CURVATURE-TORSION ALGEBRAS AND THEIR APPLICATIONS

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Abstract

In this dissertation we begin by investigating the problem of characterizing three dimensional Riemannian manifolds with the property that the total scalar curvature of a tube about an arbitrary curve depends only on the radius and length of the tube. The motivation for this problem was to extend earlier results about characterizing harmonic manifolds geometrically through studying the volume of tubes in these manifolds or studying the volume of the intersection of geodesics balls. One of the approaches attempted to solve this problem was to reduce it to an algebraic problem, in the same way that the study of simply connected Lie groups is reduced to the study of Lie algebras. The rest of the dissertation is devoted to developing the theory of how to reduce from the geometric problem to the algebraic problem.

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Sections and Chapters

Contents

1 Introduction:

In differential geometry, the problem of classifying manifolds carrying a geometric structure satisfying certain constraints comes up a lot in the literature. We list some some familiar classes of manifolds carrying a connection or a metric with some special properties. We state the definitions of each of these classes and briefly state what's known about them.

1. A Lie group is a manifold equipped with a group structure such that the group operations are smooth. In fact an application of the inverse function theorem shows that it suffices to only require that the group multiplication is smooth, and it will follow that the group inverse is smooth([1]). The natural question that arises is how to classify Lie groups up to isomorphism. It turns out that in the simply connected case, this question is equivalent to the question of classifying finite dimensional Lie algebras (defined below for convenience) up to isomorphism. We quickly outline the central elements of how this reduction from geometry to algebra happens in the simply connected case:

   Definition 1.1. A Lie algebra over a field F is a vector space L over F equipped with a skew symmetric bilinear map $[,]: L \times L \to L$ that satisfies the Jacobi identity below for all x, y, z:

   $$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

   ([2])

   Definition 1.2. Let G be a lie group and X be a smooth vector field over G. We call X left invariant iff for every $a \in G$ we have $X(a) = DL_a|_e X(e)$ (Where $L_a : G \to G$ is the map of left multiplication by a).

   Fact 1.3. *Let G be a Lie group, then the set of left invariant vector fields when equipped with natural addition,scalar multiplication, and the lie derivative will form a finite dimensional lie algebra (which is called the Lie algebra of G) over $\mathbb{R}$ whose dimension will equal dim(G)([1]).*

   Theorem 1.4 (Lie's First Correspondence)**.** *Let G be a Lie group with Lie algebra g. Let h be a lie subalgebra of g, then there exists a unique Lie subgroup H of G such that the Lie algebra of H coincides with h.([3])([1])*

   Theorem 1.5 (Lie's Second Correspondence)**.** *Let G, H be Lie groups with corresponding Lie algebras g, h. It is given that G is simply connected and that we have a Lie algebra homomorphism $f : g \to h$. Then there exists a unique Lie group homomorphism $F : G \to H$ such that $DF|_e = f$([3])([1])*

   Theorem 1.6 (Lie's Third Correspondence)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$, then there exists a unique simply connected Lie group G whose Lie algebra is $\mathfrak{g}$([3])([1]).*

   For purposes of this thesis, we would like to point out another characterization of simply connected Lie groups that uses a differential equation imposed on a connection as its formulation:

   Theorem 1.7. *A simply connected Lie group can be defined as a simply connected geodesically complete manifold equipped with a flat connection whose Torsion tensor is parallel([4]).*

2. A symmetric space is a riemannian manifold (M, g) with the property that for every $p \in M$, there exists an isometry $s_p : M \to M$ such that $s_p(p) = p$, and $T_p s_p = -Id$. We call the map $s_p : M \to M$ a geodesic involution as it sends $exp_p(v)$ to $exp_p(-v)$. It follows easily from this definition that all symmetric spaces are necessarily geodesically complete and homogeneous. Analogous to Theorem 1.7, we have another characterization of symmetric spaces by imposing a differential equation on the manifold's connection:

Theorem 1.8. *A simply connected geodesically complete riemannian manifold is symmetric iff it's curvature tensor is parallel(5).*

As a matter of fact, there exists an algebraic definition of symmetric spaces that defines symmetric spaces as manifolds equipped with a smooth multiplication operation satisfying some axioms(6). However, we focus on the approach that imposes a differential equation on a connection like Theorem 1.8 for purposes of this thesis.

As in the case of Lie groups, the geometric problem of classification of simply connected symmetric spaces can be reduced to to the algebraic problem of classifying Lie triple systems. This is achieved via the correspondence theorem .

Definition 1.9. A Lie Triple system, as defined in (5)(6)(7), is an $\mathbb{R}-$vector space V equipped with a tri-linear map $R : V \times V \times V \to V$ such that:

1) R is skew symmetric in its leftmost two arguments

2) $R(x, y, z) + R(y, z, x) + R(z, x, y) = 0$ for all $x, y, z \in V$

3)$R(x, y, R(u, v, w)) = R(R(x, y, u), v, w) + R(u, R(x, y, v), w) + R(u, v, R(x, y, w))$, for all $x, y, u, v, w \in V$

Analogous to the Lie correspondence theorem, we have the following correspondence theorem(See (5)(6)) for symmetric spaces:

Theorem 1.10. *Simply connected Symmetric spaces and Lie triple systems are in one to one correspondence*

The complete classification of Symmetric spaces was given by Cartan about a hundred years ago (8)(9)(10).

