}}(x)| \, dx \leq m^4 \cdot \gamma_0 \sqrt[18]{\frac{2\varepsilon}{m\lambda_m}} \cdot \left|\log \frac{2\varepsilon}{m\lambda_m}\right|^n \tag{3.104}$$

for $i = 1, \dots, m$.

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To replace $h_{\tilde p}$ by h_λ in (3.104), we claim that

$$\int_{\mathbb{R}^n} |h_\lambda - h_{\tilde p}| \le 3m^5 \cdot \gamma_0 \sqrt[18]{\frac{2\varepsilon}{m\lambda_m}} \cdot \left|\log \frac{2\varepsilon}{m\lambda_m}\right|^n. \tag{3.105}$$

To prove (3.105), we consider

$$\begin{aligned} X_- &= \{x \in \mathbb{R}^n : h_\lambda(x) \le h_{\tilde p}(x)\} \\ X_+ &= \{x \in \mathbb{R}^n : h_\lambda(x) > h_{\tilde p}(x)\}. \end{aligned}$$

It follows from the definition of h_λ that for any $x \in X_-$, we have

$$h_\lambda(x) \ge \prod_{i=1}^m f_i(x)^{\lambda_i} \ge \min\{f_1(x), \ldots, f_m(x)\},$$

or in other words, if $x \in X_-$, then

$$0 \le h_{\tilde p}(x) - h_\lambda(x) \le \sum_{i=1}^m |f_i(x) - h_{\tilde p}(x)|.$$

In particular, (3.104) implies

$$\begin{aligned} \int_{X_-} |h_\lambda - h_{\tilde p}| &= \int_{X_-} (h_{\tilde p} - h_\lambda) \le \sum_{i=1}^m \int_{X_-} |f_i(x) - h_{\tilde p}(x)| \\ &\le m^5 \cdot \gamma_0 \sqrt[18]{\frac{2\varepsilon}{m\lambda_m}} \cdot \left|\log \frac{2\varepsilon}{m\lambda_m}\right|^n. \end{aligned} \tag{3.106}$$

On the other hand, $\int_{\mathbb{R}^n} h_\lambda < 1+\varepsilon$ and the Prékopa-Leindler inequality yields $\int_{\mathbb{R}^n} h_{\tilde p} \ge 1$, thus (3.106) implies

$$\begin{aligned} \int_{X_+} |h_\lambda - h_{\tilde p}| &= \int_{X_+} (h_\lambda - h_{\tilde p}) = \int_{\mathbb{R}^n} h_\lambda - \int_{\mathbb{R}^n} h_{\tilde p} + \int_{X_-} (h_{\tilde p} - h_\lambda) \\ &\le \varepsilon + \int_{X_-} (h_{\tilde p} - h_\lambda) \le 2m^5 \cdot \gamma_0 \sqrt[18]{\frac{2\varepsilon}{m\lambda_m}} \cdot \left|\log \frac{2\varepsilon}{m\lambda_m}\right|^n. \end{aligned} \tag{3.107}$$

We conclude (3.105) by (3.106) and (3.107).

Finally, combining (3.104) and (3.105) prove Theorem 3.9.4. $\square$

Proof of Theorem 3.1.5 We may assume that $\int_{\mathbb{R}^n} f_i = 1$ for $i = 1, \ldots, m$ in Theorem 3.1.5 for the log-concave functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

Let $\tau \in (0, \frac{1}{m}]$ for $m \ge 2$, and let $\lambda_1, \ldots, \lambda_m \in [\tau, 1-\tau]$ satisfy $\sum_{i=1}^m \lambda_i = 1$ such that

$$\int_{\mathbb{R}^n} \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f_i(x_i)^{\lambda_i}\, dz \le 1 + \varepsilon$$

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for $\varepsilon \in (0,1]$.

For the absolute constant $\tilde{\gamma} > 1$ of Theorem 3.9.4, if

$$0 < \varepsilon < \tau \cdot \tilde{\gamma}^{-n} n^{-n} / m^4, \tag{3.108}$$

then for the log-concave $h(z) = \sup_{z=\sum_{i=1}^m \lambda_i x_i} \prod_{i=1}^m f(x_i)^{\lambda_i}$, there exist $w_1, \ldots, w_m \in \mathbb{R}^n$ such that $\sum_{i=1}^m \lambda_i w_i = o$ and for $i = 1, \ldots, m$, we have

$$\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)|\, dx \le \tilde{\gamma} m^5 n^8 \sqrt[18]{\frac{\varepsilon}{m\tau}} \cdot \left|\log \frac{\varepsilon}{m\tau}\right|^n.$$

We deduce from Lemma 3.8.1 that

$$\int_{\mathbb{R}^n} |f_i(x) - h(x + w_i)|\, dx \le \tilde{\gamma}_0^n n^n m^5 \sqrt[19]{\frac{\varepsilon}{m\tau}} \tag{3.109}$$

for $i = 1, \ldots, m$ and some absolute constant $\tilde{\gamma}_0 \ge \max\{\tilde{\gamma}, 3\}$, proving Theorem 3.1.5 if (3.108) holds. Finally if $\varepsilon \ge \tau \cdot \tilde{\gamma}^{-n} n^{-n} / m^4$, then (3.109) readily holds as the left hand side is at most $2 + \varepsilon \le 3$. $\square$

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Chapter 4

Stability of log-Brunn-Minkowski and log-Minkowski inequality under n hyperplane symmetries

4.1 Introduction

In this chapter, we establish stability versions of the log-Minkowski and log-Brunn-Minkowski conjectures under many hyperplane symmetries. First, recall that for $\lambda \in (0,1)$, the L_0 or logarithmic sum of two origin symmetric convex bodies K and C in $\mathbb{R}^n$ is defined by

$$(1-\lambda)\cdot K +_0 \lambda \cdot C = \left\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u)^{1-\lambda} h_C(u)^{\lambda} \ \forall u \in S^{n-1}\right\}.$$

It is linearly invariant in the sense that $A((1-\lambda)\cdot K +_0 \lambda \cdot C) = (1-\lambda)\cdot AK +_0 \lambda \cdot AC$ for $A \in \mathrm{GL}(n)$. We recall here the long-standing and highly investigated equivalent conjectures - the log-Brunn-Minkowski and log-Minkowski conjecture for centered convex-bodies.

Conjecture 4.1.1 (Logarithmic Brunn-Minkowski conjecture)**.** *If $\lambda \in (0,1)$ and K and C are convex bodies in $\mathbb{R}^n$ whose centroid is the origin, then*

$$V((1-\lambda)\cdot K +_0 \lambda \cdot C) \geq V(K)^{1-\lambda} V(C)^{\lambda}, \tag{4.1}$$

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one where $\sum_{i=1}^m \dim K_i = n$ and K_i and C_i are dilates, $i = 1, \ldots, m$.

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Conjecture 4.1.2 (Logarithmic Minkowski conjecture)**.** *If K and C are convex bodies in $\mathbb{R}^n$ whose centroid is the origin, then*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K \geq \frac{V(K)}{n} \log \frac{V(C)}{V(K)} \tag{4.2}$$

with the same equality conditions as in Conjecture 4.1.1.

We call a linear transformation $A \in \mathrm{GL}(n)$ a linear reflection associated to the linear hyperplane H (subspace of $\mathbb{R}^n$ of dimension $n-1$) if all the points of H are fixed under A and $\det A = -1$. In this case, there exists $u \in \mathbb{R}^n \backslash H$ such that $Au = -u$ where the invariant subspace $\mathbb{R}u$ is uniquely determined (see Davis [60], Humphreys [98], Vinberg [149]). A linear reflection A is a classical "orthogonal" reflection if and only if $A \in O(n)$.

Böröczky, Kalantzopoulos [38] verified the logarithmic Brunn-Minkowski and Minkowski conjectures under n hyperplane symmetry assumption, following Saroglou's [137] results on unconditional convex bodies.

THEOREM 4.1.3 (Böröczky, Kalantzopoulos)**.** *If the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1, \ldots, A_n$ through n hyperplanes $H_1, \ldots, H_n$ with $H_1 \cap \ldots \cap H_n = \{o\}$, then*

$$\begin{aligned} V((1-\lambda)\cdot K +_0 \lambda \cdot C) &\geq V(K)^{1-\lambda} V(C)^{\lambda} && (4.3) \\ \int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K &\geq \frac{V(K)}{n} \log \frac{V(C)}{V(K)}, && (4.4) \end{aligned}$$

with equality in either inequality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one and invariant under $A_1, \ldots, A_n$ where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$.

Barthe, Fradelizi [19] and Barthe, Cordero-Erausquin [18] first considered geometric inequalities under n-independent hyperplane symmetries where they verified the classical Mahler conjecture and Slicing conjecture, respectively for convex bodies with said symmetry.

The main result of this chapter is the following stability version of Theorem 4.1.3.

THEOREM 4.1.4. *If $\lambda \in [\tau, 1-\tau]$ for $\tau \in (0, \frac{1}{2}]$, the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1, \ldots, A_n$ through n hyperplanes $H_1, \ldots, H_n$ with $H_1 \cap \ldots \cap H_n = \{o\}$, and*

$$V((1-\lambda)\cdot K +_0 \lambda \cdot C) \leq (1+\varepsilon) V(K)^{1-\lambda} V(C)^{\lambda}$$

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for $\varepsilon > 0$, then for some $m \geq 1$, there exist compact convex sets $K_1, C_1, \ldots, K_m, C_m$ of dimension at least one and invariant under $A_1, \ldots, A_n$ where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$ such that

$$\begin{aligned} K_1 + \ldots + K_m \subset \;\; K \;\; &\subset \left(1 + c^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(K_1 + \ldots + K_m) \\ C_1 + \ldots + C_m \subset \;\; C \;\; &\subset \left(1 + c^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(C_1 + \ldots + C_m) \end{aligned}$$

where $c > 1$ is an absolute constant.

We note here that the bound in Theorem 4.1.4 is not far from being optimal in the sense that the exponent should be at least $1/(95n)$ should be at least $1/n$. The following example illustrates this. Consider the box $K_0 = [\frac{-1}{2^{n-1}}, \frac{1}{2^{n-1}}] \times [-2, 2]^{n-1}$. We cut off corners of K_0 of size of the order of $\varepsilon^{\frac{1}{n}}$, for small $\varepsilon > 0$. And we get C from the box $C_0 = [-2^{n-1}, 2^{n-1}] \times [\frac{-1}{2}, \frac{1}{2}]^{n-1}$ by cutting off corners of suitable size of order $\varepsilon^{\frac{1}{n}}$. Then we have that $\frac{1}{2} \cdot K +_0 \frac{1}{2} \cdot C = [-1, 1]^n$, and

$$V\left(\frac{1}{2} \cdot K +_0 \frac{1}{2} \cdot C\right) \leq (1 + \varepsilon) V(K)^{\frac{1}{2}} V(C)^{\frac{1}{2}},$$

But if $\eta K_0 \subset K$ for $\eta > 0$, then $\eta \leq 1 - \gamma\, \varepsilon^{\frac{1}{n}}$ for some $\gamma > 0$ depending on n.

We also derive the following stability version of the logarithmic-Minkowski inequality (4.4) for convex bodies with many hyperplane symmetries, which follows from Theorem 4.1.4.

THEOREM 4.1.5. *If the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1, \ldots, A_n$ through n hyperplanes $H_1, \ldots, H_n$ with $H_1 \cap \ldots \cap H_n = \{o\}$, and*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \frac{dV_K}{V(K)} \leq \frac{1}{n} \cdot \log \frac{V(C)}{V(K)} + \varepsilon$$

for $\varepsilon > 0$, then for some $m \geq 1$, there exist compact convex sets $K_1, C_1, \ldots, K_m, C_m$ of dimension at least one and invariant under $A_1, \ldots, A_n$ where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$ such that

$$\begin{aligned} K_1 + \ldots + K_m \subset \;\; K \;\; &\subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(K_1 + \ldots + K_m) \\ C_1 + \ldots + C_m \subset \;\; C \;\; &\subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(C_1 + \ldots + C_m) \end{aligned}$$

where $c > 1$ is an absolute constant.

Ivaki [99], Theorem 2.1 gives an improved version of Theorem 4.1.5 where K is a ball centered at the origin (and hence $m = 1$) and in this case, C need not satisfy any

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symmetry assumption (only translated in a suitable way). The error term is of the order of $\varepsilon^{\frac{1}{n+1}}$ instead of $\varepsilon^{\frac{1}{95n}}$.

Theorem 4.1.5 implies the stability of the solution of the Monge-Ampére equation Logarithmic-Minkowski Problem on S^{n-1} for unconditional data according to Böröczky, De [35]. We discuss it in Chapter 5.

In order to prove Theorem 4.1.4, we first verify it in the case of unconditional convex bodies, these partial results are presented in Section 4.2. First we use the recent stability version of the Prekopa-Leindler inequality to derive a stability result involving coordinatewise product of unconditional convex bodies (see Section 4.3). Then the unconditional case Theorem 4.2.3 of Theorem 4.1.4 is verified in Sections 4.4 and 4.5. Section 4.6 and Section 4.7 reviews some some fundamental properties of Weyl chambers and Coxeter groups in general. We prove Theorem 4.1.4 in Section 4.8 and finally, Theorem 4.1.5 is verified in Section 4.9.

4.2 The case of unconditional convex bodies

On our way to proving Theorem 4.1.4, we first consider the case of unconditional convex bodies. Note here that unconditional convex bodies are a particular case of symmetry with respect to n independent hyperplanes, that is, when $A_1, \ldots, A_n$ are orthogonal reflections and $H_1, \ldots, H_n$ are coordinate hyperplanes. The coordinatewise product of two convex bodies K and C is defined as follows: if $\lambda \in (0,1)$, then

$$\begin{aligned} K^{1-\lambda} \cdot C^{\lambda} \;&=\; \{(\pm|x_1|^{1-\lambda}|y_1|^{\lambda}, \ldots, \pm|x_n|^{1-\lambda}|y_n|^{\lambda}) \in \mathbb{R}^n : \\ &\quad\; (x_1, \ldots, x_n) \in K \;\text{ and }\; (y_1, \ldots, y_n) \in C\}. \end{aligned}$$

$K^{1-\lambda} \cdot C^{\lambda}$ is known to be an unconditional convex body (see for example Saroglou [137]). Further Hölder's inequality gives us the following containment relation between the coordinatewise product and the L_0 sum

$$K^{1-\lambda} \cdot C^{\lambda} \subset (1-\lambda) \cdot K +_0 \lambda \cdot C.$$

This containment relation together with its equality characterization was verified by [137] which also verified that for a positive definite diagonal matrix Φ, $\lambda \in (0,1)$ and unconditional convex body K in $\mathbb{R}^n$, we have

$$K^{1-\lambda} \cdot (\Phi K)^{\lambda} = \Phi^{\lambda} K \tag{4.5}$$

Here, Φ is a diagonal matrix of the form $\Phi = (t_1, \ldots, t_n)$ where $t_i > 0$, and $\Phi^{\eta} = (t_1^{\eta}, \ldots, t_n^{\eta})$ for $\eta \in \mathbb{R}$.

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The Logarithmic Brunn-Minkowski Conjecture 4.1.1 was verified for unconditional convex bodies by several authors, as Bollobas, Leader [28], Uhrin [147] and Cordero-Erausquin, Fradelizi, Maurey [56] verified the inequality $V((1-\lambda)\cdot K +_0 \lambda \cdot C) \geq V(K)^{1-\lambda}V(C)^{\lambda}$ in (4.6) about the coordinatewise product, even before the log-Brunn-Minkowski conjecture was stated, and the containment relation between the coordinatewise product and the L_0-sum and the description of the equality case are due to Saroglou [137]. For $X, Y \subset \mathbb{R}^n$, we write $X \oplus Y$ to denote $X+Y$ if lin$X$ and lin$Y$ are orthogonal.

THEOREM 4.2.1 (Saroglou)**.** *If K and C are unconditional convex bodies in $\mathbb{R}^n$ and $\lambda \in (0,1)$, then*

$$V((1-\lambda)\cdot K +_0 \lambda \cdot C) \geq V(K^{1-\lambda}\cdot C^{\lambda}) \geq V(K)^{1-\lambda}V(C)^{\lambda}. \tag{4.6}$$

(i) *$V(K^{1-\lambda}\cdot C^{\lambda}) = V(K)^{1-\lambda}V(C)^{\lambda}$ if and only if $C = \Phi K$ for a positive definite diagonal matrix Φ.*

(ii) *$V((1-\lambda)\cdot K +_0 \lambda \cdot C) = V(K)^{1-\lambda}V(C)^{\lambda}$ if and only if $K = K_1 \oplus \ldots \oplus K_m$ and $L = L_1 \oplus \ldots \oplus L_m$ for unconditional compact convex sets $K_1, \ldots, K_m, L_1, \ldots, L_m$ of dimension at least one where K_i and L_i are dilates, $i = 1, \ldots, m$.*

The second inequality in (4.6) about the coordinatewise product follows from the Prékopa-Leindler inequality (see Section 4.3). And as such the stability version Proposition 4.3.2 of the Prékopa-Leindler inequality leads to the following stability version of the aforementioned inequality.

THEOREM 4.2.2. *If $\lambda \in [\tau, 1-\tau]$ for $\tau \in (0, \frac{1}{2}]$, and the unconditional convex bodies K and C in $\mathbb{R}^n$ satisfy*

$$V(K^{1-\lambda}\cdot C^{\lambda}) \leq (1+\varepsilon)V(K)^{1-\lambda}V(C)^{\lambda}$$

for $\varepsilon > 0$, then there exists positive definite diagonal matrix Φ such that

$$V(K\Delta(\Phi C)) < c^n n^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{19}} V(K) \quad \textit{and} \quad V((\Phi^{-1}K)\Delta C) < c^n n^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{19}} V(C)$$

where $c > 1$ is an absolute constant.

In the case of the logarithmic-Brunn-Minkowski inequality for unconditional convex bodies, we have a different type stability estimate:

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THEOREM 4.2.3. *If $\lambda \in [\tau, 1-\tau]$ for $\tau \in (0, \frac{1}{2}]$, and the unconditional convex bodies K and C in $\mathbb{R}^n$ satisfy*

$$V((1-\lambda)\cdot K +_0 \lambda \cdot C) \leq (1+\varepsilon)V(K)^{1-\lambda}V(C)^{\lambda}$$

for $\varepsilon > 0$, then for some $m \geq 1$, there exist $\theta_1, \ldots, \theta_m > 0$ and unconditional compact convex sets $K_1, \ldots, K_m$ such that $\operatorname{lin} K_i$, $i = 1, \ldots, m$, are complementary coordinate subspaces, and

$$\begin{aligned} K_1 \oplus \ldots \oplus K_m \subset \quad K \quad &\subset \left(1 + c^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(K_1 \oplus \ldots \oplus K_m) \\ \theta_1 K_1 \oplus \ldots \oplus \theta_m K_m \subset \quad C \quad &\subset \left(1 + c^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(\theta_1 K_1 \oplus \ldots \oplus \theta_m K_m) \end{aligned}$$

where $c > 1$ is an absolute constant.

4.3 Coordinatewise product

The primary tool here is the Prékopa-Leindler inequality, a functional form of the Brunn-Minkowski inequality, due to Prékopa [131] and Leindler [108] in one dimension, and to Prékopa [132], C. Borell [29] and Brascamp, Lieb [42] in higher dimensions (see Artstein-Avidan, Florentin, Segal [9] for a recent variant). Various applications are explored and summarized in Ball [10], Barthe [17] and Gardner [80]. The following multiplicative version from [10] is particularly well-suited for geometric applications.

THEOREM 4.3.1 (Prékopa-Leindler)**.** *If $\lambda \in (0,1)$ and $h, f, g : \mathbb{R}^n \to [0, \infty)$ are integrable functions satisfying $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \cdot \left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{4.7}$$

Dubuc [63] characterized the equality case of Theorem 4.3.1 and showed that in the case of equality f, g and h should be essentially log-concave. Recall that a non-negative function φ on $\mathbb{R}^n$ is log-concave if $\varphi((1-\lambda)x + \lambda y) \geq \varphi(x)^{1-\lambda}\varphi(y)^{\lambda}$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$. We will make use of the following stability version of the Prékopa-Leindler inequality for log-concave functions established by Böröczky, De [34].

THEOREM 4.3.2. *If $\lambda \in (0,1)$ and f, g are log-concave probability densities on $\mathbb{R}^n$ satisfying*

$$\int_{\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}\, dz \leq 1 + \varepsilon$$

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for $\varepsilon > 0$, then there exists $w \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x) - g(x+w)|\, dx \le \omega_\lambda(\varepsilon) \tag{4.8}$$

where $\omega_\lambda(\varepsilon) = c^n n^n \left(\frac{\varepsilon}{\min\{\lambda, 1-\lambda\}}\right)^{\frac{1}{19}}$ for some absolute constant $c > 1$.

THEOREM 4.3.3. *If $\lambda \in (0,1)$ and unconditional convex bodies K and C in $\mathbb{R}^n$ satisfy*

$$V(K^{1-\lambda} \cdot C^\lambda) \le (1+\varepsilon)V(K)^{1-\lambda}V(C)^\lambda$$

for $\varepsilon > 0$, then there exists a positive definite diagonal matrix Φ such that

$$V(K\Delta(\Phi C)) < 8\omega_\lambda(\varepsilon)V(K) \quad \text{and } V((\Phi^{-1}K)\Delta C) < 8\omega_\lambda(\varepsilon)V(C) \tag{4.9}$$

where $\omega_\lambda(\varepsilon)$ is taken from (4.8).

Proof: For any unconditional convex body L, let's denote

$$L_+ = L \cap \mathbb{R}^n_+.$$

We may assume that

$$V(K_+) = V(C_+) = 1.$$

If $\omega_\lambda(\varepsilon) \ge \frac{1}{4}$, then choosing Φ to be any linear map with $\det\Phi = 1$, we have

$$\begin{aligned} V(K\Delta(\Phi C)) &= V(K) + V(\Phi C) - 2V(K \cap \Phi C) \\ &= V(K) + |\det\Phi| \cdot V(C) - 2V(K \cap \Phi C) \\ &= 2 - 2 \cdot V(K \cap \Phi C) \\ &< 2 < 8\omega_\lambda(\varepsilon)V(K) \end{aligned}$$

Similarly, we have $V((\Phi^{-1}K)\Delta C) < 8\omega_\lambda(\varepsilon)V(C)$. And hence, in this case, (4.9) is established. Therefore, we may also assume that $\varepsilon > 0$ is small enough to ensure

$$\omega_\lambda(\varepsilon) < \frac{1}{4}. \tag{4.10}$$

We set $M = K^{1-\lambda} \cdot C^\lambda$, and consider the log-concave functions $f, g, h : \mathbb{R}^n \to [0, \infty)$ defined by $f(x_1, \ldots, x_n) = \mathbf{1}_{K_+}(e^{x_1}, \ldots, e^{x_n})e^{x_1 + \ldots + x_n}$, $g(x_1, \ldots, x_n) = \mathbf{1}_{C_+}(e^{x_1}, \ldots, e^{x_n})e^{x_1 + \ldots + x_n}$ and $h(x_1, \ldots, x_n) = \mathbf{1}_{M_+}(e^{x_1}, \ldots, e^{x_n})e^{x_1 + \ldots + x_n}$. In particular,

$$h(z) = \sup_{z = (1-\lambda)x + \lambda y} f(x)^{1-\lambda} g(y)^\lambda$$

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holds for any $z \in \mathbb{R}^n$ by the definition of the coordinatewise product. In addition,

$$\begin{aligned}\int_{\mathbb{R}^n} h &= V(M_+) = \frac{V(M)}{2^n} \leq (1+\varepsilon)\left(\frac{V(K)}{2^n}\right)^{1-\lambda}\left(\frac{V(C)}{2^n}\right)^{\lambda} \\ &= (1+\varepsilon)\cdot V(K_+)^{1-\lambda}\cdot V(C_+)^{\lambda} \\ &= (1+\varepsilon)\left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g\right)^{\lambda} = 1+\varepsilon.\end{aligned}$$

Therefore Theorem 4.3.2 yields that there exists $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x) - g(x+w)|\, dx \leq \omega_\lambda(\varepsilon)\int_{\mathbb{R}^n} f.$$

Let $\Phi \in \mathrm{GL}(n)$ be the diagonal transformation $\Phi(t_1, \dots, t_n) = (e^{-w_1}t_1, \dots, e^{-w_n}t_n)$; therefore, for $a = e^{w_1+\cdots+w_n}$, we have

$$\begin{aligned}g(x+w) &= \mathbf{1}_{C_+}(e^{x_1+w_1}, \dots, e^{x_n+w_n})\cdot e^{x_1+\cdots+x_n}\cdot a \\ &= a\mathbf{1}_{(\Phi C)_+}(e^{x_1}, \dots, e^{x_n})e^{x_1+\ldots+x_n} \\ &= a\tilde{g}(x)\end{aligned}$$

It follows that

$$\begin{aligned}\omega_\lambda(\varepsilon)V(K_+) &\geq \int_{\mathbb{R}^n} |f(x) - a\tilde{g}(x)|\, dx \\ &= \int_{\mathbb{R}^n} |\mathbf{1}_{K_+}(e^{x_1}, \dots, e^{x_n}) - a\mathbf{1}_{(\Phi C)_+}(e^{x_1}, \dots e^{x_n})|e^{x_1+\cdots+x_n}\, dx \\ &= \int_{\mathbb{R}^n_+} |\mathbf{1}_{K_+} - a\mathbf{1}_{(\Phi C)_+}|\, dx \\ &= \int_{K_+\cup(\Phi C)_+} |\mathbf{1}_{K_+} - a\mathbf{1}_{(\Phi C)_+}|\, dx \\ &= \int_{K_+\cap(\Phi C)_+} |1-a|\, dx + \int_{K_+\backslash\Phi C_+} 1\, dx + a\int_{\Phi C_+\backslash K_+} 1\, dx \\ &= |a-1|\cdot V(K_+\cap(\Phi C)_+) + V(K_+\backslash(\Phi C)_+) + aV((\Phi C)_+\backslash K_+)\end{aligned}$$

In particular, using (4.10), we have

$$V(K_+\backslash(\Phi C)_+) \leq \omega_\lambda(\varepsilon)V(K_+) \leq \frac{1}{4}V(K_+), \tag{4.11}$$

and hence

$$\begin{aligned}V(K_+\cap(\Phi C)_+) &= V(K_+) - V(K_+\backslash(\Phi C)_+) \\ &\geq V(K_+) - \frac{1}{4}V(K_+) = \frac{3}{4}V(K_+)\end{aligned}$$

In turn, we deduce

$$|a-1| \leq \frac{\omega_\lambda(\varepsilon)V(K_+)}{V(K_+\cap(\Phi C)_+)} \leq \frac{4}{3}\omega_\lambda(\varepsilon) < \frac{1}{3},$$

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thus $\frac{2}{3} < a < \frac{4}{3}$. It follows that

$$V((\Phi C)_+ \backslash K_+) \leq \frac{\omega_\lambda(\varepsilon) V(K_+)}{a} < \frac{3}{2}\omega_\lambda(\varepsilon) V(K_+). \tag{4.12}$$

Combining (4.11) and (4.12) yields

$$\begin{aligned} V(K_+ \Delta (\Phi C)_+) &= V(K_+ \backslash (\Phi C)_+) + V((\Phi C)_+ \backslash K_+) \\ &< \frac{5}{2}\omega_\lambda(\varepsilon) V(K_+) < 3\omega_\lambda(\varepsilon) V(K_+) \end{aligned}$$

and hence, $V(K\Delta(\Phi C)) < 3\omega_\lambda(\varepsilon)V(K)$. Note that $|\det \Phi| = e^{-w_1 + \cdots + w_n} = 1/a$, thus

$$V(\Phi C) = |\det \Phi| \cdot V(C) = \frac{1}{a} V(C) = \frac{1}{a} V(K)$$

Finally,

$$\begin{aligned} V((\Phi^{-1}K)\Delta C) &= V(\Phi^{-1}(K\Delta(\Phi C))) \\ &= |\det \Phi^{-1}| \cdot V(K\Delta(\Phi C)) = V(K\Delta(\Phi C)) \\ &< 3\omega_\lambda(\varepsilon) V(K) = 3a\omega_\lambda(\varepsilon) V(\Phi C) \\ &< 4\omega_\lambda(\varepsilon) V(C) \end{aligned}$$

□

4.4 Linear images of unconditional convex bodies

In this section, the main purpose is to strengthen the containment relation

$$K^{1-\lambda} \cdot C^\lambda \subset (1-\lambda) \cdot K +_0 \lambda \cdot C.$$

and derive a stability verison when C is a linear image of K. We denote by $e_1, \ldots, e_n$ the fixed orthonormal basis of $\mathbb{R}^n$. For a proper subset $J \subset \{1, \ldots, n\}$, we define the linear span L_J by

$$L_J = \mathrm{lin}\{e_i\}_{i \in J}.$$

We write $T = (t_1, \ldots, t_n)$ for a diagonal matrix, where the t_i's are the diagonal entries. Then

$$\|T\|_\infty = \max_{i=1,\ldots,n} |t_i|.$$

And as usual B^n denotes the unit ball centered at the origin.