3. A third class of Riemannian manifolds that was studied in the literature is the class of harmonic manifolds. As usual we are only interested in the geodesically complete and simply connected case. Harmonic manifolds may be defined in several equivalent ways, we present one of these ways below:

Definition 1.11. Let (M, g) be a simply connected complete Riemannian manifold. Denote the volume element of (M, g) by $\overline{d}V$. (M, g) is simply connected and so orientable. Equip M with an orientation. Fix a point $m \in M$ and let $e_1, e_2, \ldots, e_n$ be any positively oriented orthonomal basis for T_mM. Choose $r > 0$ small enough so that $exp_m|B_r(0)$ gives a normal coordinates chart centered at m. Consider a function $\theta_m : B_r(0) \to \mathbb{R}$ defined by $\theta_m(q) = \overline{d}V(Dexp_m|_qe_1, Dexp_m|_qe_2, \ldots, Dexp_m|_qe_n)$. We say (M, g) is harmonic iff the volume density function θ_m is radially symmetric for all $m \in M$. (The definition can be seen easily to be independent of the choice of the orthonormal basis $e_1, e_2, \ldots, e_n$.)

Further equivalent defintions of harmonic manifolds as well as other notions of harmonicity can be found in (11)(12)(13)(14). By using Jacobi fields, one can obtain power series expansion for the Riemannian metric g in normal coordinates , and so compute a power series expansion for the volume density function θ_m in Definition 1.11. The constraint of radial symmetry on the volume density function restricts the coefficients of its power series, thus giving rise to the so called Ledger conditions(15). These are infinite sequence of constraints on the curvature tensor of (M, g) and its higher covariant derivatives $\{\nabla^k R\}_{k\geq 0}$. We list the first few Ledger conditions (L_k) for $k \in \{2, 3, 4, 5\}$ below (ρ will denote the3 Ricci tensor):

$$(L2) : \rho_{xx} = \sum_{a=1}^{n} R_{xaxa} = \lambda x \circ x \tag{1}$$

$$(L3) : \nabla_x \rho_{xx} = 0 \tag{2}$$

$$(L4) : \sum_{a,b=1}^{n} R_{xaxb} R_{xaxb} = \lambda x \circ x \tag{3}$$

$$(L5) : \sum_{a,b=1}^{n} \nabla_x R_{xaxb} R_{xaxb} = 0 \tag{4}$$

For some global constants $\lambda, \mu \in \mathbb{R}$.

Z.I. Szabo proved that all compact simply connected harmonic spaces are rank 1 symmetric spaces, thus settling the famous Lichnerowicz conjecture in the compact case(16). Damek and Ricci constructed their family of "Damek Ricci spaces" that are all harmonic but contain noncompact homogenous harmonic Riemannian manifolds, thus showing that the Lichnerowicz conjecture does not hold in the noncompact case (17). Heber, see (18), then showed that a simply connected homogeneous harmonic space is either flat or symmetric of rank 1 (rank r symmetric space means that the maximal dimension of a flat submanifold is r) or a member of the family of harmonic spaces constructed by Damek and Ricci. While a complete classification of all simply connected harmonic spaces of all dimensions is still an open problem, however Nikolayevsky showed all five dimensional harmonic spaces are of constant curvature after tedious algebraic computations using the Ledger conditions(19).

4. A fourth class of Riemannian manifolds that was studied in the literature is the class of D'Atri spaces(20)(21). As usual we are only interested in the geodesically complete and simply connected case. D'Atri spaces are defined by:

   Definition 1.12. A (simply connected, complete) D'Atri space is a riemannian manifold such that the local geodesic involution at any point p (given by $exp_p(v) \mapsto exp_p(-v)$) is volume preserving.

   Once again by using Jacobi fields to obtain power series expansion for the Riemannian metric in normal coordinates, we get the condition of being "D'Atri" restricts the coefficients of the power series of the volume density function. The resulting constraints are the vanishing of all odd order Ledger conditions mentioned earlier(12). It is easily seen that the class of all simply connected complete harmonic spaces lives inside the class of all simply connected complete D'Atri spaces. Thus, a complete classification of D'Atri spaces is still an open problem as the easier problem of classifying harmonic spaces is open as well. However, a classification of three dimensional harmonic spaces is known(22). Furthermore, a classification of all homogeneous four dimensional D'Atri spaces is also known(23)(24). There exists a strong conjecture proposing that all D'Atri spaces are homogenous (12).

The main point of view that distinguishes modern differential geometry from classical differential geometry is the abstract notion of a manifold as opposed to working with embedded submanifolds in a Euclidean space. The two point of views later turn out to be equivalent as was seen by the embedding theorems of Whitney and then by the embedding theorems of Nash (25)(26). However, the abstract notion of a manifold offers two advantages: Firstly, It forces the mathematician to think about the right notions and ignore any other distracting details. The idea which pops up a lot in mathematics is that the right objects to study are those which remain invariant under the symmetries of the problem. More intuitively, these are the objects which have nothing to do with how the problem is presented. For example, the important ideas in spherical geometry should not be dependent on which parametrization is used to present the metric of the sphere. Secondly, the right objects which the abstract point of view forces the mathematician to focus on also turn out to be shorter to express as formulas. Shorter formulas offer more clarity and thus more understanding if one accepts the philosophy that understanding is a form of data compression.