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PROPOSITION 4.4.1. *If $\tau \in (0, \frac{1}{2}]$, $\lambda \in (\tau, 1-\tau)$, K is an unconditional convex body in $\mathbb{R}^n$ and Φ is a positive definite diagonal matrix satisfying*

$$V((1-\lambda)\cdot K +_0 \lambda \cdot (\Phi K)) \leq (1+\varepsilon) V(K^{1-\lambda}\cdot (\Phi K)^{\lambda})$$

for $\varepsilon > 0$, then either $\|s\Phi - I_n\|_\infty \leq 16n^4 \cdot \frac{\varepsilon^{\frac{1}{5n}}}{\tau^{\frac{1}{5}}}$ for $s = (\det \Phi)^{\frac{-1}{n}}$, or there exist $s_1, \ldots, s_m > 0$ and a partition of $\{1,\ldots,n\}$ into proper subsets $J_1,\ldots,J_m$, $m \geq 2$, such that

$$\bigoplus_{k=1}^{m} (L_{J_k} \cap K) \subset \left(1 + 16n^4 \cdot \frac{\varepsilon^{\frac{1}{5n}}}{\tau^{\frac{1}{5}}}\right) K$$

where for $k = 1,\ldots,m$, we have

$$s_k \cdot (L_{J_k} \cap K) \subset \Phi(L_{J_k}\cap K) \subset \left(1 + 16n^4 \cdot \frac{\varepsilon^{\frac{1}{5n}}}{\tau^{\frac{1}{5}}}\right) s_k \cdot (L_{J_k}\cap K).$$

Proof: First we assume that

$$\varepsilon < \frac{\tau^n}{2^{20n} n^{15n}}. \tag{4.13}$$

Let $\Phi = (\alpha_1, \ldots, \alpha_n)$. Since we may apply a positive definite diagonal transform to K, we may also assume that

$$e_i \in \partial \Phi^{\lambda} K = \partial (K^{1-\lambda}\cdot(\Phi K)^{\lambda}) \ \text{ for } i = 1,\ldots,n.$$

Let

$$\theta = 8n^2 \cdot \frac{\varepsilon^{\frac{1}{5n}}}{\tau^{\frac{1}{5}}} < \frac{1}{2n}.$$

We write $i \bowtie j$ for $i, j \in \{1,\ldots,n\}$ if

$$\exp(-\theta) \leq \frac{\alpha_i}{\alpha_j} \leq \exp(\theta).$$

In addition, we write $\sim$ to denote the the equivalence relation on $\{1,\ldots,n\}$ induced by $\bowtie$; namely, for $i,j \in \{1,\ldots,n\}$, we have $i \sim j$ if and only if there exist pairwise different $i_0,\ldots,i_l \in \{1,\ldots,n\}$ with $i_0 = i$, $i_l = j$, and $i_{k-1} \bowtie i_k$ for $k = 1,\ldots,l$. We may readily assume that

$$l \leq n \ \text{ in the definition of } i \sim j. \tag{4.14}$$

Let $J_1,\ldots,J_m$, $m \geq 1$ be the equivalence classes with respect to $\sim$. The reason behind introducing $\sim$ are the estimates (4.15), (i) and (ii). Let $\beta_k = \min\{\alpha_i : i \in J_k\}$, $\gamma_k = \max\{\alpha_i : i \in J_k\}$ for $k = 1,\ldots,m$. We claim that any $x \in L_{J_k}$ satisfies

$$\beta_k \|x\| \leq \|\Phi x\| \leq e^{n\theta} \beta_k \|x\|. \tag{4.15}$$

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To prove (4.15), lets denote $\{i_0, \ldots, i_l\} = J_k$ such that $\alpha_{i_0} \geq \ldots \geq \alpha_{i_l} = \beta_k$. Then for any $i = i_j \in J_k$, we have

$$\frac{\alpha_i}{\beta_k} = \frac{\alpha_{i_j}}{\alpha_{i_{j+1}}} \cdots \frac{\alpha_{i_{l-1}}}{\alpha_{i_l}} \leq e^{l\theta} \leq e^{n\theta}$$

and hence

$$\beta_k \leq \alpha_i \leq e^{n\,\theta}\beta_k \tag{4.16}$$

holds for $i \in J_k$, proving (4.15).

Next, if $k \neq l$ holds for $k, l \in \{1, \ldots, m\}$, then

$$\text{either } \frac{\beta_k}{\gamma_l} > e^{\theta}, \text{ or } \frac{\gamma_k}{\beta_l} < e^{-\theta}$$

Note that $\gamma_k/\beta_l \geq \beta_k/\gamma_l$. Since J_k, J_l are distinct equivalence classes with respect to $\sim$,

$$\frac{\beta_k}{\gamma_l}, \frac{\gamma_k}{\beta_l} \notin [e^{-\theta}, e^{\theta}]$$

So we have

$$\text{either } \frac{\gamma_k}{\beta_l} \geq \frac{\beta_k}{\gamma_l} > e^{\theta}\ , \text{ or} \frac{\beta_k}{\gamma_l} \leq \frac{\gamma_k}{\beta_l} \leq e^{-\theta}.$$

Then noting that

$$\beta_k \|x\| \leq \|\Phi x\| \leq \gamma_k \|x\|$$

we have

(i) either $\frac{\|\Phi x\|}{\|x\|} \geq e^{\theta} \cdot \frac{\|\Phi y\|}{\|y\|}$ for any $x \in L_{J_k} \backslash o$ and $y \in L_{J_l} \backslash o$;

(ii) or $\frac{\|\Phi x\|}{\|x\|} \leq e^{-\theta} \cdot \frac{\|\Phi y\|}{\|y\|}$ for any $x \in L_{J_k} \backslash o$ and $y \in L_{J_l} \backslash o$.

Step 1 $m = 1$

Here we have $m = 1$, and hence $J_1 = \{1, \ldots, n\}$, and $\beta_1 = \min\{\alpha_i : i \in \{1, \ldots, n\}\}$. For any $u \in S^{n-1} \cap \mathbb{R}^n_{\geq 0}$ and $x \in \mathbb{R}^n_{\geq 0}$, (4.16) implies

$$\langle u, \beta_1 x\rangle \leq \langle u, \Phi x\rangle \leq \langle u, e^{n\theta}\beta_1 x\rangle$$

By the unconditionality of $K, \Phi K$, for any $u \in S^{n-1} \cap \mathbb{R}^n_{\geq 0}$, the points on $\partial K, \partial \Phi K$ where u is an exterior unit normal lie in $\mathbb{R}^n_{\geq 0}$. Hence, we have for any $u \in S^{n-1} \cap \mathbb{R}^n_{\geq 0}$, and $x \in \mathbb{R}^n_{\geq 0}$,

$$\langle u, \beta_1 x\rangle \leq \langle u, \Phi x\rangle \leq \max\{\langle u, \Phi x\rangle : x \in K \cap \mathbb{R}^n_{\geq 0}\} = h_{\Phi K}(u)$$

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which gives us

$$h_{\beta_1 K}(u) = \max\{\langle u, \beta_1 x\rangle : x \in K \cap \mathbb{R}^n_{\geq 0}\} \leq h_{\Phi K}(u).$$

Similarly, we have,

$$h_{\Phi K}(u) \leq h_{e^{n\theta}\beta_1 K}(u).$$

Therefore,

$$\beta_1(K \cap \mathbb{R}^n_{\geq 0}) \subset \Phi K \cap \mathbb{R}^n_{\geq 0} \subset e^{n\theta}\beta_1(K \cap \mathbb{R}^n_{\geq 0})$$

and hence, the unconditionality of K and noting that $e^{n\theta} \leq 1 + 2n\theta$ for $n\theta \leq 1$, yields

$$\beta_1 K \subset \Phi K \subset (1 + 2n\theta)\beta_1 K. \tag{4.17}$$

Now, $\det \Phi = (\alpha_1 \cdots \alpha_n) \geq \beta_1^n$ implies

$$\beta_1^{-1} \geq (\det \Phi)^{-\frac{1}{n}} = s$$

Next choosing $x = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$, (4.15) yields that

$$\begin{aligned} e^{n\theta}\beta_1 &= e^{n\theta}\beta_1 \|x\| \geq \|\Phi x\| \\ &= \sqrt{\frac{\alpha_1^2 + \cdots + \alpha_n^2}{n}} \geq (\alpha_1 \cdots \alpha_n)^{\frac{1}{n}} = s^{-1}. \end{aligned}$$

which gives us

$$e^{-n\theta}\beta_1^{-1} \leq s \leq \beta_1^{-1}$$

Therefore, we have

$$e^{-n\theta} \leq e^{-n\theta} \cdot \frac{\alpha_i}{\beta_1} \leq s\alpha_i \leq \frac{\alpha_i}{\beta_1} \leq e^{n\theta}$$

and using $n\theta \leq 1$ implies $e^{n\theta} \leq 1 + 2n\theta$ and $e^{-n\theta} \geq 1 - 2n\theta$, we have

$$1 - 2n\theta \leq s\alpha_1 \leq 1 + 2n\theta$$

which yields

$$\|s\Phi - I_n\|_\infty \leq 2n\theta$$

Step 2 *If $m \geq 2$ and K is not close to $M = \oplus_{k=1}^m (L_{J_k} \cap \Phi^\lambda K)$, then we find an $x_0 \in \partial(\Phi^\lambda K) \cap \mathbb{R}^n_{\geq 0}$ sitting "deeply" in $(1 - \lambda) \cdot K +_0 \lambda \cdot (\Phi K)$ (cf. Claim 4.4.2)*

Therefore we assume that $m \geq 2$. Here again using a similar argument as in (4.17) from Step 1 yields for $k = 1, \ldots, m$,

$$\beta_k \cdot (L_{J_k} \cap K) \subset \Phi(L_{J_k} \cap K) \subset (1 + 2n\theta)\beta_k \cdot (L_{J_k} \cap K). \tag{4.18}$$

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For
$$M = \bigoplus_{k=1}^{m} (L_{J_k} \cap \Phi^\lambda K),$$
the condition $e_i \in \partial \Phi^\lambda K$, $i = 1, \dots, n$, yields that
$$\frac{1}{\sqrt{n}} B^n \subset \Phi^\lambda K \subset M \subset \sqrt{n}\, B^n. \tag{4.19}$$

We prove indirectly that
$$(1 - 2\sqrt{n}\theta) M \subset \Phi^\lambda K, \tag{4.20}$$
which would complete the proof of Proposition 4.4.1. Indeed if (4.20) holds we would have
$$M \subset \Phi^\lambda K + 2\sqrt{n}\theta M$$
From (4.19) we have $\frac{1}{n} M \subset \Phi^\lambda K$ which yields
$$2\sqrt{n}\theta M = 2n\sqrt{n}\theta \cdot \frac{1}{n} M \subset 2n\sqrt{n}\theta \cdot \Phi^\lambda K$$
And that in turn would give us
$$\begin{aligned} M &\subset 2\sqrt{n}\theta M + \Phi^\lambda K \subset 2n\sqrt{n}\theta \cdot \Phi^\lambda K + \Phi^\lambda K \\ &\subset (1 + 2n^2\theta)\Phi^\lambda K \end{aligned}$$
which proves Proposition 4.4.1. Now, to prove (4.20), we suppose that
$$(1 - 2\sqrt{n}\theta) M \not\subset \Phi^\lambda K, \tag{4.21}$$
and seek a contradiction. Let $\eta > 0$ be maximal such that
$$\eta(M + \theta B^n) \subset \Phi^\lambda K.$$
We deduce that
$$\frac{1}{2n} \leq \eta < 1 - 2\sqrt{n}\theta. \tag{4.22}$$
For, the upper bound, indeed if $\eta \geq 1 - 2\sqrt{n}\theta$, then
$$\eta(M + \theta B^n) \subset \Phi^\lambda K$$
implies
$$(1 - 2\sqrt{n}\theta) M \subset (1 - 2\sqrt{n}\theta)(M + \theta B^n) \subset \eta(M + \theta B^n) \subset \Phi^\lambda K$$
which can't hold under (4.21). So indeed in this case, we have $\eta < 1 - 2\sqrt{n}\theta$. For the lower bound, $\frac{1}{n} M \subset \Phi^\lambda K$ (as $\Phi^\lambda K$ unconditional), $\theta < \frac{1}{2n}$ and $\frac{1}{\sqrt{n}} B^n \subset M$ gives us
$$\frac{1}{2n}(M + \theta B^n) \subset \frac{1}{2}\Phi^\lambda K + \frac{1}{4n^2} B^n \subset \frac{1}{2}\Phi^\lambda K + \frac{1}{2n} M \subset \Phi^\lambda K.$$

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Let $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$. The maximality of η and the unconditionality of K yield that there exists an

$$x_0 \in \eta(M + \theta B^n) \cap \partial(\Phi^\lambda K) \cap \mathbb{R}^n_{\geq 0},$$

and hence there exists a unique exterior normal $w \in S^{n-1} \cap \mathbb{R}^n_{\geq 0}$ to $\partial(\Phi^\lambda K)$ at x_0. Then it follows that (*cf.* (4.22))

$$x_0 - \frac{\theta}{2n} \cdot w + \frac{\theta}{2n} \cdot B^n \subset \Phi^\lambda K. \tag{4.23}$$

In addition, we have from $\frac{1}{\sqrt{n}} B^n \subset M$

$$\begin{aligned} x_0 + \theta B^n \in \eta(M + \theta B^n) + \theta B^n &\subset \eta M + \eta\theta B^n + \theta B^n \\ &\subset (1 - 2\sqrt{n}\theta)M + 2\theta B^n \\ &\subset (1 - 2\sqrt{n}\theta)M + 2\sqrt{n}\theta M = M. \end{aligned}$$

That is,

$$x_0 + \theta\, B^n \in M. \tag{4.24}$$

Writing $x|L$ to denote the orthogonal projection of $x \in \mathbb{R}^n$ to a linear subspace L, we claim that

$$\|w|L_{J_k}\|^2 \leq 1 - \frac{\theta^2}{2n} \quad \text{for } k = 1, \ldots, m. \tag{4.25}$$

Let $v \in S^{n-1} \cap L_{J_k}$ be such that $w|L_{J_k} = \|w|L_{J_k}\|\, v$, and hence

$$\|w|L_{J_k}\| = \langle w, v \rangle.$$

Note that $\left\|w|L_{J_k}^\perp\right\| = \sqrt{1 - \langle w, v\rangle^2}$, $x_0 - (x_0|L_{J_k})$ is orthogonal to v and $\|x_0\| \leq \sqrt{n}$ by (4.19). It follows that

$$\begin{aligned} |\langle w, x_0 - (x_0|L_{J_k})\rangle| &= |\langle w - \langle w, v\rangle v, x_0 - (x_0|L_{J_k})\rangle| \\ &\leq \|w - \langle w, v\rangle\| \cdot \|x_0 - (x_0|L_{J_k})\| \\ &\leq \left\|w|L_{J_k}^\perp\right\| \cdot \|x_0\| \\ &\leq \sqrt{n} \cdot \sqrt{1 - \langle w, v\rangle^2} \end{aligned}$$

It follows from (4.24) that

$$(x_0|L_{J_k}) + \theta v \in \Phi^\lambda K \cap L_{J_k}.$$

Since w is an exterior normal to $\Phi^\lambda K$ at x_0, we have $\langle w, x_0\rangle \geq \langle w, (x_0|L_{J_k}) + \theta v\rangle$, thus

$$\begin{aligned} \sqrt{n} \cdot \sqrt{1 - \langle w, v\rangle^2} &\geq \langle w, x_0 - x_0|L_{J_k}\rangle = \langle w, x_0\rangle - \langle w, (x_0|L_{J_k})\rangle \\ &\geq \langle w, (x_0|L_{J_k}) + \theta v\rangle - \langle w, (x_0|L_{J_k})\rangle \\ &= \theta\langle w, v\rangle \end{aligned}$$

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We deduce that

$$\|w|L_{J_k}\|^2 = \langle w, v\rangle^2 \le \frac{n}{n+\theta^2} = 1 - \frac{\theta^2}{n+\theta^2} < 1 - \frac{\theta^2}{2n},$$

proving (4.25). In turn, we conclude from $\sum_{k=1}^m \|w|L_{J_k}\|^2 = 1$, $m \le n$ and (4.25) that there exist $p \neq q$ satisfying

$$\|w|L_{J_p}\|^2 \ge \frac{\theta^2}{2n^2} \text{ and } \|w|L_{J_q}\|^2 \ge \frac{\theta^2}{2n^2}. \tag{4.26}$$

To deduce it, denote $a_k = \|w|L_{J_k}\|$. Assume for the sake of contradiction that that, for $k = 1, \dots, m$, $a_k^2 < \frac{\theta^2}{2n^2}$. Then

$$1 = \sum_{k=1}^m a_k^2 < m \cdot \frac{\theta^2}{2n^2} \le \frac{n\theta^2}{2n^2} = \frac{\theta}{2n} < 1$$

which is absurd. So indeed, there exists $p \in \{1, \dots, m\}$ such that

$$a_p^2 \ge \frac{\theta^2}{2n^2}$$

Next suppose a_p is the only such among the a_k's. Then from (4.25),

$$1 - \left(1 - \frac{\theta^2}{2n^2}\right) \le 1 - a_p^2 = \sum_{k=1}^m a_k^2 - a_p^2 < (m-1) \cdot \frac{\theta^2}{2n^2}$$

that is, $\frac{\theta^2}{2n} < (m-1) \cdot \frac{\theta^2}{2n^2}$ which leads to $n < m - 1 < n$ which is absurd. So, there exists at least one other a_q with

$$a_q^2 \ge \frac{\theta^2}{2n^2}$$

This verifies (4.26). Possibly after reindexing, we may assume that

$$\|w|L_{J_1}\| \ge \frac{\theta}{2n} \text{ and } \|w|L_{J_2}\| \ge \frac{\theta}{2n}. \tag{4.27}$$

For any $u \in S^{n-1} \cap \mathbb{R}_{\ge 0}$, it follows from $\Phi^{-\lambda}x_0 \in K$ and $\Phi^{1-\lambda}x_0 \in \Phi K$ that

$$\langle u, \Phi^{-\lambda}x_0\rangle \le h_K(u) \quad \text{and} \quad \langle u, \Phi^{1-\lambda}x_0\rangle \le h_{\Phi K}(u); \tag{4.28}$$

Then using Hölder's inequality we have

$$\begin{aligned}\langle u, x_0\rangle = \sum_j x_{0j}u_j &= \sum_j \left(\alpha_j^{-\lambda}x_{0j}u_j\right)^{1-\lambda} \cdot \left(\alpha_j^{1-\lambda}x_{0j}u_j\right)^{\lambda} \\ &\le \left(\sum_j \alpha_j^{-\lambda}x_{0j}u_j\right)^{1-\lambda} \cdot \left(\sum_j \alpha_j^{1-\lambda}x_{0j}u_j\right)^{\lambda} \\ &= \langle u, \Phi^{-\lambda}x_o\rangle^{1-\lambda}\langle u, \Phi^{1-\lambda}x_0\rangle^{\lambda}\end{aligned}$$

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which along with (4.28) gives us

$$\langle u, x_0\rangle \leq h_K(u)^{1-\lambda} h_{\Phi K}(u)^{\lambda}. \tag{4.29}$$

In particular, (4.29) implies that $x_0 \in (1-\lambda)\cdot K +_0 \lambda \cdot (\Phi K)$.

In order to prove (4.20); more precisely, to prove that (4.21) is false, the next step is the following stability version of (4.29).

Step 3 *x_0 sits "deeply" in* $(1-\lambda)\cdot K +_0 \lambda \cdot (\Phi K)$

CLAIM 4.4.2. *For any $u \in S^{n-1} \cap \mathbb{R}^n_{\geq 0}$, we have*

$$\langle u, x_0\rangle \left(1 + \frac{\tau\theta^5}{2^{10} n^{5.5}}\right) \leq h_K(u)^{1-\lambda} h_{\Phi K}(u)^{\lambda}. \tag{4.30}$$

Proof: We observe that $\langle u, \Phi^{-\lambda} x_0\rangle = \langle \Phi^{-\lambda} u, x_0\rangle$, $\langle u, \Phi^{1-\lambda} x_0\rangle = \langle \Phi^{1-\lambda} u, x_0\rangle$,

$$\begin{aligned} h_K(u) &= h_{\Phi^\lambda K}(\Phi^{-\lambda} u); \\ h_{\Phi K}(u) &= h_{\Phi^\lambda K}(\Phi^{1-\lambda} u), \end{aligned}$$

and hence it follows from (4.28) and (4.29) that it is sufficient to prove that if $u \in S^{n-1} \cap \mathbb{R}^n_{\geq 0}$, then either

$$\left(\tfrac{h_{\Phi^\lambda K}(\Phi^{-\lambda} u)}{\langle \Phi^{-\lambda} u, x_0\rangle}\right)^{1-\lambda} \geq 1 + \tfrac{\tau\theta^5}{2^{10} n^{5.5}}, \text{ or } \left(\tfrac{h_{\Phi^\lambda K}(\Phi^{1-\lambda} u)}{\langle \Phi^{1-\lambda} u, x_0\rangle}\right)^{\lambda} \geq 1 + \tfrac{\tau\theta^5}{2^{10} n^{5.5}}. \tag{4.31}$$

Let us write $w = \oplus_{k=1}^m w_k$ and $u = \oplus_{k=1}^m u_k$ for $w_k = w|L_{J_k}$ and $u_k = u|L_{J_k}$, and prove that (*cf.* (4.27)) there exists $i \in \{1,2\}$ such that

$$\text{either } \left|\frac{\|\Phi^{-\lambda} u_i\|}{\|\Phi^{-\lambda} u\|} - \|w_i\|\right| \geq \frac{\theta^2}{16n^2}, \quad \text{or } \left|\frac{\|\Phi^{1-\lambda} u_i\|}{\|\Phi^{1-\lambda} u\|} - \|w_i\|\right| \geq \frac{\theta^2}{16n^2}. \tag{4.32}$$

We prove (4.32) by contradiction; thus, we suppose that if $i \in \{1,2\}$, then

$$\left|\frac{\|\Phi^{-\lambda} u_i\|}{\|\Phi^{-\lambda} u\|} - \|w_i\|\right| < \frac{\theta^2}{16n^2} \quad \text{and} \quad \left|\frac{\|\Phi^{1-\lambda} u_i\|}{\|\Phi^{1-\lambda} u\|} - \|w_i\|\right| < \frac{\theta^2}{16n^2}.$$

and seek a contradiction.