A usual calculus trick for solving for an unknown function satisfying a differential equation is to use the differential equation to get a recursive formula for the coefficients of the function's power series . A similar technique applies to understanding Riemannian manifolds satisfying a given differential equation on the metric. One can pick a random chart at any point, rewrite the given differential equation of the metric as a partial differential equation on the components of the metric under the chosen chart, finally use the partial differential equation to get an iterative formula for the coefficients of the Taylor series of the metric components. However, doing that would not be optimal for reasons presented in the previous paragraph. A better approach would be to take a natural example of a coordinate system, like normal coordinates around a map, instead of a random chart. This way all attention is devoted to the "right" objects.

The next important observation is that the differential of the exponential map can be expressed using Jacobi fields(27). Jacobi fields satisfy a differential equation (the Jacobi equation) formulated using the curvature tensor. This allows for a recursive formula for the coefficients of the power series of Jacobi fields using the curvature tensor at a point and its higher covariant derivatives. It follows that the pullback of the metric along a suitable restriction of the exponential map has a power series with coefficients expressed using the curvature tensor and its higher covariant derivatives(28). These ideas are what gives rise to the Ledger conditions mentioned earlier.

Kowalski used the Ledger conditions $(L3),(L5)$ in (12) to obtain a classification of three dimensional simply connected complete D'Atri spaces (M,g). We briefly outline the strategy behind his approach as his classification theorem will be used later on in this dissertation. Firstly, the curvature tensor R of a three dimensional Riemannian metric can be expressed using the Ricci tensor ρ. This is partially due to the coincidence that in dimension 3 ,we have:

$$\text{dimension of algebriac curvature tensors on a 3 dimesnional inner produce space} = \frac{3^2(3^2-1)}{12} = 6$$

$$\text{dimension of symmetric bilinear forms on a 3 dimensional inner product space} = \frac{3(3+1)}{2} = 6$$

For every $p \in M$, the ricci tensor at p: $\rho|_p$ will be a symmetric bilinear form on the inner product space T_pM ,and thus can be diagonalized by an orthonormal basis. Hence, one gets orthonormal vector fields (not necessarily smooth) E_1, E_2, E_3 such that $\rho(E_i, E_j) = 0$ for distinct i, j. Next, Kowalski makes use of the following result of K.Sekigawa (29):

Theorem 1.13 (K.Sekigawa)**.** *: Let (M,g) be a three dimensional simply connected and complete Riemannian manifold such that for every $x, y \in M$ there exists an isomorphism of algebraic structures:*

$$\phi : (T_xM, g(x), R|_x, \nabla_R|_x) \to (T_yM, g(y), R|_y, \nabla R|_y)$$

Then (M,g) is homogeneous Riemannian manifold.

For higher dimesnional analouges of Theorem 1.13, refer to (30)(31). K.Sekigawa also classified all Riemannian homognous 3-manifolds. Thus, all one has to do now is show that the ledger conditions $(L3),(L5)$ imply the hypothesis of the above theorem, then computationally check all candidates of Sekigawa's classification for the property of being D'Atri. Since in dimension three the Ricci tensor can be used to recover the curvature tensor, thus to use Sekigawa's theorem it suffices to show that for every $x, y \in M$ we have an isomorphism of algebraic structures $\phi : (T_xM, g(x), \rho|_x, \nabla_\rho|_x) \to (T_yM, g(y), \rho|_y, \nabla\rho|_y)$. To show that two symmetric bilinear forms on two inner product spaces are isomorphic it suffices by diagonalizability to prove that they have the same set of eigenvalues. While the eigenvalues might not be smooth functions on M, however all symmetric polynomials of the eigenvalues $\lambda_i := \rho(E_i, E_i)$ will be polynomials of the coefficients of the characteristic equation of ρ, and hence will be smooth. Thus, Kowalski, begins by focusing attention on two symmetric polynomials of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$: $\tau := \lambda_1 + \lambda_2 + \lambda_3, |\rho|^2 := \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. These polynomials are smooth and so can be differentiated which allows for using the condition $(L3)$. After lots of algebraic manipulations, Kowalski succeeds with using the conditions $(L3), (L5)$ to show that $\rho(E_i, E_i), \nabla\rho(E_i, E_j, E_k)$ are constant on M which allows for the application of Sekigawa's theorem.

This thesis is divided into three chapters:

1. **Chapter 2** In this chapter we recall the definition of tubes around a submanifold of a Riemannian manifolds. Recall the Generalized Gauss lemma. Relate Jacobi fields to the shape operator of a tube, the most important geometric invariant of a submanifold of codimension 1 of a manifold.

2. **Chapter 3** In this chapter we consider the problem of classifying Riemannian 3-manifolds with the tube property, i.e.: manifolds with the property that the total scalar curvature of tubular hypersurface about an arbitrary curve depends only on radius and length. B. Csikos and M.Horvath showed in ([32]) that a Riemannian n-manifold ($n \geq 4$) is harmonic if and only if it has the tube property. Their proof breaks for the case $n = 3$ due to a division by $n - 3$ in one of the steps. The main result we show in this chapter that unlike the higher dimesnional case, a Riemannian 3-manifold has the tube property iff it is D'Atri. This combined with Kowalski's classification completes the classification of 3-manifolds possesing the tube property. The work of this chapter is based on joint work between B.Csikos, M.Horvath, and myself([33]).