Denote $a = \left\|\Phi^{-\lambda} u\right\|$, $a_i = \left\|\Phi^{-\lambda} u_i\right\|$, $b = \left\|\Phi^{1-\lambda} u\right\|$, $b_i = \left\|\Phi^{1-\lambda} u_i\right\|$, $c_i = \|w_i\|$. Then we have,

$$\left|\frac{a_i}{a} - c_i\right| < \frac{\theta^2}{16n^2}, \left|\frac{b_i}{b} - c_i\right| < \frac{\theta^2}{16n^2}, c_i \geq \frac{\theta}{2n} \tag{4.33}$$

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From (4.33), we have

$$c_i \geq \frac{\theta}{2n} > \frac{\theta}{4n}, \text{ and } \frac{a_i}{a} > c_i - \left(\frac{\theta}{4n}\right)^2 > \frac{\theta}{4n}$$

First we claim that if $x \geq y \geq \frac{\theta}{4n}$, and $|x-y| \leq \left(\frac{\theta}{4n}\right)^2$, then

$$e^{-\frac{\theta}{4}} \leq \frac{x}{y} \leq e^{\frac{\theta}{4}} \tag{4.34}$$

First note that

$$\begin{aligned} e^{\frac{4n^2}{\theta}(x-y)} &\geq 1 + \frac{4n^2}{\theta}(x-y) = 1 + \frac{4n^2}{\theta} \cdot y\left(\frac{x}{y} - 1\right) \\ &\geq 1 + \frac{4n^2}{\theta} \cdot \frac{\theta}{4n} \cdot \left(\frac{x}{y} - 1\right) = 1 + n\left(\frac{x}{y} - 1\right) \\ &\geq \frac{x}{y} \end{aligned}$$

Then,

$$\log \frac{x}{y} \leq \frac{4n^2}{\theta}(x-y) \leq \frac{4n^2}{\theta} \cdot \left(\frac{\theta}{4n}\right)^2 = \frac{\theta}{4} \tag{4.35}$$

that is,

$$e^{-\frac{\theta}{4}} < 1 \leq \frac{x}{y} \leq e^{\frac{\theta}{4}}$$

Applying (4.34) first to $\frac{a_i}{a}$ and c_i, and then to $\frac{b_i}{b}$, c_i, we have

$$e^{-\frac{\theta}{4}} < \frac{a_i}{ac_i} < e^{\frac{\theta}{4}} \text{ and } e^{-\frac{\theta}{4}} < \frac{b_i}{bc_i} < e^{\frac{\theta}{4}} \tag{4.36}$$

It follows that

$$e^{-\frac{\theta}{2}} < \frac{b_i}{bc_i} \cdot \frac{ac_i}{a_i} < e^{\frac{\theta}{2}}$$

that is,

$$e^{-\frac{\theta}{2}} < \frac{b_i}{a_i} \cdot \frac{a}{b} < e^{\frac{\theta}{2}}$$

And in turn, we have

$$e^{-\theta} < \frac{b_1}{a_1} \cdot \frac{a}{b} \cdot \frac{a_2}{b_2} \cdot \frac{b}{a} < e^{\theta}$$

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which we can write as

$$e^{-\theta} < \frac{b_1}{a_1} : \frac{b_2}{a_2} < e^{\theta} \tag{4.37}$$

That is,

$$e^{-\theta} < \frac{\left\|\Phi(\Phi^{-\lambda}u_1)\right\|}{\|\Phi^{-\lambda}u_1\|} : \frac{\left\|\Phi(\Phi^{-\lambda}u_2)\right\|}{\|\Phi^{-\lambda}u_2\|} < e^{\theta} \tag{4.38}$$

Since $\Phi^{-\lambda}u_i \in L_{J_i}$ for $i = 1, 2$, the last inequalities contradict (i) and (ii), and in turn verify (4.32).

Based on (4.32), and the triangle inequality we have the existence of $i \in \{1, 2\}$ such that either

$$\left\| \frac{\Phi^{-\lambda}u_i}{\|\Phi^{-\lambda}u\|} - w_i \right\| \geq \left| \frac{\left\|\Phi^{-\lambda}u_i\right\|}{\|\Phi^{-\lambda}u\|} - \|w_i\| \right| \geq \frac{\theta^2}{16n^2}$$

or

$$\left\| \frac{\Phi^{1-\lambda}u_i}{\|\Phi^{1-\lambda}u\|} - w_i \right\| \geq \left| \frac{\left\|\Phi^{1-\lambda}u_i\right\|}{\|\Phi^{1-\lambda}u\|} - \|w_i\| \right| \geq \frac{\theta^2}{16n^2}$$

In turn, we have for some $i \in \{1, 2\}$, either

$$\left\| \frac{\Phi^{-\lambda}u}{\|\Phi^{-\lambda}u\|} - w \right\|^2 = \sum_{i=1}^{m} \left\| \frac{\Phi^{-\lambda}u_j}{\|\Phi^{-\lambda}u\|} - w_j \right\|^2 \geq \left\| \frac{\Phi^{-\lambda}u_j}{\|\Phi^{-\lambda}u\|} - w_i \right\|^2 \geq \left(\frac{\theta^2}{16n^2} \right)^2$$

or , similarly,

$$\left\| \frac{\Phi^{1-\lambda}u}{\|\Phi^{1-\lambda}u\|} - w \right\|^2 \geq \left(\frac{\theta^2}{16n^2} \right)^2$$

that is,

$$\text{either} \quad \left\| \frac{\Phi^{-\lambda}u}{\|\Phi^{-\lambda}u\|} - w \right\| \geq \frac{\theta^2}{16n^2} \quad \text{or} \quad \left\| \frac{\Phi^{1-\lambda}u}{\|\Phi^{1-\lambda}u\|} - w \right\| \geq \frac{\theta^2}{16n^2}. \tag{4.39}$$

First, we assume that out of the two possibilities in (4.39), we have

$$\left\| \frac{\Phi^{-\lambda}u}{\|\Phi^{-\lambda}u\|} - w \right\| \geq \frac{\theta^2}{16n^2}. \tag{4.40}$$

According to (4.23), we have

$$\widetilde{B} = x_0 - \frac{\theta}{2n} \cdot w + \frac{\theta}{2n} \cdot B^n \subset \Phi^{\lambda}K,$$

which in turn yields (using (4.40) and $\|x_0\| \leq \sqrt{n}$ at the end) that

$$\begin{aligned} h_{\Phi^{\lambda}K}(\Phi^{-\lambda}u) - \langle \Phi^{-\lambda}u, x_0 \rangle &\geq h_{\widetilde{B}}(\Phi^{-\lambda}u) - \langle \Phi^{-\lambda}u, x_0 \rangle \\ &= \left\langle \Phi^{-\lambda}u, x_0 - \frac{\theta}{2n} \cdot w + \frac{\theta}{2n} \cdot \frac{\Phi^{-\lambda}u}{\|\Phi^{-\lambda}u\|} \right\rangle - \langle \Phi^{-\lambda}u, x_0 \rangle \end{aligned}$$

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$$
\begin{aligned}
&= \left\langle \Phi^{-\lambda}u, \frac{\theta}{2n}\cdot\left(\frac{\Phi^{-\lambda}u}{\|\Phi^{-\lambda}u\|} - w\right)\right\rangle \\
&= \left\|\Phi^{-\lambda}u\right\|\cdot\frac{\theta}{2n}\cdot\left\langle \frac{\Phi^{-\lambda}u}{\|\Phi^{-\lambda}u\|}, \left(\frac{\Phi^{-\lambda}u}{\|\Phi^{-\lambda}u\|} - w\right)\right\rangle \\
&= \left\|\Phi^{-\lambda}u\right\|\cdot\frac{\theta}{2n}\cdot\frac{1}{2}\left\|\frac{\Phi^{-\lambda}u}{\|\Phi^{-\lambda}u\|} - w\right\|^2 \\
&\geq \frac{\theta}{4n}\cdot\left\|\Phi^{-\lambda}u\right\|\cdot\left(\frac{\theta^2}{16n^2}\right)^2 \\
&\geq \frac{\theta^5}{2^{10}n^5}\cdot\frac{\langle\Phi^{-\lambda}u, x_0\rangle}{\|x_0\|} \geq \frac{\theta^5}{2^{10}n^5}\cdot\frac{\langle\Phi^{-\lambda}u, x_0\rangle}{\sqrt{n}} \\
&\geq \frac{\theta^5}{2^{10}n^{5.5}}\cdot\langle\Phi^{-\lambda}u, x_0\rangle.
\end{aligned}
$$

We conclude using $1-\lambda\geq\tau$ that

$$
\left(\frac{h_{\Phi^\lambda K}(\Phi^{-\lambda}u)}{\langle\Phi^{-\lambda}u, x_0\rangle}\right)^{1-\lambda} \geq \left(1+\frac{\theta^5}{2^{10}n^{5.5}}\right)^{\tau} \geq 1+\frac{\tau\theta^5}{2^{10}n^{5.5}}. \tag{4.41}
$$

Secondly, if

$$
\left\|\frac{\Phi^{1-\lambda}u}{\|\Phi^{1-\lambda}u\|} - w\right\| \geq \frac{\theta^2}{16n^2}
$$

holds in (4.39), then a similar argument yields

$$
\left(\frac{h_{\Phi^\lambda K}(\Phi^{1-\lambda}u)}{\langle\Phi^{1-\lambda}u, x_0\rangle}\right)^{\lambda} \geq 1+\frac{\tau\theta^5}{2^{10}n^{5.5}}.
$$

proving (4.31). In turn, we conclude (4.30) in Claim 4.4.2. $\square$

Step 4 *Claim 4.4.2 contradicts* (4.21)

Let $\varrho\geq 0$ be maximal with the property that

$$
x_0+\varrho B^n \subset (1-\lambda)\cdot K +_0 \lambda\cdot(\Phi K). \tag{4.42}
$$

Since x_0 sits deeply in $(1-\lambda)\cdot K +_0 \lambda\cdot(\Phi K)$ by Claim 4.4.2, it follows that $\varrho>0$

We claim that

$$
\varrho \geq \frac{\tau\theta^5}{2^{11}n^6}. \tag{4.43}
$$

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To prove (4.43), we may assume for the sake of contradiction that

$$\varrho < \frac{\tau\theta^5}{2^{11}n^6} < \frac{1}{2n}. \tag{4.44}$$

Maximality of ϱ and the unconditionality of $(1-\lambda)\cdot K +_0 \lambda\cdot(\Phi K)$ imply the existence of a

$$y_0 \in (x_0 + \varrho B^n) \cap \partial\Big((1-\lambda)\cdot K +_0 \lambda\cdot(\Phi K)\Big) \cap \mathbb{R}^n_{\geq 0},$$

Let $u \in S^{n-1} \cap \mathbb{R}^n_{\geq 0}$ be the exterior unit normal to

$$\widetilde{M} = (1-\lambda)\cdot K +_0 \lambda\cdot(\Phi K)$$

at y_0, and hence $y_0 = x_0 + \varrho u$. Now, $\pm e_i \in \Phi^\lambda K \subset \widetilde{M}$ for $i = 1,\dots,n$ implies that $Q = \operatorname{conv}\{\pm e_i : i = 1,\dots,n\} \subset \widetilde{M}$ and

$$\frac{1}{\sqrt{n}}B^n \subset Q \subset \widetilde{M}$$

and hence, $h_{\widetilde{M}}(u) \geq \frac{1}{\sqrt{n}}$, and (4.44) implies

$$\begin{aligned}\langle u, x_0\rangle &= \langle u, y_0\rangle - \varrho = h_{\widetilde{M}}(u) - \varrho\\ &\geq \frac{1}{\sqrt{n}} - \frac{1}{2n} \geq \frac{1}{2\sqrt{n}}\end{aligned} \tag{4.45}$$

On the other hand, $h_{\widetilde{M}}(u) = h_K(u)^{1-\lambda}h_{\Phi K}(u)^\lambda$ holds because y_0 is a smooth boundary point of $\widetilde{M}$; therefore, it follows from (4.30), and (4.45) that

$$\begin{aligned}\varrho &= h_{\widetilde{M}}(u) - \langle u, x_0\rangle = h_K(u)^{1-\lambda}h_{\Phi K}(u)^\lambda - \langle u, x_0\rangle\\ &\geq \left(1 + \frac{\tau\theta^5}{2^{10}n^{5.5}}\right)\langle u, x_0\rangle - \langle u, x_0\rangle = \frac{\tau\theta^5}{2^{10}n^{5.5}}\cdot\langle u, x_0\rangle\\ &\geq \frac{\tau\theta^5}{2^{10}n^{5.5}}\cdot\frac{1}{2\sqrt{n}} \geq \frac{\tau\theta^5}{2^{11}n^6}\end{aligned}$$

which contradicts (4.44), hence proving (4.43).

From $\Phi^\lambda K \subset \sqrt{n}B^n$, we have

$$V(\Phi^\lambda K) \leq n^{\frac{n}{2}}\kappa_n \tag{4.46}$$

Also note that the supporting hyperplane, H_0 at x_0 to $\Phi^\lambda K$ cuts $x_0 + \varrho B^n$ in half, and

$$\Phi^\lambda K \cup (x_0 + \varrho B^n \backslash \Phi^\lambda K) \subset \widetilde{M} \tag{4.47}$$

Finally, using (4.43), (4.46) and (4.47), we deduce

$$V(\widetilde{M}) \geq V(\Phi^\lambda K) + \frac{\varrho^n\kappa_n}{2} \geq V(\Phi^\lambda K) + \frac{1}{2}\left(\frac{\tau^n\theta^{5n}}{2^{11n}n^{6n}}\right)\cdot\kappa_n$$

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$$
\begin{aligned}
&= V(\Phi^\lambda K)\left(1+\frac{\tau^n\theta^{5n}}{2^{11n+1}\cdot n^{6n}}\cdot\frac{\kappa_n}{V(\Phi^\lambda K)}\right)\\
&\geq V(\Phi^\lambda K)\left(1+\frac{\tau^n\theta^{5n}}{2^{11n+1}n^{6n}}\cdot\frac{1}{n^{\frac{n}{2}}}\right)\\
&= V(\Phi^\lambda K)\left(1+\frac{\tau^n\theta^{5n}}{2^{11n+1}n^{6.5n}}\right)\\
&> V(\Phi^\lambda K)\left(1+\frac{\tau^n\theta^{5n}}{2^{15n}n^{10n}}\right)\\
&> V(\Phi^\lambda K)(1+\varepsilon)\\
&> (1+\varepsilon)V(K)^{1-\lambda}V(\Phi K)^\lambda
\end{aligned}
$$

which is absurd. This contradicts (4.21), and verifies $(1-2\sqrt{n}\theta)M\subset\Phi^\lambda K$, completing the proof of Proposition 4.4.1 under the condition $\varepsilon<\frac{\tau^n}{2^{20n}n^{15n}}$ (*cf.* (4.13)). On the other hand, if $\varepsilon\geq\frac{\tau^n}{2^{20n}n^{15n}}$, then

$$16n^4\cdot\frac{\varepsilon^{\frac{1}{5n}}}{\tau^{\frac{1}{5}}}\geq n,$$

thus Proposition 4.4.1 readily holds. □

4.5 Proof of Theorem 4.2.3

The main tools used to prove Theorem 4.2.3 are Theorem 4.3.3 and Proposition 4.4.1. But first we state some simple lemmas that we are going to use. The first lemma is a corollary of the logarithmic Brunn-Minowski inequality for unconditional convex bodies (see Lemma 3.1 of Kolesnikov, Milman [106]).

LEMMA 4.5.1. *If K and C are unconditional convex bodies in $\mathbb{R}^n$, then*

$$\varphi(t)=V((1-t)\cdot K+_0 t\cdot C)$$

is log-concave on $[0,1]$.

The next claim provides some simple estimates for log-concave functions.

LEMMA 4.5.2. *Let φ be a log-concave function on* $[0,1]$.

(i) *If $\lambda\in(0,1)$, $\eta\in(0,2\cdot\min\{1-\lambda,\lambda\})$ and $\varphi(\lambda)\leq(1+\eta)\varphi(0)^{1-\lambda}\varphi(1)^\lambda$, then*

$$\varphi\left(\tfrac{1}{2}\right)\leq\left(1+\frac{\eta}{\min\{1-\lambda,\lambda\}}\right)\sqrt{\varphi(0)\varphi(1)}$$

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(ii) *If $\varphi(0)=\varphi(1)=1$ and $\varphi'(0)\le 2$, then $\varphi\left(\frac{1}{2}\right)\le 1+\varphi'(0)$.*

Proof: For (i), we may assume that $0<\lambda<\frac{1}{2}$, and hence $\lambda=(1-2\lambda)\cdot 0+2\lambda\cdot\frac{1}{2}$, $\varphi(\lambda)\le(1+\eta)\varphi(0)^{1-\lambda}\varphi(1)^{\lambda}$ and the log-concavity of φ yield

$$(1+\eta)\varphi(0)^{1-\lambda}\varphi(1)^{\lambda}\ge\varphi(\lambda)\ge\varphi(0)^{1-2\lambda}\varphi\left(\tfrac{1}{2}\right)^{2\lambda}.$$

Thus $(1+\eta)^{\frac{1}{2\lambda}}\le e^{\frac{\eta}{2\lambda}}\le 1+\frac{\eta}{\lambda}$ implies

$$\varphi\left(\tfrac{1}{2}\right)\le(1+\eta)^{\frac{1}{2\lambda}}\sqrt{\varphi(0)\varphi(1)}\le\left(1+\frac{\eta}{\lambda}\right)\sqrt{\varphi(0)\varphi(1)}.$$

For (ii), we write $\varphi(t)=e^{W(t)}$ for a concave function W with $W(0)=W(1)=0$. Thus $W(\frac{1}{2})\le\frac{1}{2}W'(0)$, which in turn yields using $W'(0)=\varphi'(0)\le 2$ that

$$\varphi\left(\tfrac{1}{2}\right)=e^{W(\frac{1}{2})}\le e^{W'(0)/2}\le 1+W'(0)=1+\varphi'(0).\quad\square$$

We will also be using the following result about volume difference.

LEMMA 4.5.3. *If $M\subset K$ are o-symmetric convex bodies with $V(K\backslash M))\le\frac{1}{2^{n+1}}V(K)$, then*

$$K\subset\left(1+4\cdot\left(\frac{V(K\backslash M)}{V(M)}\right)^{\frac{1}{n}}\right)M.$$

Let $t\ge 0$ be minimal such that

$$K\subset(1+t)M$$

Then there exists $z\in\partial K$ and $y\in\partial M$ such that $z=(1+t)y$. Note that

$$\frac{2}{t+2}\cdot z+\frac{t}{t+2}\cdot x\in K,\ \forall\, x\in M\subset K$$

that is,

$$\frac{2}{t+2}\cdot z+\frac{t}{t+2}\cdot M\subset K. \tag{4.48}$$

Now using $z=(1+t)y$,

$$\begin{aligned}\frac{2}{t+2}\cdot z+\frac{t}{t+2}\cdot M&=\frac{2}{t+2}\cdot(1+t)y+\frac{t}{t+2}\cdot M\\&=y+\frac{t}{t+2}\cdot y+\frac{t}{t+2}\cdot M\in K\backslash\mathrm{int}M\end{aligned}$$

Therefore,

$$\begin{aligned}&V\left(\frac{t}{t+2}\cdot M\right)\le V(K\backslash M)\\\Longrightarrow&\left(\frac{t}{t+2}\right)^{n}V(M)\le V(K\backslash M)\end{aligned}\tag{4.49}$$

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From $V(K\backslash M) \le \frac{1}{2^{n+1}} \cdot V(K)$, we have

$$\begin{aligned} V(K)\left(1-\frac{1}{2^{n+1}}\right) &\le V(M) \\ \Longrightarrow V(K) &\le \frac{1}{1-\frac{1}{2^{n+1}}} \cdot V(M) < \left(1+\frac{1}{2^n}\right) V(M) \\ \Longrightarrow V(K\backslash M) &< \frac{1}{2^n} V(M) \end{aligned} \tag{4.50}$$

Combining (4.49) and (4.50), we get

$$\left(\frac{t}{t+2}\right) < \frac{1}{2} \Longrightarrow t < 2 \tag{4.51}$$

Finally using (4.51) and (4.49), we have

$$\frac{t^n}{4^n} \le \left(\frac{t}{t+2}\right)^n \le \frac{V(K\backslash M)}{V(M)} \Longrightarrow t \le 4 \cdot \left(\frac{V(K\backslash M)}{V(M)}\right)^{\frac{1}{n}}$$

□

The next two statements are the case $\lambda = \frac{1}{2}$ of Theorem 4.3.3 and Proposition 4.4.1 respectively that we will be using in the proof of Theorem 4.2.3.

COROLLARY 4.5.4. *If the unconditional convex bodies K and C in $\mathbb{R}^n$ satisfy*

$$V(K^{\frac{1}{2}} \cdot C^{\frac{1}{2}}) \le (1+\varepsilon)V(K)^{\frac{1}{2}}V(C)^{\frac{1}{2}}$$

for $\varepsilon > 0$, then there exists positive definite diagonal matrix Φ such that

$$V(K\Delta(\Phi C)) < c^n n^n \varepsilon^{\frac{1}{19}} V(K) \tag{4.52}$$

where $c > 1$ is an absolute constant.

COROLLARY 4.5.5. *If K is an unconditional convex body in $\mathbb{R}^n$ and Φ is a positive definite diagonal matrix satisfying*

$$V\left(\frac{1}{2} \cdot K +_0 \frac{1}{2} \cdot (\Phi K)\right) \le (1+\varepsilon)V(K^{\frac{1}{2}} \cdot (\Phi K)^{\frac{1}{2}})$$

for $\varepsilon > 0$, then either $\|s\Phi - I_n\|_\infty \le 20n^4 \cdot \varepsilon^{\frac{1}{5n}}$ for $s = (\det \Phi)^{\frac{-1}{n}}$, or there exist $s_1, \ldots, s_m > 0$ and a partition of $\{1, \ldots, n\}$ into proper subsets $J_1, \ldots, J_m$, $m \ge 2$, such that

$$\bigoplus_{k=1}^{m} (L_{J_k} \cap K) \subset \left(1 + 20n^4 \cdot \varepsilon^{\frac{1}{5n}}\right) K$$

$$s_k(L_{J_k} \cap K) \subset \Phi(L_{J_k} \cap K) \subset \left(1 + 20n^4 \cdot \varepsilon^{\frac{1}{5n}}\right) s_k(L_{J_k} \cap K), \quad k = 1, \ldots, m.$$

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Proof of Theorem 4.2.3 **Step 1** First we consider the case $\lambda = \frac{1}{2}$, and hence prove that if the unconditional convex bodies K and C in $\mathbb{R}^n$ satisfy

$$V\left(\frac{1}{2}\cdot K +_0 \frac{1}{2}\cdot C\right) \le (1+\varepsilon)V(K)^{\frac{1}{2}}V(C)^{\frac{1}{2}} \tag{4.53}$$

for $\varepsilon > 0$, then for $m \ge 1$, there exist $\theta_1, \ldots, \theta_m > 0$ and unconditional compact convex sets $K_1, \ldots, K_m > 0$ such that $\operatorname{lin} K_i$, $i = 1, \ldots, m$, are complementary coordinate subspaces, and

$$\begin{aligned} K_1 \oplus \ldots \oplus K_m \subset \quad K \quad &\subset \left(1 + c_0^n \varepsilon^{\frac{1}{95n}}\right)(K_1 \oplus \ldots \oplus K_m) && (4.54)\\ \theta_1 K_1 \oplus \ldots \oplus \theta_m K_m \subset \quad C \quad &\subset \left(1 + c_0^n \varepsilon^{\frac{1}{95n}}\right)(\theta_1 K_1 \oplus \ldots \oplus \theta_m K_m) && (4.55)\end{aligned}$$

where $c_0 > 1$ is an absolute constant. We have

$$V(K^{\frac{1}{2}}\cdot C^{\frac{1}{2}}) \le V\left(\frac{1}{2}\cdot K +_0 \frac{1}{2}\cdot C\right) \le (1+\varepsilon)V(K)^{\frac{1}{2}}V(C)^{\frac{1}{2}};$$

therefore, Corollary 4.5.4 yields a positive definite diagonal matrix Φ such that

$$V((\Phi K)\Delta C) < \tilde{c}^n n^n \varepsilon^{\frac{1}{19}} V(C) \quad \text{and} \quad V(K\Delta(\Phi^{-1}C)) < \tilde{c}^n n^n \varepsilon^{\frac{1}{19}} V(K) \tag{4.56}$$

where $\tilde{c} > 1$ is an absolute constant.

First we assume that

$$\varepsilon < \gamma^{-n} n^{-19n} \tag{4.57}$$

for a suitable absolute constant $\gamma > 1$ where γ is a chosen in a way such that

$$\tilde{c}^n n^n \varepsilon^{\frac{1}{19}} < \frac{1}{2^{n+1}} \tag{4.58}$$

for the constant $\tilde{c}$ obtained above.

Let

$$M = K \cap (\Phi^{-1}C),$$

Denoting $\rho_1 = \tilde{c}^n n^n \varepsilon^{\frac{1}{19}}$, from (4.56), we have

$$\begin{aligned} V(C) - V(\Phi K \cap C) = V(C\backslash \Phi K) < V(C\Delta\Phi K) < \rho_1 V(C)\\ V(K) - V(K\cap \Phi^{-1}C) = V(K\backslash \Phi^{-1}C) < V(K\Delta\Phi^{-1}C) < \rho_1 V(K)\end{aligned}$$

which yields

$$\begin{aligned} V(M) \quad &> \quad (1-\rho_1)\cdot V(K) && (4.59)\\ V(\Phi M) \quad &> \quad (1-\rho_1)\cdot V(C) && (4.60)\end{aligned}$$

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where $\rho_1 = \tilde{c}^n n^n \varepsilon^{\frac{1}{19}}$. As $M \subset K$ and $\Phi M \subset C$, it follows that

$$\begin{aligned}
V\left(\tfrac{1}{2}\cdot M +_0 \tfrac{1}{2}\cdot(\Phi M)\right) &\le V\left(\tfrac{1}{2}\cdot K +_0 \tfrac{1}{2}\cdot C\right)\\
&\le (1+\varepsilon)V(K)^{\frac{1}{2}}V(C)^{\frac{1}{2}}\\
&\le (1+\varepsilon)\cdot\left[\frac{V(M)}{1-\rho_1}\right]^{\frac{1}{2}}\cdot\left[\frac{V(\Phi M)}{1-\rho_1}\right]^{\frac{1}{2}}\\
&= \left(\frac{1+\varepsilon}{1-\rho_1}\right)\cdot V(M)^{\frac{1}{2}}\cdot V(\Phi M)^{\frac{1}{2}}\\
&\le (1+2\rho_1)\cdot V(M)^{\frac{1}{2}}\cdot V(\Phi M)^{\frac{1}{2}}\\
&\le (1+2\rho_1)\cdot V\left(M^{\frac{1}{2}}\cdot(\Phi M)^{\frac{1}{2}}\right).
\end{aligned}$$

Here we have used $\left(\frac{1+\varepsilon}{1-\rho_1}\right) < 1+2\rho_1$ which follows from $\rho_1 < \frac{1}{2^{n+1}}$.

Now we can apply Corollary 4.5.5 to M and ΦM. Here from Corollary 4.5.5, we get the term

$$20n^4\cdot(2\rho_1)^{\frac{1}{5n}} = 20\cdot 2^{\frac{1}{5n}}\cdot\tilde{c}^{\frac{1}{5}}\cdot n^{4+\frac{1}{5}}\cdot\varepsilon^{\frac{1}{95n}} < c_1 n^5\cdot\varepsilon^{\frac{1}{95n}} \tag{4.61}$$

for an absolute constant $c_1 > 1$.