3. **Chapter 4** In this chapter we try to extend the Lie group-Lie Algebras correspondence to all manifolds. In order to achive that, we introduce a new algebraic structure, Curvature-Torsion Algebras. Curvature-Torsion Algebras include Lie algebras as a special case. This theorem generalizes an earlier result due to Kowalski and Belger to the non-torsion free case. We also use the methods developed in this chapter to prove some side results like a more general version of the Hausdorff Campbell formula, the analyticity of manifolds possessing the geodesic tube property, and analyticity of Ricci cyclic parallel 3-manifolds. The importance of generalizing Belger and Kowalski's result to the non-torsion free case is to have an abstraction that handles the case of Lie groups and Riemannian manioflds all in one setting.

2 Chapter 2: Background Material on Tubes

In this chapter, we introduce some well known background material about tubes around submanifolds. Further information regarding tubes can be found in Gray's book on Tubes ([34]). Let S be a k dimensional boundaryless submanifold of an n dimensional boundaryless Riemannian manifold (M, g). Let $\pi : TM \to M$ denote the natural projection of the tangent bundle onto its base, ∇ denote the Levi Civita connection of (M, g), and let $exp : TM \to M$ denote the exponential map. For any submanifold A of M, we denote the normal bundle of A by NA. Let's fix some notation for this section. For every $t > 0$, we introduce the notation below:

$$T_\bullet(S,t) = \{\mathbf{v} \in NS \mid \|\mathbf{v}\| \leq t\}.$$

$$T_\circ(S,t) = \{\mathbf{v} \in NS \mid \|\mathbf{v}\| \leq t\}.$$

$$\mathcal{T}_\bullet(S,t) = exp(T_\bullet(S,t)).$$

$$\mathcal{T}_\circ(S,t) = exp(T_\circ(S,t)).$$

Assume throughout the rest of this section that there exists a k dimensional boundaryless submanifold $\overline{S}$ of M such that $S \subseteq \overline{S}$, and such that S is a precompact subspace of $\overline{S}$. $\mathcal{T}_\bullet(S,t)$ should be visualized as the solid tube of radius t around S, while $\mathcal{T}_\circ(S,t)$ should be visualized as the boundary of $\mathcal{T}_\bullet(S,t)$. . This intuition is justified by Fact 2.2. Let's recall one of the versions of the Tubular neighborhood theorem below:

Fact 2.1 (Tubular Neighborhood Theorem)**.** *Let A be a boundary less manifold of (M, g). Then there exists a positive valued smooth map $f : A \to \mathbb{R}$, $\{v \in NA : |v| < f(\pi(v))\}$ is subset of $dom(exp)$ and is mapped diffeomorphically by exp to an open subset of M.*

Proof. Can be found in ([35]),([28]) □

Fact 2.2. *For sufficiently small $t > 0$, we have that $T_\bullet(S,t) \subseteq dom(exp)$. We will also have that $\mathcal{T}_\bullet(S,t)$ is an n dimensional submanifold with boundary of M, and $\mathcal{T}_\circ(S,t)$ is an $n-1$ dimensional boundaryless submanifold of M. Furthermore, $\mathcal{T}_\circ(S,t) = \partial\mathcal{T}_\bullet(S,t)$*

Proof. Use Fact 2.1 to get a smooth map $f : \overline{S} \to \mathbb{R}$ such that exp maps $\{v \in N\overline{S} : |v| < f(\pi(v))\}$ diffeomorphically to an open subset of M. By precompactness of S as a subspace of $\overline{S}$, we can set get $\delta > 0$ such that $f(x) > \delta$ for all $x \in S$. For every $t \in]0, \delta[$ we have $\{v \in NS : |v| < t\}$ is open subset of $\{v \in N\overline{S} : |v| < f(\pi(v))\}$. $\lambda : NS \to$ given by $v \mapsto v \circ v$ is smooth map that has t as a regular value. Thus, by baisic transversality facts (See ([36])) we get $\lambda^{-1}(]-\infty, t])$ is n dimensional submanifold of NS whose boundary equals $\lambda^{-1}(\{t\})$. By choice of δ, we have $\lambda^{-1}(]-\infty, t]) \subseteq \{v \in N\overline{S} : |v| < f(\pi(v))\} \subseteq$ for every $t \in]0, \delta[$. Combining all what we have so far, we get that $T_\bullet(S,t) = \lambda^{-1}(]-\infty, t])$ is an n dimensional manifold living in $dom(exp)$ that will be mapped diffeomorphically by exp to an n submanifold of M. Furthermore:

$$\mathcal{T}_\circ(S,t) = exp(T_\circ(S,t)) = exp(\lambda^{-1}(\{t\}) = exp(\partial\lambda^{-1}(]-\infty,t]) = \partial exp(\lambda^{-1}(]-\infty,t]) = \partial exp(T_\bullet(S,t)) = \partial\mathcal{T}_\bullet(S,t)$$

□

Let $v_0 \in T_\circ(S, 1)$ be arbitrary. Set x_0 to be $\pi(v_0) \in S$. Define the geodesic γ to be given by $t \mapsto exp_{x_0}(tv_0)$.