So Corollary 4.5.5 yields that either $\|s\Phi - I_n\|_\infty \le c_1 n^5\cdot\varepsilon^{\frac{1}{95n}}$ for $s = (\det\Phi)^{\frac{-1}{n}}$, or there exist $s_1,\dots,s_m > 0$ and a partition of $\{1,\dots,n\}$ into proper subsets $J_1,\dots,J_m$, $m \ge 2$, such that

$$\bigoplus_{k=1}^{m}(L_{J_k}\cap M) \subset \left(1 + c_1 n^5\cdot\varepsilon^{\frac{1}{95n}}\right)M \tag{4.62}$$

where for $k = 1,\dots,m$, we have

$$s_k\cdot(L_{J_k}\cap M) \subset \Phi(L_{J_k}\cap M) \subset \left(1+c_1 n^5\cdot\varepsilon^{\frac{1}{95n}}\right)s_k\cdot(L_{J_k}\cap M). \tag{4.63}$$

From $\rho_1 < \frac{1}{2^{n+1}}$, (4.59), (4.60), we have

$$\begin{aligned}
V(K\backslash M) &= V(K) - V(M) < \rho_1\cdot V(K) < \frac{V(K)}{2^{n+1}}\\
V(C\backslash(\Phi M)) &= V(C) - V(\Phi M) < \rho_1\cdot V(C) < \frac{V(C)}{2^{n+1}}
\end{aligned}$$

Now, applying Lemma 4.5.3, we have

$$\begin{aligned}
K &\subset \left(1 + 4\cdot\left(\frac{V(K\backslash M)}{V(M)}\right)^{\frac{1}{n}}\right)M\\
C &\subset \left(1 + 4\cdot\left(\frac{V(K\backslash(\Phi M))}{V(\Phi M)}\right)^{\frac{1}{n}}\right)\Phi M.
\end{aligned}$$

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Then,

$$\begin{aligned}\frac{V(K\backslash M)}{V(M)} &= \frac{V(K)-V(M)}{V(M)} = \frac{V(K)}{V(M)} - 1\\ &< \frac{1}{\rho_1 - 1} - 1 = \frac{\rho_1}{1-\rho_1} < 2\rho_1.\end{aligned}$$

Similarly,

$$\frac{V(C\backslash \Phi M)}{V(\Phi M)} < 2\rho_1$$

We find an absolute constant $c_2 > 1$ such that $4\cdot(2\rho_1)^{\frac{1}{n}} = 4\cdot 2^{\frac{1}{n}}\cdot \tilde{c}\cdot n\cdot \varepsilon^{\frac{1}{19n}} \leq c_2\cdot n\cdot \varepsilon^{\frac{1}{19n}}$ which gives us

$$M \subset K \subset (1 + c_2 n\varepsilon^{\frac{1}{19n}})M \tag{4.64}$$

$$\Phi M \subset C \subset (1 + c_2 n\varepsilon^{\frac{1}{19n}})\Phi M. \tag{4.65}$$

We have $\rho_1 = \tilde{c}^n n^n \varepsilon^{\frac{1}{19}}$, and denote $\rho_2 = c_1 n^5 \varepsilon^{\frac{1}{95n}}$, $\rho_3 = c_2 n\varepsilon^{\frac{1}{19n}}$. Note that $n\varepsilon^{\frac{1}{19n}} < \rho_1^{\frac{1}{n}} < 1$ and $n^5\varepsilon^{\frac{1}{95n}} < \rho_1^{\frac{1}{5n}} < 1$. We deduce

$$\begin{aligned}(1+\rho_2)(1+\rho_3) &= 1 + \rho_2 + \rho_3 + \rho_2\rho_3\\ &= 1 + c_1 n^5 \varepsilon^{\frac{1}{95n}} + c_2 n \varepsilon^{\frac{1}{19n}} + c_1 c_2 n \varepsilon^{\frac{1}{19n}}\cdot n^5 \varepsilon^{\frac{1}{95n}}\\ &\leq 1 + (c_1 + c_2 + c_1 c_2) n^5 \varepsilon^{\frac{1}{95n}}\end{aligned}$$

that is, for $c_3 = c_1 + c_2 + c_1 c_2$

$$(1+\rho_2)(1+\rho_3) \leq 1 + c_3 n^5 \varepsilon^{\frac{1}{95n}} \tag{4.66}$$

Using the last inequality, we further deduce

$$\begin{aligned}(1+\rho_3)(1+\rho_2)^2 &\leq (1 + c_3 n^5 \varepsilon^{\frac{1}{95n}})(1+\rho_2)\\ &\leq 1 + c_1 n^5 \varepsilon^{\frac{1}{95n}} + c_3 n^5 \varepsilon^{\frac{1}{95n}} + c_1 c_3 n^{10} \varepsilon^{\frac{1}{95n}}\cdot \varepsilon^{\frac{1}{95n}}\\ &\leq 1 + (c_1 + c_3 + c_1 c_3) n^{10} \varepsilon^{\frac{1}{95n}}\end{aligned}$$

that is, for $c_4 = c_1 + c_3 + c_1 c_3$

$$(1+\rho_3)(1+\rho_2)^2 \leq 1 + c_5 n^{10} \varepsilon^{\frac{1}{95n}} \tag{4.67}$$

In the case $m = 1$, from Step 1 of the proof of Proposition 4.4.1, in particular (4.17), we have, for $\beta_1 = \min\{\alpha_1, \ldots, \alpha_n\}$ where $\Phi = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, $\theta = 8n^2\cdot(2\rho_1)^{\frac{1}{5n}}$ and using (4.61),

$$\begin{aligned}\beta_1 M \subset \Phi M &\subset (1 + 2n\theta)\beta_1 M\\ &\subset (1 + 2n^2\theta)\beta_1 M\\ &\subset (1 + c_1 n^5 \varepsilon^{\frac{1}{5n}})\beta_1 M\end{aligned} \tag{4.68}$$

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Combining (4.65), (4.68), and (4.66) we have

$$\begin{aligned}\beta_1 M \subset C &\subset (1+\rho_3)\Phi M \\ &\subset (1+\rho_3)(1+\rho_2)\beta_1 M \\ &\subset \ (1+c_3 n^5 \varepsilon^{\frac{1}{95n}})\beta_1 M\end{aligned}$$

That is,

$$\beta_1 M \subset C \subset (1+c_3 n^5 \varepsilon^{\frac{1}{95n}})\beta_1 M$$

From (4.64), we get

$$M \subset K \subset (1+\rho_3)M \subset (1+c_3 n^5 \varepsilon^{\frac{1}{95n}})M$$

Therefore, in the case $\|s\Phi - I_n\|_\infty \leq c_1 n^5 \varepsilon^{\frac{1}{95n}}$, that is, when $m = 1, choosing K_1 = M$ and $\theta = \beta_1$ establishes Theorem 4.2.3 On the other and, if $\|s\Phi - I_n\|_\infty > c_1 n^5 \cdot \varepsilon^{\frac{1}{95n}}$, then we choose

$$K_k = (1+\rho_2)^{-1}(L_{J_k} \cap M) \ \text{ for } k = 1, \ldots, m.$$

Note that

$$M \subset \bigoplus_{k=1}^{m} (L_{J_k} \cap M) \tag{4.69}$$

Using (4.64), (4.62), (4.69), (4.66) we deduce

$$\begin{aligned}\bigoplus_{k=1}^{m} K_k = \bigoplus_{k=1}^{m} (1+\rho_2)^{-1}(L_{J_k} \cap M) &\subset M \subset K \subset (1+\rho_3)M \\ &\subset (1+\rho_3)\bigoplus_{k=1}^{m}(L_{J_k} \cap M) \\ &\subset (1+\rho_3)(1+\rho_2)\bigoplus_{k=1}^{m}(1+\rho_2)^{-1}(L_{J_k} \cap M) \\ &\subset (1+\rho_3)(1+\rho_2)\bigoplus_{k=1}^{m} K_k \\ &\subset (1+c_3 n^5 \varepsilon^{\frac{1}{95n}})\bigoplus_{k=1}^{m} K_k \end{aligned} \tag{4.70}$$

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Next, using (4.65), (4.63), (4.69), (4.67) we get

$$\begin{aligned}\bigoplus_{k=1}^{m} s_k K_k &= \bigoplus_{k=1}^{m} s_k \cdot (1+\rho_2)^{-1}(L_{J_k} \cap M) \\ &\subset \Phi M \subset C \subset (1+\rho_3)\Phi M \\ &\subset (1+\rho_3)\bigoplus_{k=1}^{m} \Phi(L_{J_k} \cap M) \\ &\subset (1+\rho_3)\bigoplus_{k=1}^{m}(1+\rho_2)\cdot s_k \cdot (L_{J_k} \cap M) \\ &= (1+\rho_3)(1+\rho_2)^2 \bigoplus_{k=1}^{m} s_k K_k \\ &\subset (1+c_4 n^{10} \varepsilon^{\frac{1}{95n}}) \bigoplus_{k=1}^{m} s_k K_k \end{aligned} \tag{4.71}$$

Choosing an absolute constant $c_0 > c_4^{\frac{1}{2}} \cdot e^{\frac{10}{e}}$, we have

$$\left(1+c_3 n^5 \varepsilon^{\frac{1}{95n}}\right) < \left(1+c_4 n^{10} \varepsilon^{\frac{1}{95n}}\right) < (1+c_0^n \varepsilon^{\frac{1}{95n}})$$

and hence (4.70) and (4.71) yield (4.54) and (4.55).This proves Theorem 4.2.3 if $\lambda = \frac{1}{2}$ and $\varepsilon < \gamma^{-n} n^{-19n}$ (*cf.* (4.57)).

Still keeping $\lambda = \frac{1}{2}$, we observe that if Q is any unconditional convex body in $\mathbb{R}^n$, then $\bigoplus_{i=1}^{n}(\mathbb{R}e_i \cap Q)$ and Q share the same John ellipsoid E and as such we have

$$E \subset Q \subset \bigoplus_{i=1}^{n}(\mathbb{R}e_i \cap Q) \subset nE \subset nQ. \tag{4.72}$$

in particular,

$$\bigoplus_{i=1}^{n} \frac{1}{n}(\mathbb{R}e_i \cap Q) \subset Q \subset \bigoplus_{i=1}^{n}(\mathbb{R}e_i \cap Q) \tag{4.73}$$

Now if $\varepsilon \geq \gamma^{-n} n^{-19n}$ (*cf.* (4.57)) holds in (4.53), choosing an absolute constant

$$c_0 > \gamma^{\frac{1}{95n}} e^{\frac{6}{5e}} > \gamma^{\frac{1}{95n}} \cdot n^{\frac{1}{5n}} \cdot n^{\frac{1}{n}},$$

we get

$$c_0^n \varepsilon^{\frac{1}{95}} > \gamma^{\frac{1}{95n}} \cdot n^{\frac{1}{5}} \cdot n \cdot \gamma^{-\frac{1}{95}} \cdot n^{-\frac{1}{5}} > n.$$

Then taking $m = n$, $K_k = \frac{1}{n}(\mathbb{R}e_k \cap K)$, and choosing $\theta_k > 0$ in a way such that $\theta_k(\mathbb{R}e_k \cap K) = \mathbb{R}e_k \cap C$ for $k = 1, \ldots, n$, we have

$$\bigoplus_{k=1}^{n} \frac{1}{n}(\mathbb{R}e_k \cap K) \subset K \subset n \cdot \bigoplus_{k=1}^{n} \frac{1}{n}(\mathbb{R}e_k \cap K) \subset (1+c_0^n \varepsilon^{\frac{1}{95n}}) \bigoplus_{k=1}^{n} \frac{1}{n}(\mathbb{R}e_k \cap K) \tag{4.74}$$

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$$\bigoplus_{k=1}^{n}\theta_k\cdot\frac{1}{n}(\mathbb{R}e_k\cap C)\subset C\subset(1+c_0^n\varepsilon^{\frac{1}{95n}})\bigoplus_{k=1}^{n}\theta_k\cdot\frac{1}{n}(\mathbb{R}e_k\cap C) \tag{4.75}$$

And thus Theorem 4.2.3 has been verified if $\lambda=\frac{1}{2}$.

Next, we assume that $\lambda\in[\tau,1-\tau]$ holds for some $\tau\in(0,\frac{1}{2}]$ in Theorem 4.2.3. First let $\varepsilon\le\tau$. From Lemma 4.5.1, we have

$$\varphi(t)=V((1-t)\cdot K+_0 t\cdot C)$$

is log-concave on $[0,1]$. Note that the condition in Theorem 4.2.3 gives us

$$\varphi(\lambda)\le(1+\varepsilon)V(K)^{1-\lambda}V(C)^{\lambda}$$

Then Lemma 4.5.2 yields that

$$\varphi\left(\tfrac{1}{2}\right)\le\left(1+\frac{\varepsilon}{\min\{1-\lambda,\lambda\}}\right)\sqrt{\varphi(0)\varphi(1)};$$

and since $\lambda,1-\lambda\in[\tau,1-\tau]$,

$$V\left(\frac{1}{2}\cdot K+_0\frac{1}{2}\cdot C\right)\le\left(1+\frac{\varepsilon}{\tau}\right)V(K)^{\frac{1}{2}}V(C)^{\frac{1}{2}}.$$

Then (4.54) and (4.55) imply that for $m\ge1$, there exist $\theta_1,\dots,\theta_m>0$ and unconditional compact convex sets $K_1,\dots,K_m>0$ such that $\operatorname{lin}K_i$, $i=1,\dots,m$, are complementary coordinate subspaces, and

$$K_1\oplus\ldots\oplus K_m\subset\quad K\quad\subset\left(1+c_0^n\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(K_1\oplus\ldots\oplus K_m) \tag{4.76}$$

$$\theta_1K_1\oplus\ldots\oplus\theta_mK_m\subset\quad C\quad\subset\left(1+c_0^n\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right)(\theta_1K_1\oplus\ldots\oplus\theta_mK_m). \tag{4.77}$$

Finally, if $\lambda\in[\tau,1-\tau]$ holds for some $\tau\in(0,\frac{1}{2}]$ in Theorem 4.2.3 and $\varepsilon\ge\tau$, then taking an absolute constant $c_0>e^{\frac{1}{e}}>n^{\frac{1}{n}}$, we have $c_0^n\left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}>n$. Then choosing again $m=n$, $K_k=\frac{1}{n}(\mathbb{R}e_k\cap K)$, and $\theta_k>0$ in a way such that $\theta_k(\mathbb{R}e_k\cap K)=\mathbb{R}e_k\cap C$ for $k=1,\dots,n$, (4.73) yields (4.76) and (4.77). $\square$

4.6 Convex bodies and simplicial cones

Here we consider and state some results about the part of a convex body in a Weyl chamber, that we will use later. Recall that for a convex body M, we write $\partial' M$ to

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denote the smooth boundary points of M, that is, those boundary points with unique exterior unit normals. It is a well-known fact that the $n-1$-dimensional Hausdorff measure of $\partial M \backslash \partial' M$ is 0, that is, $\mathcal{H}^{n-1}(\partial M \backslash \partial' M) = 0$.

LEMMA 4.6.1. *Let $H_1, \ldots, H_n$ be independent $(n-1)$-dimensional linear subspaces, and let W be the closure of a connected component of $\mathbb{R}^n \backslash (H_1 \cup \ldots \cup H_n)$.*

(i) If M is a convex body in $\mathbb{R}^n$ symmetric through $H_1, \ldots, H_n$, then $\nu_{M,q} \in W$ for any $q \in W \cap \partial' M$, and in turn

$$M \cap W = \{x \in W : \langle x, u \rangle \leq h_M(u)\ \forall u \in W\}.$$

(ii) If $\lambda \in (0,1)$ and K and C are convex bodies in $\mathbb{R}^n$ symmetric through $H_1, \ldots, H_n$, then

$$W \cap ((1-\lambda)K +_0 \lambda C) = \{x \in W : \langle x, u \rangle \leq h_K(u)^{1-\lambda} h_C(u)^{\lambda}\ \forall u \in W\}.$$

Proof: For (i), it is sufficient to prove the first statement; namely, if $q \in \operatorname{int} W \cap \partial' K$, then $\nu_{M,q} \in W$.

Let $u_i \in S^{n-1}$, $i = 1, \ldots, n$, such that $W = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \geq 0\}$, and hence $\langle q, u_i \rangle > 0$, $i = 1, \ldots, n$, and (i) is equivalent with the statement that if $i = 1, \ldots, n$, then

$$\langle u_i, \nu_{K,q} \rangle \geq 0. \tag{4.78}$$

Since $q' = q - 2\langle q, u_i \rangle u_i$ is the reflected image of q through H_i, we have $q' \in M$; therefore,

$$0 \leq \langle \nu_{K,q}, q - q' \rangle = \langle \nu_{K,q}, 2\langle q, u_i \rangle u_i \rangle = 2\langle q, u_i \rangle \cdot \langle \nu_{K,q}, u_i \rangle.$$

As $\langle q, u_i \rangle > 0$, we conclude (4.78), and in turn (i).

For (ii), let $M = (1-\lambda)K +_0 \lambda C$, and let

$$M_+ = \{x \in W : \langle x, u \rangle \leq h_K(u)^{1-\lambda} h_C(u)^{\lambda}\ \forall u \in W\}.$$

Readily, $W \cap M \subset M_+$. Therefore, (ii) follows if for any $q \in \partial' M \cap \operatorname{int} W$, we have $q \in \partial M_+$. As $q \in \partial M \cap \operatorname{int} W$, there exists $u \in S^{n-1}$ such that $\langle q, u \rangle = h_K(u)^{1-\lambda} h_C(u)^{\lambda}$. Since $q \in \partial' M \cap W$, we have $u = \nu_{M,q}$, and hence (i) yields that $\nu_{M,q} \in W$. Therefore $q \in \partial M_+$, proving Lemma 4.6.1 (ii). $\square$

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In order to use already established results for unconditional convex bodies to the case of convex bodies with n independent hyperplane symmetries, our main idea is to linearly transfer a Weyl chamber W into the co-ordinate corner $\mathbb{R}^n_{\geq 0}$.

LEMMA 4.6.2. *Let K be a convex body in $\mathbb{R}^n$ with $o \in \operatorname{int} K$, let independent $v_1, \ldots, v_n \in \mathbb{R}^n$ be such that $\langle v_i, v_j \rangle \geq 0$ for $1 \leq i \leq j \leq n$, let $W = \operatorname{pos}\{v_1, \ldots, v_n\}$, and let $\Phi \in \mathrm{GL}(n, \mathbb{R})$ such that $\Phi W = \mathbb{R}^n_{\geq 0}$, then:*

(i) $\Phi^{-t} W \subset \mathbb{R}^n_{\geq 0}$.

(ii) If $\nu_{K,x} \in W$ for any $x \in W \cap \partial' K$, then

$$\nu_{\Phi K, z} \in \mathbb{R}^n_{\geq 0} \quad \text{for any } z \in \mathbb{R}^n_{\geq 0} \cap \partial' \Phi K; \tag{4.79}$$

(iii) and there exists an unconditional convex body K_0 such that

$$K_0 \cap \mathbb{R}^n_{\geq 0} = \Phi(K \cap W).$$

Proof: Let $e_1, \ldots, e_n$ be the standard orthonormal basis of $\mathbb{R}^n$ indexed in a way such that $e_i = \Phi v_i$. First we claim that for $i = 1, \ldots, n$

$$\langle \Phi^{-t} v, e_i \rangle \geq 0. \tag{4.80}$$

Since $v \in W = \operatorname{pos}\{v_1, \ldots, v_n\}$, we can write $v = \sum_{j=1}^n \lambda_j v_j$ for $\lambda_j \geq 0$. Now, $\langle v_j, v_i \rangle \geq 0$ for $j = 1, \ldots, n$ gives us

$$0 \leq \sum_{j=1}^n \lambda_j \langle v_j, v_i \rangle = \left\langle \sum_{j=1}^n \lambda_j v_j, v_i \right\rangle = \langle v, v_i \rangle = \langle \Phi^{-t} v, \Phi v_i \rangle = \langle \Phi^{-t} v, e_i \rangle,$$

proving (4.80) and hence (i) holds.

Take any $x \in W \cap \partial' K$, then from the condition in (ii), $\nu_{K,x} \in W$. Note that $\Phi^{-t} \nu_{K,x}$ is an exterior normal to ΦK at Φx. We have $\Phi x \in \mathbb{R}^n_{\geq 0} \cap \partial' \Phi K$ and from (i), $\nu_{\Phi K, \Phi x} = \Phi^{-t} \nu_{K,x} \in \mathbb{R}^n_{\geq 0}$ and hence, (ii) holds.

Now (4.79) yields that if $z = (z_1, \ldots, z_n) \in \mathbb{R}^n_{\geq 0} \cap \partial' \Phi K$ and $0 \leq y_i \leq z_i$, $i = 1, \ldots, n$, then $y = (y_1, \ldots, y_n) \in \Phi K$. Therefore repeatedly reflecting $\mathbb{R}^n_{\geq 0} \cap \Phi K$ through the coordinate hyperplanes, we obtain the unconditional convex body K_0 such that $\mathbb{R}^n_{\geq 0} \cap K_0 = \mathbb{R}^n_{\geq 0} \cap \Phi K = \Phi(W \cap K)$. $\square$

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4.7 Some properties of Coxeter groups

We note here that if a convex body K is invariant under a linear map A, then the minimal volume Löwner ellipsoid of K is also invariant under A. And hence, according to the following result from Barthe, Fradelizi [19], it is sufficient to consider only orthogonal reflections in our case.

LEMMA 4.7.1 (Barthe, Fradelizi). *If the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1, \ldots, A_n$ through n independent linear $(n-1)$-planes $H_1, \ldots, H_n$, then there exists $B \in \mathrm{SL}(n)$ such that $BA_1B^{-1}, \ldots, BA_nB^{-1}$ are orthogonal reflections through $BH_1, \ldots, BH_n$ and leave BK and BC invariant.*

Here, we briefly discuss some theory concerning Coxeter groups following Humpreys [98]. Let V be an n dimensional vector space with the usual Euclidean structure. Let $p_1, \ldots, p_n$ be n independent vectors in V. Then we denote by G the closure of the Coxeter group generated by orthogonal reflections through the independent hyperplanes $p_1^\perp, \ldots, p_n^\perp$. We note that a linear subspace $L \subset V$ is invariant under the action of G if and only if $p_1, \ldots, p_n \in L \cup L^\perp$. We call an invariant subspace $L \subset V$ irreducible if $L \neq \{o\}$ and any invariant subspace $L' \subset L$ satisfies either $L' = L$ or $L' = \{o\}$. So the action of G on an irreducible invariant subspace is irreducible. We note that the intersection and the orthogonal complement of invariant subspaces is invariant and hence, the irreducible subspaces $L_1, \ldots, L_m$, $m \geq 1$ are pairwise orthogonal, and

$$L_1 \oplus \ldots \oplus L_m = V. \tag{4.81}$$

Then for any $A \in G$, we can write $A = A|_{L_1} \oplus \ldots \oplus A|_{L_m}$. If $L \subset V$ is an invariant subspace, we denote $G|_L = \{A|_L : A \in G\}$ and $O(L)$ as the group of isometries of L fixing the origin. In this section our main task is to understand some properties of irreducible Coxeter groups.

LEMMA 4.7.2 (Barthe, Fradelizi). *Let G be closure of the Coxeter group generated by the orthogonal reflections through $p_1^\perp, \ldots, p_n^\perp$ for independent $p_1, \ldots, p_n \in \mathbb{R}^n$. If $L \subset \mathbb{R}^n$ is an irreducible invariant subspace, and $G|_L$ is infinite, then $G|_L = O(L)$.*

Let L be a d-dimensional irreducible invariant linear subspace of V with respect to the closure G of a Coxeter group. In the case when $G|_L$ is finite, we need a more detailed analysis. We write $G' = G|_L$ to denote a finite Coxeter group generated by some orthogonal reflections acting on L. Denote by $H_1, \ldots, H_d$ the $(d-1)$ dimensional subspaces of L such that the reflections through $H_1, \ldots, H_d$ generate G'. let $u_1, \ldots, u_{2d} \in L \backslash \{o\}$ be a system of roots for G'. Here, there are exactly two roots orthogonal to each H_i,

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and these two roots are opposite. For algebraic purposes, it's customary to normalize the roots such that $\frac{2\langle u_i,u_j\rangle}{\langle u_i,u_i\rangle}$ is an integer but we drop this condition since we are only concerned with the cones determined by the roots.

We denote by W the closure of a Weyl chamber, that is, a connected component of $L\backslash\{H_1,\ldots,H_d\}$. It is known (see say [98]) that

$$W=\mathrm{pos}\{v_1,\ldots,v_d\}=\left\{\sum_{i=1}^{d}\lambda_i v_i:\forall\,\lambda_i\geq 0\right\}$$

where $v_1,\ldots,v_d\in L$ are independent. Moreover, for any $x\in L\backslash\{H_1,\ldots,H_d\}$, there exists a unique $A\in G'$ such that $x\in AW$. Thus there is a natural bijective correspondence between the Weyl chambers and G'. We may reindex $H_1,\ldots,H_d$ and $u_1,\ldots,u_{2d}$ in a way such that $H_i=u_i^\perp$ for $i=1,\ldots,d$ are the "walls" of W, and

$$\begin{aligned}\langle u_i,v_i\rangle &> 0 \quad \text{for } i=1,\ldots,d;\\ \langle u_i,v_j\rangle &= 0 \quad \text{for } 1\leq i<j\leq d.\end{aligned}\tag{4.82}$$

In this case, reflections $L\to L$ through $H_1,\ldots,H_d$ generate G', and $u_1,\ldots,u_d$ is called a simple system of roots. The order we list simple roots is not related to the corresponding Dynkin diagram.

LEMMA 4.7.3. *Let G be the Coxeter group generated by the orthogonal reflections through $p_1^\perp,\ldots,p_n^\perp$ for independent $p_1,\ldots,p_n\in\mathbb{R}^n$. If $L\subset\mathbb{R}^n$ is an irreducible invariant d-dimensional subspace with $d\geq 2$, and $G|_L$ is finite, and $W=\mathrm{pos}\{v_1,\ldots,v_d\}\subset L$ is the closure of a Weyl chamber for $G|_L$, then*

$$\langle v_i,v_j\rangle\geq\frac{1}{d}\cdot\|v_i\|\cdot\|v_j\|.\tag{4.83}$$

Proof: Let $G'=G|_L$. We use the classification of finite irreducible Coxeter groups. For the cases when G' is either of D_d,E_6,E_7,E_8 (see Adams [1] about E_6,E_7,E_8), we use the known simple systems of roots in terms of the orthonormalt basis $e_1,\ldots,e_d$ of L to construct $v_1,\ldots,v_d$ *via* (4.82). However, there is a unified construction for the other finite irreducible Coxeter groups because they are the symmetries of some regular polytopes.