Using the immersion theorem followed by Gram Schmidt orthogonalization one can get an open subset U of M containing x_0, and get orthonormal smooth vector fields $E_1, E_2, ..., E_n$ with domain U with the property that: For every $x \in S \cap U$, we have that $\{E_j(x)\}_{j\in[k]}$ is an orthonormal basis for T_xS. Assume WLOG that E_{k+1} is selected so that $E_{k+1}(x_0) = v_0$

Consider the map $FER : (S \cap U) \times]-L, L[^{n-k} \to M$ given by

$$(x, t_{k+1}, t_{k+2}, \dots, t_n) \mapsto exp(\sum_{a=k+1}^{n} t_a E_a(x))$$

By selecting the open set U, and the positive number L to be sufficently small, we may assume WLOG that FER gives a diffeomorphism between its domain and an open subset of M. For every $j \in [n]$, we define a smooth vector field C_j locally defined at x_0 ($dom(C_j) = Im(FER)$) and is given by:

$$C_j FER(x, t_{k+1}, t_{k+2}, \dots, t_n) = \begin{cases} T_{(x,t_{k+1},t_{k+2},\dots,t_n)} FER(E_j(x), 0, 0, \dots, 0) & j \le k \\ T_{(x,t_{k+1},t_{k+2},\dots,t_n)} FER(0_x, \overrightarrow{e_{j-k}}) & j \ge k+1 \end{cases}$$

Where 0_x is the zero tangent at the point x in manifold S. One checks easily that for all $x \in S, j \in [n]$,we have that $C_j(x) = E_j(x)$.

The next fact generalizes the well known Gauss lemma Citation ([37]).

Fact 2.3 (Generalized Gauss Lemma). *Let $t > 0$ be sufficently small so that $\mathcal{T}_\circ(S,t)$ is an $n-1$ dimensional submanifold of M, then $T_{\gamma(t)}\mathcal{T}_\circ(S,t) = \{\gamma'(t)\}^\perp$*

Proof. Let $j \in \{k+2, k+3, \dots, n\}$ be arbitrary. Consider a variation through geodesics Γ defined by:

$$\Gamma(\theta, t) = exp_{x_0}(t(cos(\theta)E_{k+1}(x_0) + sin(\theta)E_j(x_0))) = FER(x_0, t(cos(\theta)\overrightarrow{e_1} + sin(\theta)\overrightarrow{e_{j-k}}))$$

Note that $\Gamma(0,t) = \gamma(t)$ for all t. Differentiate to get :

$$\partial_1 \Gamma(\theta, t) = -t sin(\theta) C_{k+1}(\Gamma(\theta,t)) + t cos(\theta) C_j(\Gamma(\theta,t))$$

$$\partial_2 \Gamma(\theta, t) = cos(\theta) C_{k+1}(\Gamma(\theta,t)) + sin(\theta) C_j(\Gamma(\theta,t))$$

Since Γ is a variation through geodesics, thus we get $t \mapsto \partial_1\Gamma(0,t) = tC_j(\gamma(t))$ is a Jacobi field along γ. In fact, it is a normal jacobi field since one can check that the inital conditions are orthogonal to $\gamma'(0)$:

$$[(tC_j(\gamma(t))]_{t=0} \circ \gamma'(0) = 0$$

$$\frac{d}{dt}\Big|_{t=0}[(tC_j(\gamma(t))] \circ \gamma'(0) = C_j(x_0) \circ \gamma'(0) = E_j(x_0) \circ E_{k+1}(x_0) = 0$$

By definition of Γ, we have $\Gamma(\theta,t) \in \mathcal{T}_\circ(S,t)$ for every θ. Thus:

$$tC_j(\gamma(t)) = \partial_1\Gamma(0,t) = \frac{d}{d\theta}\Big|_{\theta=0}\Gamma(\theta,t) \in T_{\gamma(t)}\mathcal{T}_\circ(S,t)$$

. Furthermore $tC_j(\gamma(t)) \in \{\gamma'(t)\}^\perp$, as $t \mapsto tC_j(\gamma(t))$ is normal Jacobi field along γ. Hence:

$$\therefore \bigwedge_{i=k+1}^{n} C_j(\gamma(t)) \in T_{\gamma(t)}\mathcal{T}_\circ(S,t) \cap \{\gamma'(t)\}^\perp \tag{5}$$

Now let $j \in [k]$ be arbitrary. Thus, E_j restricts to a vector field over S. Hence, we may consider the integral curve $\sigma :]-\epsilon, \epsilon[\to S$ of E_j with initial condition $\sigma(0) = x_0$.