Case 1: G' is one of the types $\mathbf{I_2(m)}$, $\mathbf{A_d}$, $\mathbf{B_d}$, $\mathbf{F_4}$, $\mathbf{H_3}$, $\mathbf{H_4}$

In this case, G' is the symmetry group of some d-dimensional regular polytope P centered at the origin. Let $F_0\subset\ldots\subset F_{d-1}$ be a tower of faces of P where $\dim F_i=i$, $i=0,\ldots,d-1$. Defining v_i to be the centroid of F_{i-1}, $i=1,\ldots,d$, we have that

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$W = \mathrm{pos}\{v_1, \ldots, v_d\}$ is the closure of a Weyl chamber because the symmetry group of P is simply transitive on the towers of faces of P.

As G' is irreducible, the John ellipsoid of P (the unique ellipsoid of largest volume contained in P) is a d-dimensional ball centered at the origin of some radius $r > 0$. It follows that $P \subset drB^n$, and hence $r \leq \|v_i\| \leq dr$ for $i = 1, \ldots, d$. In addition, v_i is the closest point of aff F_{i-1} to the origin for $i = 1, \ldots, d$, and $v_j \in F_{i-1}$ if $1 \leq j \leq i$, thus $\langle v_j, v_i \rangle = \langle v_i, v_i \rangle$ if $1 \leq j \leq i \leq d$. We conclude that if $1 \leq j \leq i \leq d$, then

$$\frac{\langle v_j, v_i \rangle}{\|v_j\| \cdot \|v_i\|} = \frac{\|v_i\|}{\|v_j\|} \geq \frac{1}{d}.$$

Case 2: $G' = D_n$
In this case, a simple system of roots is

$$\begin{aligned} u_i &= e_i - e_{i+1} \qquad \text{for } i = 1, \ldots, d-1, \\ u_d &= e_{d-1} + e_d. \end{aligned}$$

In turn, we may choose $v_1, \ldots, v_d$ as

$$\begin{aligned} v_i &= \textstyle\sum_{l=1}^{i} e_l \qquad \text{for } i = 1, \ldots, d-2 \text{ and } i = d, \\ v_{d-1} &= -v_d + \textstyle\sum_{l=1}^{d-1} e_l. \end{aligned}$$

As $\langle v_i, v_j \rangle$ is a positive integer for $i \neq j$, and $\|v_i\| \leq \sqrt{d}$ for $i = 1, \ldots, d$, we conclude (4.83).

Case 3: $G' = E_6$
In this case $d = 6$, and a simple system of roots is

$$\begin{aligned} u_i &= e_i - e_{i+1} \qquad \text{for } i = 1, 2, 3, 4, \\ u_5 &= e_4 + e_5 \\ u_6 &= \sqrt{3}\, e_6 - \textstyle\sum_{l=1}^{5} e_l. \end{aligned}$$

Using coordinates in $e_1, \ldots, e_6$, we may choose $v_1, \ldots, v_6$ as $v_1 = (\sqrt{3}, 0, 0, 0, 0, 1)$, $v_2 = (\sqrt{3}, \sqrt{3}, 0, 0, 0, 2)$, $v_3 = (\sqrt{3}, \sqrt{3}, \sqrt{3}, 0, 0, 3)$, $v_4 = (1, 1, 1, 1, -1, \sqrt{3})$, $v_5 = (1, 1, 1, 1, 1, \frac{5}{\sqrt{3}})$ and $v_6 = (0, 0, 0, 0, 0, 3)$. As $\langle v_i, v_j \rangle \geq 3$ for $i \neq j$, and $\|v_i\| \leq \sqrt{18}$ for $i = 1, \ldots, 6$, we conclude (4.83).

Case 4: $G' = E_7$

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In this case $d = 7$, and a simple system of roots is

$$\begin{aligned} u_i &= e_i - e_{i+1} && \text{for } i = 1,2,3,4,5, \\ u_6 &= e_5 + e_6 \\ u_7 &= \sqrt{2}\, e_7 - \textstyle\sum_{l=1}^{6} e_l. \end{aligned}$$

Using coordinates in $e_1, \dots, e_7$, we may choose $v_1, \dots, v_7$ as $v_1 = (2,0,0,0,0,0,\sqrt{2})$, $v_2 = (1,1,0,0,0,0,\sqrt{2})$, $v_3 = (1,1,1,0,0,0,\frac{3}{\sqrt{2}})$, $v_4 = (1,1,1,1,0,0,2\sqrt{2})$, $v_5 = (1,1,1,1,1,-1,2\sqrt{2})$, $v_6 = (1,1,1,1,1,1,3\sqrt{2})$ and $v_7 = (0,0,0,0,0,4)$. As $\langle v_i, v_j\rangle \geq 4$ for $i \neq j$, and $\|v_i\| < \sqrt{28}$ for $i = 1, \dots, 7$, we conclude (4.83).

Case 5: $G' = E_8$

In this case $d = 8$, and a simple system of roots is

$$\begin{aligned} u_i &= e_i - e_{i+1} && \text{for } i = 1,2,3,4,5,6,7, \\ u_8 &= -\textstyle\sum_{l=1}^{5} e_l + \sum_{l=6}^{8} e_l. \end{aligned}$$

Using coordinates in $e_1, \dots, e_8$, we may choose $v_1, \dots, v_8$ as

$$\begin{aligned} v_1 &= (1,-1,-1,-1,-1,-1,-1,-1) \\ v_2 &= (0,0,-1,-1,-1,-1,-1,-1) \\ v_3 &= (-1,-1,-1,-3,-3,-3,-3,-3) \\ v_4 &= (-1,-1,-1,-1,-2,-2,-2,-2) \\ v_5 &= (-1,-1,-1,-1,-1,-\tfrac{5}{3},-\tfrac{5}{3},-\tfrac{5}{3}) \\ v_6 &= (-1,-1,-1,-1,-1,-1,-2,-2) \\ v_7 &= (-1,-1,-1,-1,-1,-1,-1,-3) \text{ and} \\ v_8 &= (-1,-1,-1,-1,-1,-1,-1,-1) \end{aligned}$$

As $\langle v_i, v_j\rangle \geq 6$ for $i \neq j$, and $\|v_i\| < \sqrt{48}$ for $i = 1, \dots, 8$, we conclude (4.83). $\square$

For a convex body invariant under a Coxeter group, we can determine some exterior normals at certain points provided by the symmetries of the convex body.

LEMMA 4.7.4. *Let G be the closure of a Coxeter group generated by n independent orthogonal reflections of $\mathbb{R}^n$, let $L \subset \mathbb{R}^n$ be an irreducible linear subspace and let K be a convex body in $\mathbb{R}^n$ invariant under G.*

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(i) If $G|_L$ is finite, and $W = \mathrm{pos}\{v_1, \ldots, v_d\} \subset L$ is the closure of a Weyl chamber for $G|_L$, and $t_i v_i \in \partial K$ for $t_i > 0$, $i = 1, \ldots, d$, then v_i is an exterior normal at tv_i.

(ii) If $G|_L$ is infinite and $v \in L\backslash\{o\}$, and $tv \in \partial K$ for $t > 0$, then v is an exterior normal at tv.

Proof: Let $d = \dim L$.

For (i), first we claim that there exist independent $u_1, \ldots, u_{n-1} \in v_i^\perp$ such that the reflection through $u_j^\perp$ lies in G for $j = 1, \ldots, n-1$. To construct $u_1, \ldots, u_{n-1} \in v_i^\perp$, if $d \geq 2$, then we choose roots $u_1, \ldots, u_{d-1} \in v_i^\perp$ for $G|_L$ that corresponds to the walls of W containing v_i. In addition, if $d < n$, then we choose independent $u_d, \ldots, u_{n-1} \in L^\perp$ such that the reflection through $u_j^\perp$ lies in G for $j = d, \ldots, n-1$, completing the construction of $u_1, \ldots, u_{n-1}$.

Let $N = \{z \in \mathbb{R}^n : \langle z, t_i v_i - x\rangle \geq 0 \ \forall x \in K\}$ be the normal cone at $t_i v_i \in \partial K$. If $N = \mathbb{R}_{\geq 0} v_i$, then we are done. Since N is a cone and $o \in \mathrm{int}\, K$, if $N \neq \mathbb{R}_{\geq 0} v_i$, then there exists $w \in v_i^\perp \backslash \{o\}$ such that $z = v_i + w \in N$. Let $H \subset G$ be the closure of the subgroup generated by the reflections through $u_1^\perp, \ldots, u_{n-1}^\perp$, and hence both $\mathbb{R}v_i$ and $v_i^\perp$ are invariant under H. Since $u_1, \ldots, u_{n-1} \in v_i^\perp$ are independent, the centroid of $M = \mathrm{conv}\{Aw : A \in H\} \subset v_i^\perp$ is o. We deduce that the centroid of $v_i + M = \mathrm{conv}\{Aw : A \in H\} \subset N$ is v_i; therefore, $v_i \in N$.

For (ii), the argument is essentially same because similarly, there exist independent $\tilde{u}_1, \ldots, \tilde{u}_{n-1} \in v^\perp$ such that the reflection through $\tilde{u}_j^\perp$ lies in G for $j = 1, \ldots, n-1$. $\square$

4.8 The proof Theorem 4.1.4

Lemma 4.7.1 and the linear invariance of the L_0-sum yield that we may assume that $A_1, \ldots, A_n$ are orthogonal reflections through the linear $(n-1)$-spaces $H_1, \ldots, H_n$, respectively, with $H_1 \cap \ldots \cap H_n = \{o\}$ where K and C are invariant under $A_1, \ldots, A_n$.

Let G be the closure of the group generated by $A_1, \ldots, A_n$, and let $L_1, \ldots, L_m$ be the irreducible invariant subspaces of $\mathbb{R}^n$ of the action of G. If $t_1, \ldots, t_m > 0$ and $\Psi \in \mathrm{GL}(n, \mathbb{R})$ satisfies $\Psi x = t_i x$ for $x \in L_i$ and $i = 1, \ldots, m$, then

$$\Psi K \text{ and } \Psi C \text{ are both invariant under } G. \tag{4.84}$$

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Let E be the John ellipsoid of K, that is, the unique ellipsoid of maximal volume contained in K. Therefore, E is also invariant under G. In particular, we can choose the principal directions of E in a way such that each is contained in one of the L_i, and $L_i \cap E$ is a Euclidean ball of dimension $\dim L_i$. Therefore, after applying a suitable linear transformation like in (4.84), we may assume that $E = B^n$, and hence

$$B^n \subset K \subset nB^n. \tag{4.85}$$

For any $i = 1, \ldots, n$, let $G_i = G|_{L_i}$ if $G|_{L_i}$ is finite, and let G_i be the symmetry group of some $\dim L_i$ dimensional regular simplex in L_i centered at the origin if $G|_{L_i}$ is infinite.

We consider the finite subgroup $\widetilde{G} \subset G$ that is the direct sum of $G_1, \ldots, G_m$, acting in the natural way $\widetilde{G}|_{L_i} = G_i$ for $i = 1, \ldots, m$. Let $0 = p_0 < p_1 < \ldots < p_m = n$ satisfy that $p_i - p_{i-1} = \dim L_i$ for $i = 1, \ldots, m$. We choose a basis $v_1, \ldots, v_n \in S^{n-1}$ of $\mathbb{R}^n$, in a way such that for each $i = 1, \ldots, m$, $W_i = \mathrm{pos}\{v_{p_{i-1}+1}, \ldots, v_{p_i}\}$ is the closure of a Weyl chamber for the irreducible action of G_i on L_i.

According to Lemma 4.7.3, these $v_1, \ldots, v_n \in S^{n-1}$ satisfy that

$$\langle v_j, v_l \rangle \geq \frac{1}{n} \quad \text{if } p_{i-1} + 1 \leq j < l \leq p_i \text{ and } i = 1, \ldots, m; \tag{4.86}$$

$$\langle v_j, v_l \rangle = 0 \quad \text{if there exists } i = 1, \ldots, m-1 \text{ such that } j \leq p_i < l. \tag{4.87}$$

Let $e_1, \ldots, e_n$ be the standard orthonormal basis of $\mathbb{R}^n$, let $\Phi \in \mathrm{GL}(n)$ satisfy that $\Phi v_i = e_i$, $i = 1, \ldots, n$, and let

$$W = W_1 \oplus \ldots \oplus W_m.$$

It follows that $\Phi W = \mathbb{R}^n_{\geq 0}$ and $\mathrm{int}\, W$ is a fundamental domain for $\widetilde{G}$ in the sense that

$$\begin{aligned} \bigcup\{AW : A \in \widetilde{G}\} &= \mathbb{R}^n \\ \mathrm{int}\, AW \cap \mathrm{int}\, BW &= \emptyset \qquad \text{if } A, B \in \widetilde{G} \text{ and } A \neq B. \end{aligned} \tag{4.88}$$

If $i \in \{1, \ldots, m\}$ and $p_{i-1} + 1 \leq j \leq p_i$, then we define $u_j \in L_i \cap S^{n-1}$ by $\langle u_j, v_j \rangle > 0$ and $\langle u_j, v_l \rangle = 0$ for $l \neq j$. Therefore, $u_1^\perp, \ldots, u_n^\perp$ are the walls of W; namely, the linear hulls of the facest of the simplicial cone W, and the reflections through $u_1^\perp, \ldots, u_n^\perp$ are symmetries of both K and C (and actually generate $\widetilde{G}$). We may apply Lemma 4.6.2 to W because of Lemma 4.6.1, (4.86) and (4.87), and deduce the existence unconditional convex bodies $\widetilde{K}$ and $\widetilde{C}$ such that

$$\mathbb{R}^n_{\geq 0} \cap \widetilde{K} = \Phi(W \cap K) \text{ and } \mathbb{R}^n_{\geq 0} \cap \widetilde{C} = \Phi(W \cap C).$$

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We claim that

$$\mathbb{R}^n_{\geq 0} \cap ((1-\lambda)\widetilde{K} + \lambda\widetilde{C}) \subset \Phi\left(W \cap ((1-\lambda)K +_0 \lambda C)\right). \tag{4.89}$$

According to Lemma 4.6.1 and to $\Phi^{-t}W \subset \mathbb{R}^n_{\geq 0}$ (*cf.* Lemma 4.6.2), we have

$$\begin{aligned} \mathbb{R}^n_{\geq 0} \cap ((1-\lambda)\widetilde{K} + \lambda\widetilde{C}) &= \{x \in \mathbb{R}^n_{\geq 0} : \langle x, u\rangle \leq h_{\widetilde{K}}(u)^{1-\lambda} h_{\widetilde{C}}(u)^{\lambda}\ \forall u \in \mathbb{R}^n_{\geq 0}\} \\ &\subset \{x \in \mathbb{R}^n_{\geq 0} : \langle x, u\rangle \leq h_{\widetilde{K}}(u)^{1-\lambda} h_{\widetilde{C}}(u)^{\lambda}\ \forall u \in \Phi^{-t}W\}. \end{aligned}$$

We observe that if $u \in \Phi^{-t}W$, then there exist $y_0 \in \mathbb{R}^n_{\geq 0} \cap \partial\widetilde{K} = \mathbb{R}^n_{\geq 0} \cap \partial(\Phi K)$ and $z_0 \in \mathbb{R}^n_{\geq 0} \cap \partial\widetilde{C} = \mathbb{R}^n_{\geq 0} \cap \partial(\Phi C)$ with $h_{\widetilde{K}}(u) = \langle y_0, u\rangle$ and $h_{\widetilde{C}}(u) = \langle z_0, u\rangle$. For $v = \Phi^t u \in W$, $y = \Phi^{-1}y_0 \in W \cap \partial K$ and $y = \Phi^{-1}y_0 \in W \cap \partial K$, it follows that v is an exterior normal to K at y and to C at z, and

$$h_{\widetilde{K}}(u)^{1-\lambda} h_{\widetilde{C}}(u)^{\lambda} = \langle \Phi y, \Phi^{-t}v\rangle^{1-\lambda} \langle \Phi z, \Phi^{-t}v\rangle^{\lambda} = \langle y, v\rangle^{1-\lambda}\langle z, v\rangle^{\lambda} = h_K(v)^{1-\lambda} h_C(v)^{\lambda}.$$

We deduce from the considerations just above and from applying Lemma 4.6.1 to W that

$$\begin{aligned} \mathbb{R}^n_{\geq 0} \cap ((1-\lambda)\widetilde{K} + \lambda\widetilde{C}) &\subset \Phi\{q \in W : \langle q, v\rangle \leq h_K(v)^{1-\lambda} h_K(v)^{\lambda}\ \forall v \in W\} \\ &= \Phi\left(W \cap ((1-\lambda)K +_0 \lambda C)\right), \end{aligned}$$

proving (4.89).

Writing $|\widetilde{G}|$ to denote the cardinality of $\widetilde{G}$, (4.88) yields

$$V(M) = |\widetilde{G}| \cdot V(M \cap W)$$

where M is either K, C or $(1-\lambda)\cdot K +_0 \lambda \cdot C$. We deduce from (4.89) and the condition in Theorem 4.1.4 that

$$\begin{aligned} V((1-\lambda)\cdot \widetilde{K} +_0 \lambda \cdot \widetilde{C}) &= 2^n V\left(\mathbb{R}^n_{\geq 0} \cap ((1-\lambda)\cdot \widetilde{K} +_0 \lambda \cdot \widetilde{C})\right) \\ &\leq 2^n V\left(\Phi\left(W \cap ((1-\lambda)K +_0 \lambda C)\right)\right) \\ &\leq \frac{2^n |\det \Phi|}{|\widetilde{G}|} \cdot (1+\varepsilon) V(K)^{1-\lambda} V(C)^{\lambda} \\ &= (1+\varepsilon) V(\widetilde{K})^{1-\lambda} V(\widetilde{C})^{\lambda}. \end{aligned}$$

We apply the following equivalent form of Theorem 4.2.3 to $\widetilde{K}$ and $\widetilde{C}$ where $\lambda \in [\tau, 1-\tau]$ for $\tau \in (0, \frac{1}{2}]$. There exist absolute constant $\tilde{c} > 1$, complementary coordinate linear subspaces $\widetilde{\Lambda}_1, \ldots, \widetilde{\Lambda}_k$, $k \geq 1$, with $\oplus_{j=1}^k \widetilde{\Lambda}_j = \mathbb{R}^n$ such that

$$\oplus_{j=1}^k \left(\widetilde{K} \cap \widetilde{\Lambda}_j\right) \subset \left(1 + \tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) \widetilde{K}, \tag{4.90}$$

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and there exist $\theta_1, \ldots, \theta_k > 0$ such that

$$\oplus_{j=1}^{k} \theta_j \left(\widetilde{K} \cap \widetilde{\Lambda}_j\right) \subset \widetilde{C} \subset \left(1 + \tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) \oplus_{j=1}^{k} \theta_j \left(\widetilde{K} \cap \widetilde{\Lambda}_j\right). \tag{4.91}$$

For $\Lambda_j = \Phi^{-1}\widetilde{\Lambda}_j$, $j = 1, \ldots, k$, we deduce that

$$W \cap \sum_{j=1}^{k} (K \cap \Lambda_j) \subset \left(1 + \tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) (W \cap K), \tag{4.92}$$

and

$$W \cap \sum_{j=1}^{k} \theta_j (K \cap \Lambda_j) \subset W \cap C \subset \left(1 + \tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) \left(W \cap \sum_{j=1}^{k} \theta_j (K \cap \Lambda_j)\right). \tag{4.93}$$

We observe that each Λ_j is spanned by a subset of $v_1, \ldots, v_n$.

For the rest of the argument, first we assume that ε is small enough to satisfy

$$\tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}} < \frac{1}{n^2}. \tag{4.94}$$

We claim that if (4.94) holds, then

$$\text{each } \Lambda_j,\ j = 1, \ldots, k, \text{ is invariant under } G. \tag{4.95}$$

We suppose indirectly that the claim (4.95) does not hold, and we seek a contradiction. In this case, $k \geq 2$. Since each Λ_j is spanned by a subset of $v_1, \ldots, v_n$, after possibly reindexing $L_1, \ldots, L_m$, $\Lambda_1, \ldots, \Lambda_k$ and $v_1, \ldots, v_n$, we may assume that $v_1 \in L_1 \cap \Lambda_1$ and $v_2 \in L_1 \cap \Lambda_2$. For $i = 1, \ldots, n$, let $s_i > 0$ satisfy $s_i v_i \in \partial K$; therefore, (4.85) yields

$$1 \leq s_i \leq n, \tag{4.96}$$

and hence

$$s_1 v_1 \in L_1 \cap K \cap \Lambda_1 \text{ and } v_2 \in L_1 \cap K \cap \Lambda_2. \tag{4.97}$$

It follows from (4.86) that

$$\langle v_1, v_2 \rangle \geq \frac{1}{n}. \tag{4.98}$$

We deduce from (4.97), and then from (4.92) that

$$s_1 v_1 + v_2 \in W \cap \sum_{j=1}^{k} (K \cap \Lambda_j) \subset \left(1 + \tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) (W \cap K). \tag{4.99}$$

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Lemma 4.7.4 yields that v_1 is an exterior unit normal to ∂K at $s_1 v_1$, and hence $s_1 = h_K(v_1)$. We deduce from first (4.99) and then from assumption (4.94) and the formula (4.96) that

$$\begin{aligned} s_1 + \langle v_1, v_2\rangle &= \langle v_1, s_1 v_1 + v_2\rangle \le \left(1 + \tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) h_K(v_1) \\ &= s_1 + \tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}} s_1 < 1 + \frac{1}{n}. \end{aligned} \tag{4.100}$$

On the other hand, we have $s_1 + \langle v_1, v_2\rangle \ge 1 + \frac{1}{n}$ by (4.98), contradicting (4.100). In turn, we conclude (4.95) under the assumption (4.94).

We deduce from (4.92), (4.93), (4.95) and the symmetries of K and C that

$$\oplus_{j=1}^{k} (K \cap \Lambda_j) \subset \left(1 + \tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) K, \tag{4.101}$$

and

$$\oplus_{j=1}^{k} \theta_j (K \cap \Lambda_j) \subset C \subset \left(1 + \tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) \oplus_{j=1}^{k} \theta_j (K \cap \Lambda_j). \tag{4.102}$$

In addition, the symmetries of K and (4.95) yield that $K \cap \Lambda_j = K|\Lambda_j$ for $j = 1, \dots, k$, therefore,

$$K \subset \oplus_{j=1}^{k} (K \cap \Lambda_j).$$

Combining this relation with (4.101) and (4.102) implies Theorem 4.1.4 under the assumption (4.94).

Finally, we assume that

$$\tilde{c}^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}} \ge \frac{1}{n^2}, \tag{4.103}$$

and hence

$$(4\tilde{c})^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}} \ge n^2. \tag{4.104}$$

For $i = 1, \dots, m$, the symmetries of K and C yield that $r_i(B^n \cap L_i)$ is the John ellipsoid of $K \cap L_i$ and $\theta_i r_i(B^n \cap L_i)$ is the John ellipsoid of $C \cap L_i$ for some $r_i, \theta_i > 0$. For $K_i = \frac{r_i}{n}(B^n \cap L_i)$, $i = 1, \dots, m$, we have

$$\oplus_{i=1}^{m} K_i \subset \mathrm{conv}\{mK_1, \dots, mK_m\};$$

therefore, it follows from (4.104) that

$$\begin{aligned} \oplus_{i=1}^{m} K_i &\subset K \subset n^2 \cdot \oplus_{i=1}^{m} K_i \subset \left(1 + (4\tilde{c})^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) \oplus_{i=1}^{m} K_i \\ \oplus_{i=1}^{m} \theta_i K_i &\subset C \subset n^2 \cdot \oplus_{i=1}^{m} \theta_i K_i \subset \left(1 + (4\tilde{c})^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{95n}}\right) \oplus_{i=1}^{m} \theta_i K_i, \end{aligned}$$

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proving Theorem 4.1.4 under the assumption (4.103). □

4.9 Proof of Theorem 4.1.5

As in the case of Theorem 4.1.4, it follows from Lemma 4.7.1 and the linear invariance of the L_0-sum that we may assume that $A_1, \ldots, A_n$ are orthogonal reflections through the linear $(n-1)$-spaces $H_1, \ldots, H_n$, respectively, with $H_1 \cap \ldots \cap H_n = \{o\}$ where K and C are invariant under $A_1, \ldots, A_n$. We write G to denote the closure of the group generated by $A_1, \ldots, A_n$, and $L_1, \ldots, L_m$ to denote the irreducible invariant subspaces of $\mathbb{R}^n$ of the action of G.

For the logarithmic Minkowski Conjecture 4.1.2, replacing either K or C by a dilate does not change the difference of the two sides; therefore, we may assume that

$$V(K) = V(C) = 1.$$

In this case, the condition in Theorem 4.1.5 states that

$$\int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K < \varepsilon \tag{4.105}$$

for $\varepsilon > 0$.

First we assume that

$$n\varepsilon < 1, \tag{4.106}$$

for $t \in [0,1]$, we define

$$\varphi(t) = V((1-t)\cdot K +_0 t\cdot C).$$

According to (3.7) in Böröczky, Lutwak, Yang, Zhang [39], we have

$$\varphi'(0) = n \int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K, \tag{4.107}$$

and hence (4.105) and the assumption (4.106) yield that $\varphi'(0) < n\varepsilon$ where $n\varepsilon < 1$. We deduce from Lemma 4.5.2 (ii) that

$$V\left(\frac{1}{2}\cdot K +_0 \frac{1}{2}\cdot C\right) = \varphi\left(\frac{1}{2}\right) < 1 + n\varepsilon.$$

Now we apply Theorem 4.1.4, and conclude that for some $m \geq 1$, there exist $\theta_1, \ldots, \theta_m > 0$ and compact convex sets $K_1, \ldots, K_m > 0$ invariant under G such that $\operatorname{lin} K_i$, $i = 1, \ldots, m$, are complementary coordinate subspaces, and

$$K_1 \oplus \ldots \oplus K_m \subset \quad K \quad \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(K_1 \oplus \ldots \oplus K_m) \tag{4.108}$$

$$\theta_1 K_1 \oplus \ldots \oplus \theta_m K_m \subset \quad C \quad \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(\theta_1 K_1 \oplus \ldots \oplus \theta_m K_m) \tag{4.109}$$

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where $c > 1$ is an absolute constant. In turn, we deduce Theorem 4.1.5 under the assumption $n\varepsilon < 1$ on (4.106).

On the other hand, if $n\varepsilon \geq 1$, then Theorem 4.1.5 can be proved as Theorem 4.1.4 under the assumption (4.103). $\square$

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Chapter 5

Stability of solution of the log-Minkowski problem under the symmetries of a Coxeter group acting without fixed points

5.1 Introduction

The main goal in this chapter is to obtain the stability of solution of the logarithmic or L_0 Minkowski problem under symmetry with respect to a Coxeter group acting on $\mathbb{R}^n$ without non-zero fixed points.