Consider a new variation through geodesics Γ defined by $\Gamma(s,t) = exp_{\sigma(s)}(tE_{k+1}(\sigma(s))) = FER(\sigma(s), t\overrightarrow{e_1})$. Note that $\Gamma(0,t) = \gamma(t)$ for all t Differentiate to get that:

$$\partial_1\Gamma(s,t) = C_j(\Gamma(s,t))$$

$$\partial_2\Gamma(s,t) = C_{k+1}(\Gamma(s,t))$$

Since Γ is a variation through geodesics, thus we get $t \mapsto \partial_1\Gamma(0,t) = C_j(\gamma(t))$ is a Jacobi field along γ. In fact, it is a normal Jacobi field since one can check that the initial conditions are orthogonal to $\gamma'(0) = E_{k+1}(x_0)$ (Note: The computations below rely on Torsion freeness of the Levi Civita conenction∇):

$$\Big[(C_j(\gamma(t))\Big]_{t=0} \circ \gamma'(0) = E_j(x_0) \circ E_{k+1}(x_0) = 0$$

$$\frac{d}{dt}\Big|_{t=0}[(C_j(\gamma(t))] \circ \gamma'(0) = \nabla_{\gamma'(0)}C_j \circ \gamma'(0) = [\nabla_{C_{k+1}}C_j \circ C_{k+1}]|_{x_0} = [\nabla_{C_j}C_{k+1} \circ C_{k+1}]|_{x_0} =$$
$$\frac{1}{2} < C_j(x_0), C_{k+1} \circ C_{k+1}) >=< \sigma'(0), C_{k+1} \circ C_{k+1} >= \frac{d}{dt}\Big|_{t=0}[C_{k+1}(\sigma(t)) \circ C_{k+1}(\sigma(t))] =$$
$$\frac{d}{dt}\Big|_{t=0}[E_{k+1}(\sigma(t)) \circ E_{k+1}(\sigma(t))] = \frac{d}{dt}\Big|_{t=0}(1) = 0$$

By definition of Γ, we have $\Gamma(s,t) \in \mathcal{T}_\circ(S,t)$ for every s. Thus:

$$C_j(\gamma(t)) = \partial_1\Gamma(0,t) = \frac{d}{ds}\Big|_{s=0}\Gamma(s,t) \in T_{\gamma(t)}\mathcal{T}_\circ(S,t)$$

Furthermore $C_j(\gamma(t)) \in \{\gamma'(t)\}^\perp$, as $t \mapsto C_j(\gamma(t))$ is normal Jacobi field along γ. Hence:

$$\therefore \bigwedge_{i=1}^{k} C_j(\gamma(t)) \in T_{\gamma(t)}\mathcal{T}_\circ(S,t) \cap \{\gamma'(t)\}^\perp \tag{6}$$

Combine (5) and (6) to get that we have $n-1$ linearly independent vectors $\{C_j(\gamma(t)\}_{j\neq k+1} \in \{\gamma'(t)\}^\perp \cap T_{\gamma(t)}\mathcal{T}_\circ(S,t)$. Since $\{\gamma'(t)\}^\perp, T_{\gamma(t)}\mathcal{T}_\circ(S,t)$ are both $n-1$ dimensional linear spaces, therefore we get $T_{\gamma(t)}\mathcal{T}_\circ(S,t) = \{\gamma'(t)\}^\perp$. □

By retaining the details of the proof of Fact 2.3, we may define Normal Jacobi fields $\{Y_j\}_{j\neq k+1}$ along γ defined by:

$$Y_j(t) = \begin{cases} C_j(\gamma(t)) & j \leq k \\ tC_j(\gamma(t)) & j \geq k+2 \end{cases}$$

For every t, denote the shape operator, second fundamental form of $\mathcal{T}_\circ(S,t)$ by $Sh_t, \mathrm{I\!I}^t$ respectively.

Fact 2.4. *Let $t > 0$ be sufficently small so that $\mathcal{T}_\circ(S,t)$ is an $n-1$ dimensional submanifold of M. Let $j \in [n] - \{k+1\}$ be arbitrary, then we have that $Sh_t(Y_j(t)) = -Y_j'(t)$.*

Proof. Consider the smooth vector field N on $Im(FER) - S$ given by:

$$N(FER(x,\vec{v})) = T_{(x,\vec{v})}FER(0_x, \frac{\vec{v}}{|\vec{v}|}).$$

In the above equation $x \in S \cap U, \vec{v} \in (]-L,L[^{n-k} - \{\overrightarrow{0_{n-k}}\})$ are arbitrary. Using the Fact 2.3, one checks that the restriction of N to $\mathcal{T}_\circ(S,t)$ gives a normal vector field along $\mathcal{T}_\circ(S,t)$. For every $i \in \{k+1, k+2, \dots, n\}$, let $\pi_i : (S \cap U)\times]-L,L[^{n-k} \to \mathbb{R}$ be the smooth map given by $(x, t_{k+1}, t_{k+2}, \dots, t_n) \mapsto t_i$. Let $a \in [k]$ be arbitrary then we have(Note ∇ is torsion free):

$$Sh_t(Y_a(t)) = -\nabla_{Y_a(t)}N = -\nabla_{C_a(\gamma(t))}N =$$

$$= -\nabla_{C_a(\gamma(t))} \sum_{i=k+1}^{n} \frac{\pi_i \circ FER^{-1}}{\sqrt{(\pi_{k+1} \circ FER^{-1})^2 + (\pi_{k+2} \circ FER^{-1})^2 + \dots + (\pi_n \circ FER^{-1})^2}} C_i = -\nabla_{C_a} C_{k+1}|_{\gamma(t)} =$$

$$= -\nabla_{C_a} C_{k+1}|_{\gamma(t)} = -\nabla_{C_{k+1}} C_a|_{\gamma(t)}$$

$$-Y_a'(t) = -[C_a(\gamma(t))]' = -\nabla_{\gamma'(t)} C_a = -\nabla_{C_{k+1}} C_a|_{\gamma(t)}$$

The above two equations give us that $Sh_t(Y_a(t)) = -Y_a'(t)$. Now let $a \in \{k+2, k+3, \dots, n\}$ be arbitrary, then we have(Note ∇ is torsion free):

$$Sh_t(Y_a(t)) = -\nabla_{tC_a(\gamma(t))}N = -t\nabla_{C_a(\gamma(t))}N =$$

$$= -t\nabla_{C_a(\gamma(t))} \sum_{i=k+1}^{n} \frac{\pi_i \circ FER^{-1}}{\sqrt{(\pi_{k+1} \circ FER^{-1})^2 + (\pi_{k+2} \circ FER^{-1})^2 + \dots + (\pi_n \circ FER^{-1})^2}} C_i =$$

$$-[C_a(\gamma(t)) + t\nabla_{C_a} C_{k+1}|_{\gamma(t)}] = -[C_a(\gamma(t)) + t\nabla_{C_{k+1}} C_a|_{\gamma(t)}]$$

$$-Y_a'(t) = -[tC_a(\gamma(t))]' = -[C_a(\gamma(t)) + t\nabla_{\gamma'(t)} C_a] = -[C_a(\gamma(t)) + t\nabla_{C_{k+1}} C_a|_{\gamma(t)}]$$

It follows from the above equations that $Sh_t(Y_a(t)) = -Y_a'(t)$, and we're done. ☐

Fact 2.4 allows one in principle to obtain a power series expansion for the shape operator Sh_t by utilizing Jacobi fields, just like how one uses Jacobi fields to obtain a power series for the components of Riemannian metric in normal coordinates. Once one has a power series for the shape operator of the tube $\mathcal{T}_\circ(S,t)$, it becomes possible to get a power series (in the variable t) expansion for other intrinsic quantities of $\mathcal{T}_\circ(S,t)$ like its volume or its total scalar curvature etc. For full details of the computation was published by L.Gheysens and H.Vanhecke ([38]) .

3 Chapter 3: Manifolds Possessing The Tube Property

By H. Hotelling's theorem ([39]), in the n-dimensional Euclidean or spherical space, the volume of a solid tube of small radius about a curve depends only on the length of the curve and the radius of the tube. A. Gray and L. Vanhecke ([40]) extended Hotelling's theorem to rank one symmetric spaces. B. Csikós and M. Horváth ([41]), ([42]) showed that Hotelling's theorem is true also in harmonic manifolds, and conversely, if a Riemannian manifold has the property that the volume of a solid tube of small radius about a *geodesic segment* depends only on the radius of the tube and the length of the geodesic, then the manifold is harmonic. Using the Steiner-type formula of E. Abbena, A. Gray, and L. Vanhecke ([43]), the above characterization of harmonic spaces provided further similar characterizations of harmonicity in which the condition on the volume of solid tubes is replaced by analogous conditions either on the surface volume, or on the total mean curvature of the tubular hypersurfaces. If the dimension of the manifold is at least 4, harmonicity can also be characterized by an analogous property of the total scalar curvature of the tubular hypersurfaces. It was left open in ([42]) whether the restriction on the dimension is necessary in the case of total scalar curvature. L. Gheysens and L. Vanhecke ([38], p. 193) pointed out that the 3-dimensional case is different. They also posed the question whether vanishing of the total scalar curvature of tubes about curves in a 3-dimensional Riemannian manifold implies that the manifold is harmonic. Recall that a 3-dimensional connected Riemannian manifold is harmonic if and only if it is of constant sectional curvature.

The goal of the present chapter is to fill this gap and characterize 3-dimensional Riemannian manifolds, in which the total scalar curvature of tubular surfaces of small radii about regular curves, or only about geodesic segments depends only on the length of the central curve and the radius of the tube. One of our main theorems, Theorem 3.10 says that a 3-dimensional Riemannian manifold has this property for tubes about arbitrary regular curves if and only if the space is a D'Atri space, furthermore, the total scalar curvature of tubes in a 3-dimensional D'Atri space is constant 0.

Recall that a Riemannian manifold is said to be a D'Atri space if the local geodesic reflection with respect to an arbitrary point is volume-preserving. Every harmonic manifold is a D'Atri space, but the family of D'Atri spaces is strictly larger than that of harmonic manifolds even in dimension 3, as shown by the classification of 3-dimensional D'Atri spaces by O. Kowalski ([12]). In particular, by Theorem 3.10, the answer to the above mentioned question of L. Gheysens and L. Vanhecke is negative.

It is a natural question to ask whether the D'Atri property of a 3-dimensional Riemanian manifold is implied by the weaker assumption that the total scalar curvature of tubular surfaces of small radius about geodesic segments depends only on the length of the geodesic and the radius of the tube. In Theorem 3.11, we show that the answer is yes, if we assume additionally that the manifold is complete and has bounded sectional curvature, for example if it is compact, or homogeneous. However, the following question remains open.

Question 3.1. Can we omit the assumptions on completeness and boundedness of the sectional curvature in Theorem 3.11?