Let's recall that for a convex body K containing the origin with support function h_K ($h_K(u) = \max_{x\in K}\langle u, x\rangle$), and surface area measure S_K, it's cone volume measure is given by

$$dV_K = \frac{1}{n}\, h_K\, dS_K.$$

And the total measure then is

$$V_K(S^{n-1}) = V(K).$$

The Monge-Ampère equation on the sphere S^{n-1} corresponding to the logarithmic (or L_0-) Minkowski problem is

$$h \det(\nabla^2 h + h\,\mathrm{Id}) = nf \tag{5.1}$$

where ∇h and $\nabla^2 h$ are the gradient and the Hessian of h with respect to a moving orthonormal frame. For a given finite Borel measure μ on S^{n-1}, a positive h on S^{n-1}

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that is the restriction of a convex homogeneous function on $\mathbb{R}^n$ is the solution of (5.1) in the Alexandrov sense if the corresponding Monge-Ampère measure satisfies

$$\det(\nabla^2 h + h\,\mathrm{Id})\,d\sigma = \frac{n}{h}\cdot d\mu \tag{5.2}$$

where σ is the Lebesgue measure on S^{n-1}.

We note that the Monge-Ampère equation (5.1) is homogeneous in the sense that replacing f by λf for $\lambda > 0$ is equivalent to replacing h by $\lambda^{1/(n+1)}h$. Therefore, we may assume that $V_K(S^{n-1}) = V(K) = 1$; or in other words, the f in (5.1) is a probability density, or the measure μ in (5.2) is a probability measure.

For any group $G \subset O(n)$ acting on $\mathbb{R}^n$ without non-zero fixed points, there exist only finitely many G invariant linear subspaces of $\mathbb{R}^n$ where G is a Coxeter group if it is generated by reflections through n independent hyperplanes. Böröczky, Kalantzopoulos [38] established the following characterization of cone-volume measures under hyperplane symmetry assumption.

THEOREM 5.1.1 (Böröczky, Kalantzopoulos)**.** *Let $G \subset O(n)$ be a Coxeter group acting on $\mathbb{R}^n$ without non-zero fixed points. For a finite non-trivial Borel measure μ on S^{n-1} invariant under G, there exists a G invariant Alexandrov solution of the logarithmic Minkowski equation (5.2) if and only if*

(i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n}\cdot \mu(S^{n-1})$ *for any G-invariant proper linear subspace L;*

(ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n}\cdot \mu(S^{n-1})$ *in (i) for an invariant proper linear subspace L is equivalent to* $\operatorname{supp}\mu \subset L \cup L^\perp$.

In addition, if strict inequality holds in (i) for each G-invariant proper linear subspace L, then the G invariant solution is unique.

We note that the measure in Theorem 5.1.1 may not be even; for example, possibly $\mu = V_K$ for a regular simplex K whose centroid is the origin.

For compact convex sets M and N, we write $M \oplus N$ to denote $M + N$ if $\langle x, y\rangle = 0$ holds for $x \in M$ and $y \in N$. In addition, we say that a linear subspace L of $\mathbb{R}^n$ is proper if $1 \leq \dim L \leq n-1$. We note that [38] proved that $V_K(L \cap S^{n-1}) = \frac{\dim L}{n}\cdot V(K)$ holds in Theorem 5.1.1 (i) for a proper invariant subspace L if and only if $K = (K \cap L) \oplus (K \cap L^\perp)$.

According to [38], $V_K = V_C$ holds for convex bodies K and C in $\mathbb{R}^n$ invariant under a Coxeter group $G \subset O(n)$ acting on $\mathbb{R}^n$ without non-zero fixed points if and only if

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$V(K) = V(C)$, and $K = K_1 \oplus \ldots \oplus K_m$ and $C = C_1 \oplus \ldots \oplus C_m$ for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one and invariant under G where K_i and C_i are dilates for $i = 1, \ldots, m$. Naturally, if $m = 1$, then $K = C$.

In order to prepare for the stability version Theorem 5.1.2 of Theorem 5.1.1, for any compact $X \subset S^{n-1}$ and $\varrho \in [0, 2]$, we consider the tube

$$\Psi(X, \varrho) = \{u \in S^{n-1} : \exists x \in X, \ \|x - u\| \leq \varrho\}.$$

The cone volume measure V_K of a convex body K readily satisfies $dV_{tK} = t^n dV_K$ for $t > 0$. Therefore, when comparing the cone volume measures of convex bodies K and C, we may asssume that $V(K) = V(C) = 1$, and hence V_K and V_C are probability measures on S^{n-1}.

One natural distance to consider between two probability measures μ and ν on S^{n-1} is the l_1 Wasserstein distance. First, we consider the family of Lipschitz functions on S^{n-1}; namely, for $\theta > 0$, let

$$\mathrm{Lip}_\theta = \Big\{f : S^{n-1} \to \mathbb{R} : \forall a, b \in S^{n-1}, \ |f(a) - f(b)| \leq \theta \|a - b\|\Big\}. \tag{5.3}$$

Now the Wasserstein distance of the Borel probability measures μ and ν on S^{n-1} is

$$d_W(\mu, \nu) = \sup \left\{ \int_{S^{n-1}} f \, d\mu - \int_{S^{n-1}} f \, d\nu : \ f \in \mathrm{Lip}_1 \right\}.$$

It is known that convergence of a sequence of probability measures with respect to the Wasserstein distance is equivalent to weak convergence.

We note that as $\mu(S^{n-1}) = \nu(S^{n-1})$ in the definition of $d_W(\mu, \nu)$, we may assume that $\min f = -1$; therefore, $f \in \mathrm{Lip}_1$ implies that

$$\|f\|_\infty = \max_{u \in S^{n-1}} |f(u)| \leq 1. \tag{5.4}$$

In turn, we observe that if $d\mu(u) = \varphi(u) \, du$ and $d\nu(u) = \psi(u) \, du$, then

$$d_W(\mu, \nu) \leq \int_{S^{n-1}} |\varphi(u) - \psi(u)| \, du. \tag{5.5}$$

THEOREM 5.1.2. *Let $G \subset O(n)$ be a Coxeter group acting on $\mathbb{R}^n$ without non-zero fixed points. If μ_1 and μ_2 are G-invariant Borel probability measures on S^{n-1}, and*

$$\begin{aligned} \mu_1\Big(\Psi(L \cap S^{n-1}, \delta)\Big) &\leq (1 - \tau) \cdot \tfrac{\dim L}{n}, \\ \mu_2\Big(\Psi(L \cap S^{n-1}, \delta)\Big) &\leq (1 - \tau) \cdot \tfrac{\dim L}{n} \end{aligned} \tag{5.6}$$

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for $\delta, \tau \in (0, \frac{1}{2})$ and for any G-invariant proper subspace L, then the unique G invariant Alexandrov solution h_i of the logarithmic Minkowski problem (5.2) for $\mu = \mu_i$, $i = 1, 2$, satisfies

$$\|h_1 - h_2\|_\infty \quad \leq \quad \gamma_0 \cdot d_W(\mu_1, \mu_2)^{\frac{1}{95n}} \tag{5.7}$$

$$r_0 \leq \quad h_1, h_2 \quad \leq R_0 \tag{5.8}$$

where for some absolute constant $c > 1$, we have

- $R_0 = n$, $r_0 = \frac{1}{e}$, $\gamma_0 = c^n$ *and the condition* (5.6) *is irrelevant provided the action of G is irreducible;*
- $R_0 = \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$, $r_0 = \frac{n^{\frac{n}{2}}}{6^n}\left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}$ *and* $\gamma_0 = \frac{c^n}{\tau} \cdot \delta^{\frac{-3n}{\tau}} n^{\frac{12n}{\tau}}$ *provided the action of G is reducible.*

Actually, Theorem 5.1.2 can be extended to the case when $\mu_1(S^{n-1}) \neq \mu_2(S^{n-1})$ (see Corollary 5.1.3). In this case, we need the bounded Lipschitz distance $d_{\mathrm{bL}}(\mu, \nu)$ of two Borel measures μ and ν on S^{n-1} (see Dudley [64]); namely,

$$d_{\mathrm{bL}}(\mu, \nu) = \sup\left\{\int_{S^{n-1}} f\, d\mu - \int_{S^{n-1}} f\, d\nu : \, f \in \mathrm{Lip}_1 \text{ and } \|f\|_\infty \leq 1\right\}.$$

Using the test function constant 1 shows that

$$|\mu(S^{n-1}) - \nu(S^{n-1})| \leq d_{\mathrm{bL}}(\mu, \nu). \tag{5.9}$$

We observe that if $\mu(S^{n-1}) = \nu(S^{n-1}) = 1$, then $d_{\mathrm{bL}}(\mu, \nu) = d_W(\mu, \nu)$. On the other hand, if $\lambda > 0$ and μ is any finite non-trivial Borel measure on S^{n-1}, then

$$d_{\mathrm{bL}}(\mu, \lambda\mu) \leq |\lambda - 1| \cdot \mu(S^{n-1}). \tag{5.10}$$

COROLLARY 5.1.3. *Let $G \subset O(n)$ be a Coxeter group acting on $\mathbb{R}^n$ without non-zero fixed points. If μ_1 and μ_2 are G-invariant finite Borel measures on S^{n-1} satisfying $d_{\mathrm{bL}}(\mu_1, \mu_2) \leq M = \min\{\mu_1(S^{n-1}), \mu_2(S^{n-1})\} > 0$ and*

$$\begin{aligned} \mu_1\Big(\Psi(L \cap S^{n-1}, \delta)\Big) \quad &\leq \quad (1 - \tau) \cdot \frac{\dim L}{n}, \\ \mu_2\Big(\Psi(L \cap S^{n-1}, \delta)\Big) \quad &\leq \quad (1 - \tau) \cdot \frac{\dim L}{n} \end{aligned} \tag{5.11}$$

for $\delta, \tau \in (0, \frac{1}{2})$ and for any G-invariant proper subspace L, then the unique G invariant Alexandrov solution h_i of the logarithmic Minkowski problem (5.2) for $\mu = \mu_i$, $i = 1, 2$, satisfies

$$\|h_1 - h_2\|_\infty \quad \leq \quad \gamma_0 M^{\frac{1}{n}} \cdot d_{\mathrm{bL}}(\mu_1, \mu_2)^{\frac{1}{95n}} \tag{5.12}$$

$$r_0 M^{\frac{1}{n}} \leq \quad h_1, h_2 \quad \leq R_0 M^{\frac{1}{n}} \tag{5.13}$$

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where for some absolute constant $c > 1$, we have

- $R_0 = 2n$, $r_0 = \frac{1}{e}$, $\gamma_0 = c^n$ *if the action of G is irreducible, in which case the condition (5.11) is irrelevant;*
- $R_0 = 2\left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$, $r_0 = \frac{n^{\frac{n}{2}}}{5^n}\left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}$ *and* $\gamma_0 = \frac{c^n}{\tau} \cdot \delta^{\frac{-3n}{\tau}} n^{\frac{12n}{\tau}}$ *if the action of G is reducible.*

We observe that the error term in Theorem 5.1.2 in terms of ε is not far from being optimal. We provide an unconditional example; namely, when G is generated by the reflections through the coordinate hyperplanes. Let K be the unit cube $K = [-\frac{1}{2}, \frac{1}{2}]^n$, and the unconditional C be obtained from K by chopping off vertices of K using simplices of volume ε and rescaling (to ensure $V(C) = 1$). Then $d_W(V_K, V_C) < \gamma_1 \cdot \varepsilon$, while $(1 - \gamma_2 \varepsilon^{\frac{1}{n}})K \not\subset C$ for suitable $\gamma_1, \gamma_2 > 0$ depending on n.

The stable solution Theorem 5.1.2 of the logarithmic Minkowski problem under hyperplane symmetry does use the metric structure on S^{n-1}. The next example shows that we can't expect an "affine invariant" stability version of Theorem 5.1.2 even if the cone volume measure is affine invariant in certain sense.

Example 5.1.4. *If $e \in S^{n-1}$, and K and C are any convex bodies in $\mathbb{R}^n$ containing the origin in their interiors with $V(K) = V(C) = 1$ and $V_K(e^\perp \cap S^{n-1}) = V_C(e^\perp \cap S^{n-1}) = 0$, and Φ_s is the diagonal transformation with $\Phi_s(e) = s^{-(n-1)}e$ and $\Phi_s(x) = sx$ for $x \in e^\perp$, then both $V_{\Phi_s K}$ and $V_{\Phi_s C}$ tend weakly to μ_0 as s tends to infinity where μ_0 denotes the probability measure on S^{n-1} with $\mu_0(\{\pm e\}) = \frac{1}{2}$. In particular, $V_{\Phi_s K}$ and $V_{\Phi_s C}$ are arbitrarily close if s is large.*

Next, we consider two partial converses of Theorem 5.1.2 to show that concerning Theorem 5.1.2, both the conditions involved and the conclusion are of the right kind. The first result does not require any symmetry assumption.

THEOREM 5.1.5. *Let μ_1 and μ_2 be finite Borel measures on S^{n-1} such that there exists Alexandrov solution h_i of the logarithmic Minkowski problem (5.2) for $\mu = \mu_i$ and $i = 1, 2$. If $h_1, h_2 < R$ for $R > 0$, then*

$$d_{\mathrm{bL}}(\mu_1, \mu_2) \leq \gamma(R, n) \cdot \sqrt{\|h_1 - h_2\|_\infty}$$

where $\gamma(R, n) > 0$ depends on R and n.

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Secondly, we show that if we have almost equality in Theorem 5.1.1 (ii) for measures μ_1 and μ_2 and a proper linear subspace L invariant under reflections through independent hyperplanes $H_1, \dots, H_n$, then even if μ_1 and μ_2 are close, it is possible that the solutions h_1 and h_2 of (5.2) are arbitrarily far away.

THEOREM 5.1.6. *Let $G \subset O(n)$ be a group acting without non-zero fixed points on $\mathbb{R}^n$, and let h be a positive G-invariant Alexandrov solution of (5.2) for a probability measure μ on S^{n-1} with $h < R$ for $R > \sqrt{n}$ such that*

$$\mu\Big(\Psi(L \cap S^{n-1}, \delta)\Big) \geq (1-\varepsilon) \cdot \frac{\dim L}{n}$$

for $\varepsilon \in (0, \frac{\varepsilon_0}{R^n})$, $\delta \in (0, \varepsilon]$ and a G-invariant proper subspace L where $\varepsilon_0 > 0$ depends on n. Then for any $t > 1$, there exists a positive G-invariant Alexandrov solution h_t of (5.2) for a probability measure μ_t on S^{n-1} such that

$$\begin{aligned} \|h - h_t\|_\infty &\geq t \\ d_W(\mu, \mu_t) &\leq \gamma(R,n)\varepsilon^{\frac{1}{10n}} \end{aligned}$$

where $\gamma(R,n) > 0$ depends on R and n.

5.2 Bounding the diameter of K in terms of V_K

First we state a simple relation for balls contained in and containing a convex body. Since $\kappa_n = V(B^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, lower and upper bounds for the Γ function on positive reals are often needed. The following version of Stirling's formula due to Artin [8] (3.9) would be useful for us. For any $x \geq 1$, there exists $\theta \in (0,1)$ such that

$$\Gamma(x+1) = \left(\frac{x+1}{e}\right)^x \sqrt{2\pi(x+1)} \cdot e^{-1+\frac{\theta}{12(x+1)}}. \tag{5.14}$$

Note that $t \in (0,1]$, $\log(1+t) < t - \frac{t^2}{2} + \frac{t^3}{3}$. It follows that for $x \geq 1$, $(\frac{x+1}{x})^x e^{\frac{1}{12x}} < e$. Since $(\frac{x+1}{x})^{x+\frac{1}{2}}$ is monotone decreasing, we also have $(\frac{x+1}{x})^{x+\frac{1}{2}} > e$. Then it follows from (5.14) that

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi x} < \Gamma(x+1) < \left(\frac{x}{e}\right)^x \sqrt{2\pi(x+1)}. \tag{5.15}$$

Batir [23], Theorem 1.6 and [24] provides much more precise lower and upper bounds.

LEMMA 5.2.1. *If K is a convex body in $\mathbb{R}^n$ whose centroid is the origin, and $K \subset R\,B^n$ for $R > 0$, then $rB^n \subset K$ for some*

$$r \geq \frac{n^{\frac{n}{2}}}{6^n} \cdot \frac{V(K)}{R^{n-1}}.$$

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Proof: We set $r>0$ be maximal with the property $rB^n \subset K$. Since the origin is the centroid of K, we have $-K \subset nK$, and hence K is contained in a cylinder whose height is $(n+1)r \leq 2nr$ and base is an $(n-1)$-ball of radius R. Therefore,

$$V(K) \leq 2n\kappa_{n-1}R^{n-1}r.$$

As $\Gamma(t+1) > (\frac{t}{e})^t\sqrt{2\pi t}$ for $t \geq 1$ (see (5.15)) and $\kappa_{n-1} < \frac{\sqrt{n+1}}{\sqrt{2\pi}} \cdot \kappa_n$, we have

$$\kappa_{n-1} < \frac{\sqrt{n+1}}{\sqrt{2\pi}} \cdot \kappa_n = \frac{\sqrt{n+1}}{\sqrt{2\pi}} \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} < \frac{\sqrt{n+1}}{\sqrt{2\pi}} \cdot \frac{(2e\pi)^{\frac{n}{2}}}{n^{\frac{n}{2}}\sqrt{\pi n}} < \frac{(2e\pi)^{\frac{n}{2}}}{n^{\frac{n}{2}}\pi}.$$

In turn, we deduce

$$r \geq \frac{V(K)}{2n\kappa_{n-1}R^{n-1}} > \frac{n^{\frac{n}{2}}}{n \cdot (2e\pi)^{\frac{n}{2}}} \cdot \frac{V(K)}{R^{n-1}},$$

completing the proof of Lemma 5.2.1 as $n \cdot (2e\pi)^{\frac{n}{2}} < 6^n$. □

For a convex body K in $\mathbb{R}^n$, we write $R(K)$ to denote the minimal radius of a Euclidean ball containing K, and $r(K)$ to denote the radius of largest ball contained in K. We observe that if the convex body K is invariant under the reflections through the hyperplanes $H_1, \ldots, H_n$ with $H_1 \cap \ldots \cap H_n = \{o\}$, then its centroid is the origin, and

$$r(K)B^n \subset K \subset R(K)\, B^n.$$

For Proposition 5.2.2 and Lemma 5.2.3, $\widetilde{B}$ denotes the Euclidean ball centered at the origin with $V(\widetilde{B}) = 1$.

PROPOSITION 5.2.2. *Let $G \subset O(n)$ be a Coxeter group acting reducibly and without non-zero fixed points on $\mathbb{R}^n$ ($n \geq 2$), and let μ be a G-invariant Borel probability measure on S^{n-1} satisfying*

$$\mu(\Psi(L \cap S^{n-1}, \delta)) < (1-\tau) \cdot \frac{i}{n}$$

for $\delta, \tau \in (0, \frac{1}{2})$, and for any G-invariant linear subspace L of $\mathbb{R}^n$ of dimension i, $i = 1, \ldots, n-1$, and let C be a G-invariant convex body in $\mathbb{R}^n$ with $V(C) = 1$. Then

(i)

$$\int_{S^{n-1}} \log h_C \, d\mu \geq \log \frac{R(C)^\tau \delta}{n^5}, \quad \textit{and}$$

(ii) *if $\int_{S^{n-1}} \log h_C \, d\mu \leq \int_{S^{n-1}} \log h_{\widetilde{B}} \, d\mu$, then*

$$R(C) < \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}} \quad \textit{and} \quad r(C) > \frac{n^{\frac{n}{2}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}.$$

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Proof: Let E be the John ellipsoid of C (i.e. the maximal volume ellipsoid contained in C), and hence E is invariant under G, and

$$E \subset C \subset n\,E. \tag{5.16}$$

Let $L_1, \ldots, L_m$ be the irreducible linear subspaces invariant under G. The symmetries of E yield that there exists a set of principal directions of E that are part of $L_1 \cup \ldots \cup L_m$, and for each L_i there exists $r_i > 0$ such that $E \cap L_i = r_i(B^n \cap L_i)$, $i = 1, \ldots, m$. We may assume that $r_1 \leq \ldots \leq r_m$.

If $m = 1$, then (5.16) yields that $r_1 B^n \subset C \subset n r_1 B^n$; therefore, Proposition 5.2.2 trivially holds. In particular, let

$$m \geq 2.$$

For

$$Q = \mathrm{conv}\{r_i(B^n \cap L_i)\}_{i=1,\ldots,m},$$

E is the Loewner of Q (i.e. minimal volume ellipsoid containing Q), and hence $Q \subset E \subset \sqrt{n}\,Q$, thus (5.16) yields that $Q \subset C \subset n^2 Q$. In particular, writing $d_i = \dim L_i$ for $i = 1, \ldots, m$, $Q \subset C$ satisfies

$$n^n \prod_{i=1}^{m} r_i^{d_i} \geq \prod_{i=1}^{m} r_i^{d_i} \kappa_{d_i} \geq V(Q) \geq n^{-2n} V(C) = n^{-2n} \tag{5.17}$$

where $d_1 + \ldots + d_m = n$. We observe that for any $u \in S^{n-1}$, there exists L_i such that $\|u|L_i\| \geq \frac{1}{\sqrt{m}} > \frac{\delta}{n}$. For $i = 1, \ldots, m$, we define

$$\begin{aligned} \Lambda_i &= L_1 \oplus \ldots \oplus L_i \\ B_i &= \left\{ u \in S^{n-1} : \|u|L_i\| \geq \frac{\delta}{n} \text{ and } \|u|L_j\| < \frac{\delta}{n} \text{ for } j > i \right\}. \end{aligned}$$

It follows that S^{n-1} is partitioned into the Borel sets $B_1, \ldots, B_m$, and as $B_j \subset \Psi(\Lambda_i \cap S^{n-1}, \delta)$ for $1 \leq j \leq i \leq m-1$, we have

$$\mu(B_1) + \ldots + \mu(B_i) \leq \frac{(d_1 + \ldots + d_i)(1-\tau)}{n} \quad \text{for } i = 1, \ldots, m-1 \tag{5.18}$$

$$\mu(B_1) + \ldots + \mu(B_m) = 1. \tag{5.19}$$

For $\zeta = \frac{1-\tau}{n} > \frac{1}{2n}$, next we define

$$\beta_j = \mu(B_j) - d_j \zeta \quad \text{for } j = 1, \ldots, m-1 \tag{5.20}$$

$$\beta_m = \mu(B_m) - d_m \zeta - \tau \tag{5.21}$$

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where (5.18) and (5.19) yield

$$\begin{aligned} \beta_1 + \ldots + \beta_i &\leq 0 \quad \text{for } i = 1, \ldots, m-1 && (5.22)\\ \beta_1 + \ldots + \beta_m &= 0. && (5.23) \end{aligned}$$

It follows from $r_i B^n \cap L_i \subset Q$ and from the definition of B_i that $h_Q(u) \geq r_i \cdot \frac{\delta}{n}$ for $u \in B_i$, $i = 1, \ldots, m$. We deduce from applying (5.17), (5.19), (5.20), (5.21), (5.22), (5.23), $r_1 \leq \ldots \leq r_m$ and $\frac{1}{2n} < \zeta < \frac{1}{n}$ that

$$\begin{aligned} \int_{S^{n-1}} \log h_C \, d\mu &\geq \int_{S^{n-1}} \log h_Q \, d\mu = \sum_{i=1}^{m} \int_{B_i} \log h_Q \, d\mu \\ &\geq \sum_{i=1}^{m} \mu(B_i) \log r_i + \sum_{i=1}^{m} \mu(B_i) \log \frac{\delta}{n} = \sum_{i=1}^{m} \mu(B_i) \log r_i + \log \frac{\delta}{n} \\ &= \sum_{i=1}^{m} \beta_i \log r_i + \sum_{i=1}^{m} \zeta d_i \log r_i + \tau \log r_m + \log \frac{\delta}{n} \\ &\geq \sum_{i=1}^{m} \beta_i \log r_i + \zeta \log \frac{1}{n^{3n}} + \tau \log r_m + \log \frac{\delta}{n} \\ &= (\beta_1 + \ldots + \beta_m) \log r_m + \sum_{i=1}^{m-1} (\beta_1 + \ldots + \beta_i)(\log r_i - \log r_{i+1}) \\ &\quad -3n\zeta \log n + \tau \log r_m + \log \frac{\delta}{n} \\ &\geq -3 \log n + \tau \log r_n + \log \frac{\delta}{n} \end{aligned}$$

where we used $\zeta < \frac{1}{n}$ at the end. Now $r_m = R(E) \geq R(C)/n$ and $\tau < 1$ imply

$$\begin{aligned} -3 \log n + \tau \log r_m + \log \frac{\delta}{n} &\geq -3 \log n + \tau \log \frac{R(C)}{n} + \log \frac{\delta}{n} \\ &\geq -3 \log n + \tau \log R(C) - \log n + \log \delta - \log n \\ &= \log \frac{R(C)^\tau \delta}{n^5}, \end{aligned}$$

proving Proposition 5.2.2 (i).

For (ii), let $\tilde{r}_n$ be the radius of $\widetilde{B}$, and hence $\Gamma(\frac{n}{2}+1) < (\frac{n}{2e})^{\frac{n}{2}} \sqrt{2\pi(\frac{n}{2}+1)} < (\frac{2n}{e})^{\frac{n}{2}}$ (see (5.15)) implies

$$1 = \tilde{r}_n^n \kappa_n = \tilde{r}_n^n \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} > \tilde{r}_n^n \cdot \left(\frac{e\pi}{2n}\right)^{\frac{n}{2}},$$

and hence

$$\tilde{r}_n < \sqrt{\frac{2n}{e\pi}}. \tag{5.24}$$

We deduce from (i) and (5.24) that

$$\log \frac{R(C)^\tau \delta}{n^5} \leq \int_{S^{n-1}} \log h_C \, d\mu \leq \int_{S^{n-1}} \log h_{\widetilde{B}} \, d\mu < \log \sqrt{\frac{2n}{e\pi}},$$

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thus $R(C) < (\frac{n^6}{\delta})^{\frac{1}{\tau}}$.

In turn, the bound for $r(C)$ follows from Lemma 5.2.1, completing the proof of Proposition 5.2.2. $\square$

LEMMA 5.2.3. *Let $G \subset O(n)$ be a Coxeter group acting irreducibly, let μ be a G-invariant Borel probability measure on S^{n-1} and C be a G-invariant convex body in $\mathbb{R}^n$ with $V(C) = 1$ Then*
$$\int_{S^{n-1}} \log h_C \, d\mu \geq -1;$$
$$\frac{1}{e} < r(C) \leq R(C) < n.$$

Proof: As the action of G is irreducible, it follows that the inscribed ball of C is the John ellipsoid; namely, the ellipsoid of maximum volume contained in C. According to Ball [11], $r(C)$ is at least the inradius r_n of the regular simplex of volume one, and hence $n! > (\frac{n}{e})^n \sqrt{2\pi\, n}$ (see (5.15)) yields
$$r(C)^n \geq r_n^n = \frac{n!}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}} > \frac{(\frac{n}{e})^n \sqrt{2\pi\, n}}{2n^{n+\frac{1}{2}}} > \frac{1}{e^n}.$$
On the other hand, as the action of G is irreducible, it follows that the circumscribed ball of C is the Loewner ellipsoid; namely, the ellipsoid of minimum volume containing C. According to Barthe [16] (see also Lutwak, Yang, Zhang [119]), $R(C)$ is at most the inradius R_n of the regular simplex of volume one, and hence $n! < (\frac{n}{e})^n \sqrt{2\pi(n+1)}$ (see (5.15)) yields
$$R(C)^n \leq R_n^n = \frac{n^n \cdot n!}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}} < \frac{n^{\frac{n}{2}} \cdot (\frac{n}{e})^n \sqrt{2\pi(n+1)}}{(n+1)^{\frac{n+1}{2}}} < \frac{n^n \sqrt{2\pi}}{e^n} < n^n.$$
We conclude $\frac{1}{e} < r(C) \leq R(C) < n$.

Finally, $r(C) > \frac{1}{e}$ implies that $\log h_C(u) \geq \log r(C) > -1$ for all $u \in S^{n-1}$. $\square$

For a convex body K with $V(K) = 1$ and hyperplane symmetries the Logarithmic Minkowski Inequality Theorem 5.3.2

For a convex body K with $V(K) = 1$ and hyperplane symmetries, combining Proposition 5.2.2 with the consequence $\int_{S^{n-1}} \log h_K \, dV_K \leq \int_{S^{n-1}} \log h_{\widetilde{B}} \, dV_K$ of the Logarithmic Minkowski Inequality Theorem 5.3.2 or using Lemma 5.2.3 we have the following corollary.

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COROLLARY 5.2.4. *Let $G \subset O(n)$ be a Coxeter group acting without non-zero fixed points on $\mathbb{R}^n$, and let K be a G-invariant convex body in $\mathbb{R}^n$, $(n \geq 2)$, satisfying*

$$V_K(\Psi(L \cap S^{n-1}, \delta)) < (1-\tau) \cdot \frac{d}{n} \cdot V(K)$$

for $\delta, \tau \in (0, \frac{1}{2})$ and for any G-invariant subspace L of dimension d, $d \in \{1, \dots, n-1\}$, then

$$R(K) \quad < \quad \begin{cases} \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}} V(K)^{\frac{1}{n}} & \text{if the action of } G \text{ is reducible;} \\ nV(K)^{\frac{1}{n}} & \text{if the action of } G \text{ is irreducible;} \end{cases}$$

$$r(K) \quad > \quad \begin{cases} \frac{n^{\frac{n}{2}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}} V(K)^{\frac{1}{n}} & \text{if the action of } G \text{ is reducible;} \\ \frac{1}{e} \cdot V(K)^{\frac{1}{n}} & \text{if the action of } G \text{ is irreducible.} \end{cases}$$

Another consequence of Proposition 5.2.2 is a condition yielding that a convex body with hyperplane symmetries is not close to be the direct sum of lower dimensional invariant compact convex sets.

PROPOSITION 5.2.5. *Let $G \subset O(n)$ be a Coxeter group acting reducibly and without non-zero fixed points on $\mathbb{R}^n$ ($n \geq 2$), and let K be a G-invariant convex body in $\mathbb{R}^n$, satisfying*

$$V_K(\Psi(L \cap S^{n-1}, \delta)) < (1-\tau) \cdot \frac{\dim L}{n} \cdot V(K)$$

for $\delta, \tau \in (0, \frac{1}{2})$, and for any proper G-invariant coordinate subspace L, then for

$$\eta = \frac{\delta\tau}{4n} \cdot \frac{n^{\frac{n}{2}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n}{\tau}},$$

we have

$$(1-\eta)\big((L \cap K) \oplus (L^{\perp} \cap K)\big) \not\subset K$$

for any proper G-invariant subspace L.

Proof: We may assume that $V(K) = 1$, and define

$$R_0 = \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$$

$$r_0 = \frac{n^{\frac{n}{2}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}},$$

and hence

$$\eta = \frac{\delta\tau}{4n} \cdot \frac{r_0}{R_0} < \frac{\tau}{4n}, \tag{5.25}$$

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while Proposition 5.2.2 implies that

$$r_0B^n \subset K \subset R_0B^n.$$

We prove Proposition 5.2.5 by contradiction; therefore, we suppose that there exists a coordinate i-subspace L, $1 \le i \le n-1$, such that

$$(1-\eta)\Big((L\cap K)\oplus(L^\perp\cap K)\Big)\subset K. \tag{5.26}$$

We define

$$\Omega_0=\left\{[o,x+y]:\ x\in(1-\eta)\partial(L\cap K)\ \text{ and }\ y\in(1-\eta)\left(1-\frac{\tau}{2n}\right)(L^\perp\cap K)\right\}.$$

In addition, let

$$\begin{aligned}\Omega &= \{z\in K:\ \exists t\in(0,1],\ tz\in\Omega_0\}\\ &= \{z\in K:\ (1-\eta)z\in\Omega_0\}\\ \Xi &= \{u\in S^{n-1}:\ \exists x\in\Omega\cap\partial K,\ h_C(u)=\langle x,u\rangle\}.\end{aligned}$$

We deduce using $\eta<\frac{\tau}{2n}$ that

$$\begin{aligned}V_K(\Xi) &\ge \mathcal{H}^n(\Omega_0)\\ &= \frac{i}{n}\cdot(1-\eta)^i\mathcal{H}^i(L\cap K)\cdot(1-\eta)^{n-i}\left(1-\frac{\tau}{2n}\right)^{n-i}\mathcal{H}^{n-i}(L^\perp\cap K)\\ &> (1-\tau)\frac{i}{n}\cdot\mathcal{H}^i(L\cap K)\cdot\mathcal{H}^{n-i}(L^\perp\cap K)>(1-\tau)\frac{i}{n}.\end{aligned} \tag{5.27}$$

Therefore, we contradict (5.26) by proving

$$\Xi\subset\Psi(L\cap S^{n-1},\delta). \tag{5.28}$$

Let $u\in\Xi$ be an exterior normal at $z\in\partial K$. We observe that

$$u=v\cos\beta+w\sin\beta$$

where $v\in L\cap S^{n-1}$, $w\in L\cup S^{n-1}$ and $\beta=\angle(u,v)\in[0,\frac{\pi}{2})$. We write $z=x+y$ for $x\in L\cap K$ and $y\in L^\perp\cap K$. As $z\in\Xi$, we have

$$\begin{aligned}(1-\eta)x+(1-\eta)y &= (1-\eta)z\in\Omega_0\\ &\in (1-\eta)(L\cap K)+(1-\eta)\left(1-\frac{\tau}{2n}\right)(L^\perp\cap K).\end{aligned}$$

In turn, we deduce that

$$y\in\left(1-\frac{\tau}{2n}\right)(L^\perp\cap K). \tag{5.29}$$

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Let
$$p = (1-\eta)x + y + \frac{\tau}{4n} \cdot r_0 w,$$
which, using (5.29), $r_0 B^n \subset K$, (5.25) and (5.26) satisfies
$$\begin{aligned} p &\in (1-\eta)(L \cap K) + \left(1 - \frac{\tau}{2n}\right)(L^\perp \cap K) + \frac{\tau}{4n} \cdot (L^\perp \cap K) \\ &= (1-\eta)(L \cap K) + \left(1 - \frac{\tau}{4n}\right)(L^\perp \cap K) \\ &\subset (1-\eta)(L \cap K) + (1-\eta)(L^\perp \cap K) \subset K. \end{aligned}$$
Since u is exterior normal at $z = x + y$ where $w \in L^\perp \cap S^{n-1}$, $v \in L \cap S^{n-1}$ and $x \in L \cap R_0 B^n$, we have
$$\begin{aligned} 0 &\geq \langle u, p - z \rangle = \left\langle u, \frac{\tau r_0}{4n} \cdot w - \eta x \right\rangle \\ &= \left\langle v \cos\beta + w \sin\beta, \frac{\tau r_0}{4n} \cdot w - \eta x \right\rangle = \frac{\tau r_0}{4n} \cdot \sin\beta - \langle v, x \rangle \eta \cos\beta \\ &\geq \frac{\tau r_0}{4n} \cdot \sin\beta - R_0 \eta \cos\beta. \end{aligned}$$
We conclude that
$$\|u - v\| \leq \tan\beta \leq \frac{4n\eta}{\tau} \cdot \frac{R_0}{r_0} \leq \delta,$$
which in turn, yields (5.28) and contradicts (5.26), proving Proposition 5.2.5. $\square$

5.3 The logarithmic Minkowski conjecture

In this section, we recall the logarithmic Minkowski conjecture. Here, of particular interest to us is the case of convex bodies with n independent hyperplane symmetries in which case the conjecture has been verified by Böröczky, Kalantzopoulos [38], and even a stability version has been established by Böröczky, De [33] (see Chapter 4).

For origin symmetric convex bodies, the logarithmic Brunn-Minkowski conjecture is equivalent to the following logarithmic Minkowski conjecture (see Böröczky, Lutwak, Yang, Zhang [39]).

Conjecture 5.3.1 (Logarithmic Minkowski conjecture)**.** *If K and C are convex bodies in $\mathbb{R}^n$ whose centroid is the origin, then*
$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \, dV_K \geq \frac{V(K)}{n} \log \frac{V(C)}{V(K)} \tag{5.30}$$
with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$.

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According to Böröczky, Lutwak, Yang, Zhang [40], uniqueness of the solution of the logarithmic-Minkowski problem (5.1) for any positive even C^∞ f is equivalent to saying that the Logarithmic Minkowski conjecture (5.30) holds for any o-symmetric convex bodies K and C with C^∞_+ boundaries with equality if and only if K and C are dilates.

In $\mathbb{R}^2$, Böröczky, Lutwak, Yang, Zhang [39] verified Conjecture 5.3.1 for origin symmetric convex bodies, but the general case remains open. In higher dimensions, Conjecture 5.3.1 is proved in the case of convex bodies with n independent hyperplane symmetries (*cf.* Theorem 5.3.2) and for complex bodies (*cf.* Rotem [136]).

Conjecture 5.3.1 has been verified for origin symmetric convex bodies in the case when K is close to being an ellipsoid by a combination of the local estimates by Kolesnikov, Milman [106] and the use of the continuity method in PDE by Chen, Huang, Li, Liu [48]. Putterman [133] provides a more recent proof of the same result using Alexandrov's approach of considering the Hilbert-Brunn-Minkowski operator for polytopes. Kolesnikov, Livshyts [105] and Hosle, Kolesnikov, Livshyts [95] provide other local versions of Conjecture 5.3.1.

Following the result on unconditional convex bodies by Saroglou [137], Böröczky, Kalantzopoulos [38] verified the logarithmic Minkowski conjecture for convex bodies with n independent hyperplane symmetries.

THEOREM 5.3.2 (Böröczky, Kalantzopoulos)**.** *If the convex bodies K and C in $\mathbb{R}^n$ are invariant under linear reflections $A_1, \ldots, A_n$ through n independent linear $(n-1)$-planes $H_1, \ldots, H_n$, then*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K \geq \frac{V(K)}{n} \log \frac{V(C)}{V(K)},$$

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one and invariant under $A_1, \ldots, A_n$ where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$.

Further, Böröczky, De [33] (see Chapter 4) proved the following stability version of the logarithmic-Minkowski inequality Theorem 5.3.2 for convex bodies with many hyperplane symmetries. We will make use of this stability version in the proof of Theorem 5.1.2.

THEOREM 5.3.3. *If the convex bodies K and C in $\mathbb{R}^n$ are invariant under the*

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Coxeter group $G \subset O(n)$ acting without non-zero fixed points on $\mathbb{R}^n$, and

$$\frac{1}{V(K)} \cdot \int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K \leq \frac{1}{n} \cdot \log \frac{V(C)}{V(K)} + \varepsilon$$

for $\varepsilon > 0$, then for some $m \geq 1$, there exist G-invariant compact convex sets $K_1, C_1, \ldots, K_m, C_m$ of dimension at least one, where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$ such that

$$\begin{array}{rcl} K_1 + \ldots + K_m \subset & K & \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(K_1 + \ldots + K_m) \\ C_1 + \ldots + C_m \subset & C & \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(C_1 + \ldots + C_m) \end{array}$$

where $c > 1$ is an absolute constant.

Ivaki [99], Theorem 2.1 provides an improved version of Theorem 5.3.3 with an error term of the order of $\varepsilon^{\frac{1}{n+1}}$ instead of $\varepsilon^{\frac{1}{95n}}$ for the case when K is a ball centered at the origin (and hence $m = 1$), and in that case C does not need to satisfy any symmetry assumption (only translated in a suitable way).

5.4 Proof of Theorem 5.1.2

For compact convex sets K and C in $\mathbb{R}^n$, their Hausdorff distance is

$$d_\infty(K, C) = \|h_K - h_C\|_\infty = \min\{r \geq 0 : \ K \subset C + r\, B^n \text{ and } C \subset K + r\, B^n\}.$$

We prove Theorem 5.1.2 in the following form.

THEOREM 5.4.1. *Let $G \subset O(n)$ be a Coxeter group acting without non-zero fixed points on $\mathbb{R}^n$. If K and C are G-invariant convex bodies in $\mathbb{R}^n$ with $V(K) = V(C) = 1$ satisfying*

$$\begin{array}{rcl} V_K\Big(\Psi(L \cap S^{n-1}, \delta)\Big) & \leq & (1 - \tau) \cdot \frac{\dim L}{n}, \\ V_C\Big(\Psi(L \cap S^{n-1}, \delta)\Big) & \leq & (1 - \tau) \cdot \frac{\dim L}{n} \end{array} \tag{5.31}$$

for $\delta, \tau \in (0, \frac{1}{2})$ and for any G-invariant proper subspace L, then

$$r_0 < \quad h_K,\ h_C \quad < R_0; \tag{5.32}$$

$$d_\infty(K, C) \quad \leq \quad \gamma_0 \cdot d_W(V_K, V_C)^{\frac{1}{95n}} \tag{5.33}$$

where for some absolute constant $c > 1$, we have

- *$R_0 = n$, $r_0 = \frac{1}{e}$ and $\gamma_0 = c^n$ if the action of G is irreducible (and hence the condition (5.31) is irrelevant);*

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- $R_0 = \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$, $r_0 = \frac{n^{\frac{n}{2}}}{6^n}\left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}$ *and* $\gamma_0 = \frac{c^n}{\tau} \cdot \delta^{\frac{-3n}{\tau}} n^{\frac{12n}{\tau}}$ *if the action of* G *is reducible.*

We will use the simple statements Lemma 5.4.2, (5.34) and (5.35).

LEMMA 5.4.2. *If* K *is a convex body with* $K \subset R\,B^n$ *for* $R > 0$*, then for* $u, v \in S^{n-1}$*,* $|h_K(u) - h_K(v)| \le R\|u - v\|$.

Proof: Let $x_0 \in \partial K$ be the boundary point where $u \in S^{n-1}$ is an exterior normal, that is, $h_K(u) = \langle u, x_0\rangle$. Since $K \subset RB^n$, $\|x_0\| \le R$. Then we have for for any $v \in S^{n-1}$,

$$h_K(u) - h_K(v) \le \langle u, x_0\rangle - \langle v, x_0\rangle = \langle u - v, x_0\rangle \le \|u - v\| \cdot \|x_0\| \le \|u - v\| \cdot R.$$

A similar argument shows that $h_K(v) - h_K(u) \le \|v - u\| \cdot R$, and the lemma follows. $\square$

Let μ, ν be Borel probability measures on S^{n-1} and $f : S^{n-1} \to \mathbb{R}$ and $\theta > 0$ satisfy that $|f(u) - f(v)| \le \theta\|u - v\|$ for $u, v \in S^{n-1}$. That is, $f \in \mathrm{Lip}_\theta$. Then $\frac{1}{\theta} \cdot f \in \mathrm{Lip}_1$ and it follows from $|\int_{S^{n-1}} \frac{1}{\theta} f d\mu - \int_{S^{n-1}} \frac{1}{\theta} f d\nu| \le d_W(\mu, \nu)$ that

$$\left|\int_{S^{n-1}} f\,d\mu - \int_{S^{n-1}} f\,d\nu\right| \le \theta \cdot d_W(\mu, \nu). \tag{5.34}$$

If $x \ge y \ge r > 0$, we have from $e^{\frac{y}{r}\left(\frac{x}{y}-1\right)} \ge 1 + \frac{y}{r}\left(\frac{x}{y} - 1\right) \ge \frac{x}{y}$ that $\frac{x-y}{r} \ge \log\frac{x}{y}$, that is,

$$\log x - \log y \le \frac{x - y}{r} \tag{5.35}$$

Proof of Theorem 5.4.1 Let $d_W(V_K, V_C) = \varepsilon$. In order to apply Corollary 5.2.4, we set

$$R_0 = \begin{cases} \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}} & \text{if the action of } G \text{ reducible;} \\ n & \text{if the action of } G \text{ irreducible;} \end{cases}$$

$$r_0 = \begin{cases} \frac{n^{\frac{n}{2}}}{6^n}\left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}} & \text{if the action of } G \text{ reducible;} \\ \frac{1}{e} & \text{if the action of } G \text{ irreducible.} \end{cases}$$

Then Corollary 5.2.4 gives us that

$$r_0 B^n \subset K, C \subset R_0 B^n.$$

That is,

$$r_0 \le h_K, h_C \le R_0. \tag{5.36}$$

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Then for $u, v \in S^{n-1}$, (5.35) and Lemma 5.4.2 imply

$$|\log h_K(u) - \log h_K(v)| \leq \frac{|h_K(u) - h_K(v)|}{r_0} \leq \frac{R_0}{r_0} \cdot \|u - v\| \quad (5.37)$$

$$|\log h_C(u) - \log h_C(v)| \leq \frac{|h_C(u) - h_C(v)|}{r_0} \leq \frac{R_0}{r_0} \cdot \|u - v\| \quad (5.38)$$

where

$$\frac{R_0}{r_0} = \begin{cases} \frac{6^n}{n^{\frac{n}{2}}} \left(\frac{n^6}{\delta}\right)^{\frac{n}{\tau}} & \text{if the action of } G \text{ reducible;} \\ en & \text{if the action of } G \text{ irreducible.} \end{cases} \quad (5.39)$$

From (5.37) and (5.38), we have that $\log h_K, \log h_C \in \mathrm{Lip}_{\frac{R_0}{r_0}}$. The noting that here $d_W(V_K, V_C) = \varepsilon$, and using first (5.34), and then the Logarithmic Minkowski Inequality Theorem 5.3.2 and again (5.34), we get

$$\begin{aligned} \int_{S^{n-1}} \log h_C \, dV_K &\leq \int_{S^{n-1}} \log h_C \, dV_C + \frac{R_0}{r_0} \cdot \varepsilon \leq \int_{S^{n-1}} \log h_K \, dV_C + \frac{R_0}{r_0} \cdot \varepsilon \\ &\leq \int_{S^{n-1}} \log h_K \, dV_K + \frac{2R_0}{r_0} \cdot \varepsilon. \end{aligned}$$

It follows from Theorem 5.3.3 that for some $m \geq 1$, there exist $\theta_1, \ldots, \theta_m > 0$ and compact convex sets $K_1, \ldots, K_m > 0$ invariant under G such that $\sum_{i=1}^m \dim K_i = n$ and

$$K_1 \oplus \ldots \oplus K_m \subset K \subset \left(1 + c_0^n \left(\frac{2R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right) (K_1 \oplus \ldots \oplus K_m) \quad (5.40)$$

$$\theta_1 K_1 \oplus \ldots \oplus \theta_m K_m \subset C \subset \left(1 + c_0^n \left(\frac{2R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right) (\theta_1 K_1 \oplus \ldots \oplus \theta_m K_m)$$

where $c_0 > 1$ is an absolute constant.

If the action of G is irreducible, then $m = 1$, and hence

$$\left(1 + c_1^n \left(\frac{R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right)^{-1} K \subset C \subset \left(1 + c_1^n \left(\frac{R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right) K$$

for some absolute constant $c_1 > 1$. In turn, $K, C \subset R_0 B^n$ (*cf.* (5.32)), $R_0 = n$ and $\left(\frac{R_0}{r_0}\right)^{\frac{1}{95n}} < 2$ (*cf.* (5.39)) yield (5.33) as

$$d_\infty(K, C) \leq R_0 \cdot c_1^n \left(\frac{R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}} \leq (2c_1)^n \cdot \varepsilon^{\frac{1}{95n}}.$$

Next, let the action of G be reducible. First, we assume that

$$\varepsilon < c_2^{95n^2} (\delta\tau)^{95n} \left(\frac{\delta}{n^6}\right)^{\frac{96n^2}{\tau}} \quad (5.41)$$

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where $c_2 \in (0,1)$ is a suitably small absolute constant such that if $\varepsilon > 0$ satisfies (5.41), then

$$c_0^n \left(\frac{2R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}} < \frac{\delta\tau}{4n} \cdot \frac{r_0}{R_0} \quad (< 1) \tag{5.42}$$

holds for the c_0 in (5.40) (*cf.* (5.39)). Therefore, on the one hand, we have

$$\left(1 - c_0^n \left(\frac{2R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right) \Big((K \cap L_1) \oplus \ldots \oplus (K \cap L_m)\Big) \subset K$$

for $L_i = \operatorname{lin} K_i$, $i = 1, \ldots, m$, and, on the other hand, we deduce from (5.42) and Proposition 5.2.5 that $m = 1$. In particular,

$$\left(1 - c_3^n \left(\frac{n^6}{\delta}\right)^{\frac{1}{95\tau}} \varepsilon^{\frac{1}{95n}}\right) K \subset C \subset \left(1 + c_3^n \left(\frac{n^6}{\delta}\right)^{\frac{1}{95\tau}} \varepsilon^{\frac{1}{95n}}\right) K$$

for a suitable absolute constant $c_3 > 1$, and hence $K, C \subset R_0 B^n$ implies

$$d_\infty(K, C) \le R_0 \cdot c_3^n \left(\frac{n^6}{\delta}\right)^{\frac{1}{95\tau}} \varepsilon^{\frac{1}{95n}}.$$

We conclude Theorem 5.4.1 under the condition (5.41).

Finally, we assume that the condition (5.41) does not hold; namely,

$$\varepsilon \ge c_2^{95n^2} (\delta\tau)^{95n} \left(\frac{\delta}{n^6}\right)^{\frac{96n^2}{\tau}}.$$

Since $o \in K, C \subset R_0 B^n$, we have

$$\begin{aligned} d_\infty(K, C) &\le R_0 = \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}} \le c_2^{-n} (\delta\tau)^{-1} \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}(1 + \frac{96n}{95})} \varepsilon^{\frac{1}{95n}} \\ &\le \frac{c_2^{-n}}{\tau} \cdot \delta^{-1} \cdot \left(\frac{n^6}{\delta}\right)^{\frac{2n}{\tau}} \varepsilon^{\frac{1}{95n}} \le \frac{c_2^{-n}}{\tau} \cdot \delta^{\frac{-3n}{\tau}} n^{\frac{12n}{\tau}} \varepsilon^{\frac{1}{95n}}, \end{aligned}$$

proving Theorem 5.4.1. □

Proof of Theorem 5.1.2 According to Theorem 5.1.1, there exist convex bodies K and C invariant under G such that $h_1(u) = h_K(u)$ and $h_2(u) = h_C(u)$ for $u \in S^{n-1}$. In turn, we conclude (5.8) from (5.32), and (5.7) from (5.32) and (5.33). □

After verifying Theorem 5.1.2, we consider the case when $\mu_1(S^{n-1}) \neq \mu_2(S^{n-1})$.

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Proof of Corollary 5.1.3 We may assume that

$$1 = M = \mu_1(S^{n-1}) \leq \mu_2(S^{n-1}).$$

For $\varepsilon = d_{\rm bL}(\mu_1, \mu_2) \leq 1$, it follows from (5.9) that

$$1 \leq \mu_2(S^{n-1}) \leq 1 + \varepsilon. \tag{5.43}$$

We consider the probability measure $\tilde{\mu}_2 = \mu_2(S^{n-1})^{-1} \cdot \mu_2$. Since $d_{bL}(\mu_2, \tilde{\mu}_2) \leq \varepsilon$ by (5.10), the triangle inequality yields $d_{bL}(\mu_1, \tilde{\mu}_2) \leq 2\varepsilon$ by (5.43), and readily

$$\tilde{\mu}_2\Big(\Psi(L \cap S^{n-1}, \delta)\Big) \leq (1-\tau) \cdot \frac{\dim L}{n}$$

for any proper subspace L invariant under G. In addition,

$$\tilde{h}_2 = \mu_2(S^{n-1})^{\frac{-1}{n}} \cdot h_2$$

is the invariant Alexandrov solution of the Logarithmic Minkowski Problem (5.2).

We deduce from Theorem 5.1.2 that

$$\begin{aligned} \|h_1 - \tilde{h}_2\|_\infty \quad &\leq \quad \tilde{\gamma}_0 \cdot (2\varepsilon)^{\frac{1}{95n}} \\ r_0 \leq \quad h_1, \tilde{h}_2 \quad &\leq \widetilde{R}_0 \end{aligned}$$

where for some absolute constant $\tilde{c} > 1$, we have

- $\widetilde{R}_0 = n$, $r_0 = \frac{1}{e}$ and $\tilde{\gamma}_0 = \tilde{c}^n \cdot \varepsilon^{\frac{1}{95n}}$ provided the action of G is irreducible;
- $\widetilde{R}_0 = \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$, $r_0 = \frac{n^{\frac{n}{2}}}{6^n}\left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}$ and $\tilde{\gamma}_0 = \frac{\tilde{c}^n}{\tau} \cdot \delta^{\frac{-3n}{\tau}} n^{\frac{12n}{\tau}}$ provided the action of G is reducible.

Therefore, $h_2 = \mu_2(S^{n-1})^{\frac{1}{n}} \cdot \tilde{h}_2$ and (5.43) imply Corollary 5.1.3 with $c = 2\tilde{c}$ and $R_0 = 2\widetilde{R}_0$. $\square$

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5.5 Partial converses Theorem 5.1.5 and Theorem 5.1.6 of Theorem 5.1.2

In this section, we prove the two partial converses Theorem 5.1.5 and Theorem 5.1.6 of Theorem 5.1.2 by verifying Theorem 5.5.1 and Theorem 5.5.2.

Our argument for Theorem 5.5.1 is based on Hug, Schneider [97], which paper proved that if $R>0$ and K and C are convex bodies in $\mathbb{R}^n$ satisfying $K,C\subset RB^n$, then

$$d_{\rm bL}(S_K,S_C)\leq \tilde{\gamma}(R,n)\cdot\sqrt{d_\infty(K,C)} \tag{5.44}$$

where $\tilde{\gamma}(R,n)>0$ depends on R and n. Theorem 5.1.5 directly follows from the following theorem (see the explanation after (5.2)).

THEOREM 5.5.1. *If K and C are convex bodies in $\mathbb{R}^n$ satisfying $o\in{\rm int}K,{\rm int}C$ and $K,C\subset RB^n$ for $R>0$, then*

$$d_{\rm bL}(V_K,V_C)\leq \gamma(R,n)\cdot\sqrt{d_\infty(K,C)}$$

where $\gamma(R,n)>0$ depends on R and n.

Proof: Let $\varepsilon=d_\infty(K,C)\leq R$. By the symmetry of K and C, it is sufficient to prove that if $f\in{\rm Lip}_1$ with $\|f\|_\infty\leq 1$, then

$$\int_{S^{n-1}}f\,dV_K-\int_{S^{n-1}}f\,dV_C\leq\gamma(R,n)\cdot\sqrt{\varepsilon}$$

where $\gamma(R,n)>0$ depends on R and n, which is equivalent to say that

$$\int_{S^{n-1}}f\cdot h_K\,dS_K-\int_{S^{n-1}}f\cdot h_C\,dS_C\leq n\gamma(R,n)\cdot\sqrt{\varepsilon}. \tag{5.45}$$

It follows from $d_\infty(K,C)\leq\varepsilon$ that

$$h_K\leq h_C+\varepsilon.$$

We deduce from $C\subset R\,B^n$ and Lemma 5.4.2 that $h_C\in{\rm Lip}_R$, and hence $f\cdot h_C\in{\rm Lip}_{2R}$. For $g=\frac{1}{2R}\,f\cdot h_C$, it follows that $g\in{\rm Lip}_1$ and $\|g\|_\infty\leq 1$, thus $\|f\|_\infty\leq 1$, $K\subset R\,B^n$ and the result (5.44) by Hug, Schneider [97] yield

$$\begin{aligned}\int_{S^{n-1}}f\,h_K\,dS_K-\int_{S^{n-1}}f\,h_C\,dS_C &\leq \int_{S^{n-1}}f(h_C+\varepsilon)\,dS_K-\int_{S^{n-1}}f\,h_C\,dS_C\\ &= \varepsilon\cdot\int_{S^{n-1}}f\,dS_K+\\ &\quad 2R\left(\int_{S^{n-1}}g\,dS_K-\int_{S^{n-1}}g\,dS_C\right)\\ &\leq \varepsilon\cdot R^{n-1}n\kappa_n+2R\cdot\tilde{\gamma}(R,n)\cdot\sqrt{\varepsilon}.\end{aligned}$$

We conclude (5.45) from $\varepsilon<2R$, and in turn Theorem 5.5.1. $\square$

Convex bodies whose centroid is the origin and having almost equality in Theorem 5.1.1 (ii) were characterized by Böröczky, Henk [32]. More precisely, if $\varepsilon\in(0,\tilde{\varepsilon}_0)$ and the convex body $K\subset\mathbb{R}^n$ has its centroid at the origin, and satisfies

$$V_K(L\cap S^{n-1})\geq(1-\varepsilon)\cdot\frac{d}{n}\cdot V(K)$$

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for a linear d-space L with $1 \leq d < n$, then

$$(1 - \tilde{\gamma} \cdot \varepsilon^{\frac{1}{5n}})(C + M) \subset K \subset C + M \tag{5.46}$$

for some compact convex set $C \subset L^\perp$, and complementary d-dimensional compact convex set M where $\tilde{\varepsilon}_0, \tilde{\gamma} > 0$ depend on the dimension n.

The paper [32] also verified two observations that we need in the sequel. For a convex body Q in $\mathbb{R}^n$, we write $\sigma(Q)$ to denote the centroid, and $\|x\|_{Q-Q}$ to denote the norm of an $x \in \mathbb{R}^n$ with respect to the origin symmetric convex body $Q - Q$; namely, $\|x\|_{Q-Q} = \min\{t \geq 0 : x \in t(Q - Q)\}$.

For convex bodies $K, \widetilde{K}$ in $\mathbb{R}^n$, writing $K \Delta \widetilde{K}$ to denote the symmetric difference, Lemma 3.4 in [32] says that if $V(K \Delta \widetilde{K}) \leq t\, V(\widetilde{K})$ for $t \in (0, \frac{1}{4^n e})$, then

$$\|\sigma(\widetilde{K}) - \sigma(K)\|_{\widetilde{K}-\widetilde{K}} \leq 4nt. \tag{5.47}$$

The second observation, Lemma 3.3 in [32] states that if $z \in \mathbb{R}^n$, then

$$V(\widetilde{K}\Delta(z + \widetilde{K})) \leq 2n\|z\|_{\widetilde{K}-\widetilde{K}} V(\widetilde{K}). \tag{5.48}$$

The following statement exhibits why we need a condition of the type of (5.31) in Theorem 5.4.1.

THEOREM 5.5.2. *Let K be a convex body in $\mathbb{R}^n$ ($n \geq 2$) with centroid at the origin, $V(K) = 1$, $K \subset RB^n$ (for $R > \sqrt{n}$) satisfying*

$$V_K(\Psi(L \cap S^{n-1}, \delta)) \geq (1 - \varepsilon) \cdot \frac{d}{n}$$

for $\varepsilon \in (0, \frac{\varepsilon_0}{R^n})$, $\delta \in (0, \varepsilon]$ and a proper linear subspace L of dimension d (i.e, $1 \leq d < n$), where $\varepsilon_0 > 0$ depends on n, then

$$d_\infty(K, C + M) \leq \gamma\, R^{n+1} \varepsilon^{\frac{1}{5n}}$$

for some compact convex set $C \subset L^\perp$, and complementary d-dimensional compact convex set M, and a constant $\gamma > 0$ depedning on n.

If, in addition, K and L are G-invariant for a Coxeter group $G \subset O(n)$ acting without non-zero fixed points on $\mathbb{R}^n$, then we may assume that $C = K|L^\perp$ and $M = K|L$.

Proof: We assume that $\varepsilon \in (0, \frac{\varepsilon_0}{R^n})$ where $\varepsilon_0 > 0$ depending on n is small enough to make the argument work.

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We deduce from Lemma 5.2.1 that $r\,B^n \subset K$ for

$$r = \frac{n^{\frac{n}{2}}}{6^n R^{n-1}}.$$

We plan to cut off a rim from K in order to apply (5.46). For

$$\eta = \frac{4 \cdot 6^n R^n}{n^{\frac{n}{2}}} \cdot \delta = \frac{4R}{r} \cdot \delta,$$

we claim that if $u \in S^{n-1}$ is an exterior normal at $x \in \partial K$ with $x|L \in (1-\eta)(K|L)$, then

$$u \notin \Psi(L \cap S^{n-1}, \delta). \tag{5.49}$$

Let $\alpha \in [0, \frac{\pi}{2}]$, $v \in S^{n-1} \cap L$ and $w \in S^{n-1} \cap L^{\perp}$ such that $u|L = v\cos\alpha$ and $u|L^{\perp} = w \sin\alpha$, and hence $u = v\cos\alpha + w\sin\alpha$.

Next let $y \in \partial K$ be such that v is an exterior normal at y. Since $x|L \in (1-\eta)(K|L)$, we have

$$\langle x, v\rangle v \in (1-\eta)(K|L)$$

which gives us

$$\langle \langle x, v\rangle v, v\rangle \leq h_{(1-\eta)K}(v) = (1-\eta)\langle y, v\rangle$$

Then using $h_K \geq r$ yields

$$\begin{aligned}\langle x, v\rangle &\leq (1-\eta)\langle y, v\rangle = \langle y, v\rangle - \eta\langle y, v\rangle \\ &\leq \langle y, v\rangle - \eta r \end{aligned} \tag{5.50}$$

It follows that

$$\begin{aligned} 0 \leq h_K(u) - \langle y, u\rangle &= \langle x, u\rangle - \langle y, u\rangle \\ &= \langle x - y, v\cos\alpha + w\sin\alpha\rangle \\ &= \langle x-y, v\rangle \cos\alpha + \langle x-y, w\rangle \sin\alpha \\ &\leq -\eta r\cos\alpha + \|x-y\|\sin\alpha \\ &\leq -\eta r\cos\alpha + 2R\sin\alpha. \end{aligned}$$

And hence,

$$\tan\alpha \geq \frac{\eta r}{2R} = \frac{4R}{r} \cdot \delta \cdot \frac{r}{2R} = 2\delta.$$

Consider $u_0 \in S^{n-1}$ such that $\|u_0 - v\| = \delta$, and let β be the angle between u_0 and v. Then

$$\|u_0 - v\| = 2\sin\frac{\beta}{2} = \delta,\ \cos\frac{\beta}{2} = \frac{\sqrt{4-\delta^2}}{2}.$$

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Therefore, for $\delta < 1$, we have

$$\begin{aligned}\tan\beta &= \frac{2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}} = \frac{2\cdot\frac{\delta}{2}\cdot\frac{\sqrt{4-\delta^2}}{2}}{\frac{4-\delta^2}{4}-\frac{\delta^2}{4}}\\ &= \frac{\delta\sqrt{4-\delta^2}}{2-\delta^2} < 2\delta.\end{aligned}$$

Then $\tan\alpha > 2\delta > \tan\beta$ gives us

$$u \notin \Psi\left(L\cap S^{n-1},\delta\right)$$

We define

$$\widetilde{K} = \{x \in K : x|L \in (1-\eta)(K|L)\},$$

Note that (5.49) implies all the points of $\partial(1-\eta)K$ that have exterior unit normal in $\Psi(L\cap S^{n-1},\delta)$ lie on the extruding part cutoff by $(1-\eta)\widetilde{K}$ and hence noting that $V(\widetilde{K}) \leq V(K) = 1$, we get

$$\begin{aligned}V_{\widetilde{K}}\left(L\cap S^{n-1}\right) &\geq V_{(1-\eta)K}\left(\Psi\left(L\cap S^{n-1},\delta\right)\right) = (1-\eta)^n V_K\left(\Psi\left(L\cap S^{n-1},\delta\right)\right)\\ &\geq (1-n\eta)(1-\varepsilon)\cdot\frac{d}{n} = (1-\varepsilon-n\eta+n\eta\varepsilon)\cdot\frac{d}{n}\\ &\geq (1-((1-n\eta)\varepsilon+n\eta))\cdot\frac{d}{n}\\ &\geq (1-(\gamma_1\varepsilon+\gamma_1R^n\delta))\cdot\frac{d}{n}\\ &\geq (1-(\gamma_1R^n\varepsilon+\gamma_1R^n\varepsilon))\cdot\frac{d}{n}\\ &\geq (1-2\gamma_1R^n\varepsilon)\cdot\frac{d}{n}\cdot V(\widetilde{K}) \qquad (5.51)\end{aligned}$$

where $\gamma_1 \geq \frac{4.6^n}{n^{\frac{n}{2}}}, 1-n\eta$ depends on n. Here we first derive some estimates. We have

$$\left(\frac{1}{1-\eta}\right) < 1+c\eta$$

for some constant $c > 1$ such that $\eta < 1-\frac{1}{c}$ and $\eta < \frac{1}{c}$. We can choose c depending on ε_0. Now for $c\eta < 1$, we have

$$\begin{aligned}\left(\frac{1}{1-\eta}\right)^n &< (1+c\eta)^n < 1+(2^n-1)c\eta\\ &= 1+(2^n-1)\cdot c\cdot\frac{4\cdot 6^n\cdot R^n}{n^{\frac{n}{2}}}\cdot\varepsilon\\ &= 1+\gamma_2\cdot R^n\varepsilon \qquad (5.52)\end{aligned}$$

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for $\gamma_2 > 0$ depending on n. Since $(1-\eta)K \subset \widetilde{K}$, we have from (5.52),

$$\begin{aligned} V(K\Delta\widetilde{K}) &= V(K) - V(\widetilde{K}) \\ &\leq \left(\left(\frac{1}{1-\eta}\right)^n - 1\right) \cdot V(\widetilde{K}) \\ &\leq \gamma_2 R^n \varepsilon \cdot V(\widetilde{K}) \end{aligned}$$

for $\gamma_2 > 0$ depending on n. According to (5.47) based on [32], the centroid $\sigma(\widetilde{K})$ of $\widetilde{K}$ satisfies

$$\|\sigma(\widetilde{K})\|_{\widetilde{K}-\widetilde{K}} \leq 4n\gamma_2 R^n \cdot \varepsilon; \tag{5.53}$$

It follows from (5.48) based on [32] the convex body $K_0 = \widetilde{K} - \sigma(\widetilde{K})$ satifies that $\sigma(K_0) = o$ and

$$V(K_0 \Delta \widetilde{K}) \leq 8n^2\gamma_2 R^n V(\widetilde{K}) \cdot \varepsilon,$$

and hence

$$\begin{aligned} V_{K_0}\left(L \cap S^{n-1}\right) &\geq V_{\widetilde{K}}\left(L \cap S^{n-1}\right) - V\left(K_0 \Delta \widetilde{K}\right) \\ &\geq (1 - 2\gamma_1 R^n \varepsilon) \cdot \frac{d}{n} V(\widetilde{K}) - 8n^2\gamma_2 R^n \varepsilon \cdot V(\widetilde{K}) \\ &= \left(1 - 2\gamma_1 R^n \varepsilon - 8n^2\gamma_2 R^n \varepsilon \cdot \frac{n}{d}\right) \frac{d}{n} \cdot V(\widetilde{K}) \\ &\geq \left(1 - 2\gamma_1 R^n \varepsilon - 8n^3\gamma_2 R^n \varepsilon\right) \cdot \frac{d}{n} \cdot V(K_0) \\ &= (1 - \gamma_3 R^n \varepsilon) \cdot \frac{d}{n} \cdot V(K_0) \end{aligned}$$

for $\gamma_3 > 0$ depending on n.

We deduce from (5.46) based on [32] and $V(K_0) \leq 1$ that there exist some compact convex set $C_0 \subset L^{\perp}$, and complementary d-dimensional compact convex set M_0 such that

$$(1 - \gamma_4 R^{\frac{1}{5}} \cdot \varepsilon^{\frac{1}{5n}})(C_0 + M_0) \subset K_0 \subset C_0 + M_0 \tag{5.54}$$

where $\gamma_4 > 0$ depends on the dimension n. Note that

$$K_0 + \sigma(\widetilde{K}) \subset K \subset (1-\eta)^{-1}(K_0 + \sigma(\widetilde{K})) \tag{5.55}$$

and $\widetilde{K} - \widetilde{K} = K_0 - K_0 = 2K_0$. Denote $\sigma(\widetilde{K}) = z$ and

$$\|\sigma(\widetilde{K})\|_{\widetilde{K}-\widetilde{K}} = \|z\|_{2K_0} = t_0(\text{ say })$$

Then $z \in 2t_0K_0$. Let $z = 2t_0w_0$ for some $w_0 \in K_0$. For any $x \in K_0$

$$\begin{aligned} &(1-2t_0)\,x + 2t_0(-w) \in K_0 \\ \Rightarrow &(1-2t_0)\,x \in K_0 + 2t_0 w = K_0 + z \end{aligned}$$

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Then from (5.55), we have

$$\begin{aligned}(1-2t_0)\,K_0 \subset K_0+\sigma(\widetilde{K}) \subset K \subset (1-\eta)^{-1}\left(K_0+\sigma(\widetilde{K})\right)\\ \subset (1-\eta)^{-1}\,(K_0+2t_0\cdot K_0)\\ =(1-\eta)^{-1}\,(1+2t_0)\,K_0\\ \subset (1-\eta)^{-1}\,(1+2t_0)\,(C_0+M_0)\end{aligned}$$

that is,

$$(1-2t_0)\,K_0 \subset (1-\eta)^{-1}\,(1+2t_0)\,(C_0+M_0) \tag{5.56}$$

Denoting $\alpha=\gamma_4 R^{\frac{1}{5}}\varepsilon^{\frac{1}{5n}}$, from (5.54), we have

$$\begin{aligned}&(1-\alpha)\,(1-2t_0)\,(C_0+M_0)\subset(1-2t_0)\,K_0\\ \Rightarrow&\left[(1-\alpha)\cdot\frac{(1-2t_0)}{(1+2t_0)}(1-\eta)\right]\cdot(1-\eta)^{-1}\,(1+2t_0)\,(C_0+M_0)\subset(1-2t_0)\,K_0\end{aligned} \tag{5.57}$$

Recall $t_0\le 4n\gamma_2R^n\cdot\varepsilon$ from (5.53). Now, note that

$$\left(\frac{1-2t_0}{1+2t_0}\right)>(1-2t_0)^2>(1-4t_0)$$

and

$$\begin{aligned}(1-4t_0)\,(1-\eta)&=1-4t_0-\eta+4t_0\eta>1-4t_0-\eta\\ &>1-16n\gamma_2R^n\varepsilon-4\gamma_0R^n\varepsilon\\ &=1-\beta\end{aligned} \tag{5.58}$$

where we denoted $\eta=4\gamma_0\cdot R^n\varepsilon$, and $\beta=(16n\gamma_2+4\gamma_0)\,R^n\varepsilon=\gamma_5R^n\varepsilon$ for $\gamma_5>0$ depending on n. Finally, then from (5.58), we have

$$\begin{aligned}(1-\alpha)\frac{(1-2t_0)}{(1+2t_0)}(1-\eta)>(1-\alpha)(1-\beta)>1-\alpha-\beta\\ =1-\left(\gamma_4\cdot R^{\frac{1}{5}}\varepsilon^{\frac{1}{5n}}+\gamma_5\cdot R^n\varepsilon\right)\\ \ge 1-(\gamma_4+\gamma_5)\cdot R^n\varepsilon^{\frac{1}{5n}}\\ =1-\gamma_6\cdot R^n\varepsilon^{\frac{1}{5n}}\end{aligned} \tag{5.59}$$

where $\gamma_6>0$ depends on n. Combining (5.56), (5.57) and (5.59), and denoting $C=(1-\eta)^{-1}\,(1+2t_0)\,C_0$ and $M=(1-\eta)^{-1}\,(1+2t_0)\,M_0$, we have

$$\left(1-\gamma_6R^n\varepsilon^{\frac{1}{5n}}\right)(C+M)\subset K\subset C+M \tag{5.60}$$

for compact convex set $C\subset L^{\perp}$and complementary d-dimensional compact convex set M. From (5.60), denoting $\rho=\gamma_6R^n\varepsilon^{\frac{1}{5n}}$, we further get,

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$$C + M \subset \frac{1}{1-\rho} \cdot K \subset (1 + c_1\rho)\, K$$

for some constant $c_1 > 1$ such that $\rho < 1 - \frac{1}{c_1}$. Then using $K \subset RB^n$, we get

$$C + M \subset K + c_1\rho K \subset K + c_1\rho RB^n.$$

And since $K \subset C + M$, we have

$$d_\infty(K, C + M) \leq c_1\rho R = c_1\gamma_6 R^{n+1} \cdot \varepsilon^{\frac{1}{5n}} = \gamma_7 R^{n+1} \cdot \varepsilon^{\frac{1}{5n}} \tag{5.61}$$

for $\gamma_7 > 0$ depending on n.

Finally, if K is invariant under a group $G \subset O(n)$ leaving L (and hence also $L^\perp$) invariant and acting without fixed point on $L \cap S^{n-1}$, then let $G' \subset O(n)$ be the group whose elements are of the form $\Phi|_L \oplus \mathrm{id}_{L^\perp}$ for $\Phi \in G$ that acts without non-zero fixed point on L, and let $G" \subset O(n)$ be the group whose elements are of the form $\Phi|_{L^\perp} \oplus \mathrm{id}_L$ for $\Phi \in G$ that acts without non-zero fixed point on $L^\perp$. Now for any $x \in K|L^\perp$, the section $K \cap (x + L)$ is invariant under G', and hence the centroid $\sigma(K \cap (x + L))$ of $K \cap (x+L)$ is invariant under G', which in turn yields that $x|L^\perp = \sigma(K \cap (x+L)) \in K$. Therefore, $K|L^\perp = K \cap L^\perp$. Since similar argument implies $K|L = K \cap L$, we may choose $C = K|L^\perp$ and $M = K|L$. $\square$

Proof of Theorem 5.1.6: According to the remarks after (3.2), there exists a convex body K, invariant under G such that $h = h_K$ and $\mu = V_K$. The centroid, $\sigma(K)$ of K is invariant under G. But since G has no nonzero fixed points, it must be that $\sigma(K) = 0$, that is, K has centroid at the origin.

From theorem 5.5.2, we have

$$(1 - \alpha)\,(C_0 + M_0) \subset K \subset C_0 + M_0 \tag{5.62}$$

where $C_0 = K|L^\perp, M_0 = K|L$ and $\alpha = \gamma' R^n \cdot \varepsilon^{\frac{1}{5n}}$ for $\gamma' > 0$ depending on n. Rescaling C_0, M_0 to $C_1 = sC_0$ and $M_1 = sM_0$ such that $V\,(C_1 + M_1) = 1$, we have

$$1 = V\,(sC_0 + sM_0) = s^n V\,(C_0 + M_0) \geq s^n V(K) = s^n$$

which gives us $s \leq 1$. Then from (5.62), we have

$$(1 - \alpha)s\,(C_0 + M_0) \subset sK \subset K$$

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which gives us

$$\begin{aligned} C_1 + M_1 &\subset \frac{1}{1-\alpha} \cdot K \subset (1 + c\alpha)K \\ &\subset K + c\alpha R B^n \end{aligned} \tag{5.63}$$

where $\frac{1}{1-\alpha} < 1 + c\alpha$ for a positive constant $c > 1$ such that $\alpha < 1 - \frac{1}{c}$ (c an be chosen depending on ε_0). Now, note that from (5.62) we have

$$V((1-\alpha)(C_0 + M_0)) \leq V(K) = 1 = V(s(C_0 + M_0))$$

which gives us $1 - \alpha \leq s$, or, $\alpha \geq 1 - s$. Hence,

$$\begin{aligned} K = sK + (1-s)K &\subseteq s\,(C_0 + M_0) + (1-s)K \\ &\subseteq s\,(C_0 + M_0) + \alpha K \\ &\subseteq s\,(C_0 + M_0) + c\alpha R B^n \end{aligned} \tag{5.64}$$

Therefore, from (5.63) and (5.64), we have

$$d_\infty\,(K, C_1 + M_1) \leq c\alpha R = c\gamma' \cdot R^{n+1} \cdot \varepsilon^{\frac{1}{5n}} \tag{5.65}$$

Denoting $Q = C_1 + M_1$, and noting that $K \subset RB^n$ and $C_1 + M_1 \subset K + c\alpha R B^n \subset (1 + c\alpha)RB^n$, we can apply theorem 5.5.1 to get

$$\begin{aligned} d_W\,(V_K, V_Q) &\leq \gamma_1(R,n)\sqrt{d_\infty(K,Q)} \\ &\leq \gamma_1(R,n)\sqrt{c \cdot \gamma' \cdot R^{n+1}} \cdot \varepsilon^{\frac{1}{10n}} \\ &= \gamma(R,n) \cdot \varepsilon^{\frac{1}{10n}} \end{aligned} \tag{5.66}$$

where $\gamma(R,n) = \gamma_1(R,n)\sqrt{c \cdot \gamma' R^{n+1}} > 0$ depends on R and n.

Let $d = \dim L$ and $\rho > 0$ be the radius of the maximal ball of dimension d contained in M_1. Denote

$$a = \frac{t+\rho}{\rho s}, b = \left(\frac{\rho s}{t+\rho}\right)^{\frac{d}{n-d}}$$

Then for any $t > 1$, we have

$$\begin{aligned} d_\infty\,(aM_1, M_0) &= \|ah_{M_1} - h_{M_0}\|_\infty = \|ash_{M_0} - h_{M_0}\|_\infty \\ &= |as - 1| \cdot \|h_{M_0}\|_\infty \\ &\geq \left|\frac{t+\rho}{\rho s} \cdot s - 1\right| \cdot \rho \\ &= \frac{t}{\rho} \cdot \rho = t \end{aligned} \tag{5.67}$$

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Now, let

$$Q_t = aM_1 + bC_1$$

Let $r_1 = d_\infty(K, Q_t)$. That is, $r_1 \geq 0$ is minimal such that

$$\begin{aligned} K &\subset aM_1 + bC_1 + r_1 B^n \\ aM_1 + bC_1 &\subset K + r_1 B^n \end{aligned}$$

Projecting onto L, we have

$$\begin{aligned} M_0 = K|L &\subset aM_1 + r_1 B^d \\ aM_1 &\subset M_0 + r_1 B^d \end{aligned}$$

Then we must have

$$r_1 \geq d_\infty(aM_1, M_0) \geq t \tag{5.68}$$

that is, $d_\infty(K, Q_t) \geq t$. Note also that

$$\begin{aligned} V(Q_t) &= V(aM_1 + bC_1) \\ &= a^d V(M_1) \cdot b^{n-d} V(C_1) \\ &= \left(\frac{t+\rho}{\rho s}\right)^d \cdot \left(\frac{\rho s}{t+\rho}\right)^d \cdot V(M_1 + C_1) \\ &= 1 \\ &= V(Q) \end{aligned}$$

And since $Q_t = aM_1 + bC_1$ and $Q = M_1 + C_1$, we also have

$$V_{Q_t} = V_Q$$

Finally from (5.66) we have

$$d_W(V_K, V_{Q_t}) = d_W(V_K, V_Q) \leq \gamma(R, n)\varepsilon^{\frac{1}{10n}}$$

Therefore, we can choose $h_t = h_{Q_t}$ and $\mu_t = V_{Q_t}$. $\square$

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