The proof of Theorems 3.10 and 3.11 will be based on Theorem 3.7, which provides some characterizations of D'Atri spaces in terms of the scalar curvature functions of geodesic spheres. It claims, for example, that a Riemannian manifold is a D'Atri space if and only if any two geodesic hemispheres lying on the same geodesic sphere have the same total scalar curvature. There is a strong conjecture proposing that all D'Atri spaces are locally homogeneous. If it is true, then it would complete the classification problem of harmonic manifolds by J. Heber ([44]). The conjecture is true in dimension 3 and is supported by a theorem of P. Günther and F. Prüfer ([45]) claiming that in a D'Atri space, the volume of small balls depends only on the radius, but not on the center. A positive answer to the following question would be a further support to the conjecture and would sharpen Theorem 3.7.

Question 3.2. Do small geodesic spheres of the same radius have equal total scalar curvature in a connected D'Atri space?

3.1 Notations

All manifolds in this chapter are assumed to be smooth, connected, and of dimension at least 3.

Let $(M, \langle\, , \rangle)$ be a Riemannian manifold of dimension n. The symbols ∇, R, ρ, and τ will denote the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature function of M, respectively. For a two-dimensional linear subspace $\sigma \subset T_pM$, the sectional curvature in the direction of σ will be denoted by $K(\sigma)$. Let $\mathring{T}M \subseteq TM$ be the domain of the exponential map $\exp\colon \mathring{T}M \to M$ of M, $\exp_p\colon \mathring{T}_pM \to M$ be the restriction of exp to $\mathring{T}_pM = T_pM \cap \mathring{T}M$. The injectivity radius at p will be denoted by (p).

For $p \in M$ and $r > 0$, we shall denote by $B_p(r) \subset T_pM$ and $S_p(r) \subset T_pM$ the closed ball and the sphere of radius r centered at the origin $\mathbf{0}_p \in T_pM$, respectively. The unit sphere $S_p(1)$ will be denoted simply by S_p. Denote by $SM = \bigcup_{p \in M} S_p \subset TM$ the total space of the unit sphere bundle of the tangent bundle.

Associated to a non-zero tangent vector $\mathbf{v} \in T_pM \setminus \{\mathbf{0}_p\}$, we shall consider the hemisphere

$$S^+(\mathbf{v}) = \{\mathbf{w} \in T_pM \mid \langle \mathbf{w}, \mathbf{v} \rangle \geq 0,\ \|\mathbf{w}\| = \|\mathbf{v}\|\}.$$

When $r < (p)$ and $\|\mathbf{v}\| < (p)$, we can take the exponential images

$$\mathcal{S}_p(r) = \exp(S_p(r)), \qquad\qquad \mathcal{S}^+(\mathbf{v}) = \exp(S^+(\mathbf{v})).$$

The set $\mathcal{S}_p(r)$ is the geodesic sphere of radius r centered at p. Analogously, the set $\mathcal{S}^+(\mathbf{v})$ will be called a *geodesic hemisphere.*

For a smooth regular curve $\gamma\colon [a, b] \to M$ and $r > 0$, set

$$T_\bullet(\gamma, r) = \{\mathbf{v} \in TM \mid \exists t \in [a, b] \text{ such that } \mathbf{v} \in T_{\gamma(t)}M, \mathbf{v} \perp \gamma'(t), \text{ and } \|\mathbf{v}\| \leq r\},$$

and

$$T_\circ(\gamma, r) = \{\mathbf{v} \in TM \mid \exists t \in [a, b] \text{ such that } \mathbf{v} \in T_{\gamma(t)}M, \mathbf{v} \perp \gamma'(t), \text{ and } \|\mathbf{v}\| = r\}.$$

Assume that r is small enough to guarantee that $T_\bullet(\gamma, r) \subset \mathring{T}M$ and the exponential map is an immersion of $T_\bullet(\gamma, r)$ into M. Then we define the *solid tube of radius r about γ* by

$$\mathcal{T}_\bullet(\gamma, r) = exp(T_\bullet(\gamma, r)),$$

while the *tubular hypersurface,* or shortly the *tube of radius r about γ* is defined as

$$\mathcal{T}_\circ(\gamma, r) = exp(T_\circ(\gamma, r)).$$

Tubular hypersurfaces about geodesic segments will be called *cylinders.*

Speaking of geodesic spheres and hemispheres, tubes, and cylinders of small radius r, "small" will always mean that r satisfies the requirements given above in the definition of these geometric shapes.

The scalar curvature of the geodesic sphere $\mathcal{S}_p(r)$ at the point $\exp_p(\mathbf{v})$ for $\mathbf{v} \in S_p(r)$ will be denoted by $\tau^S(\mathbf{v})$.

The *total scalar curvature* of a compact submanifold, possibly with boundary, of a Riemannian manifold is the integral of the scalar curvature function of the submanifold over the submanifold with respect to the volume measure induced by the Riemannian metric. The definition can be extended in an obvious way to immersed submanifolds having self-intersections.

When T is a tensor field of type $(k,0)$ and $Y, X_1, \dots, X_k$ are arbitrary vector fields, then the expression $\nabla_Y T(X_1,\dots,X_k)$ can be understood in two different ways, namely as $\big(\nabla_Y T\big)(X_1,\dots,X_k)$ or as $\nabla_Y\big(T(X_1,\dots,X_k)\big)$. We shall use the convention that whenever the clarifying brackets are missing, $\nabla_Y T(X_1,\dots,X_k)$ should be understood as $\big(\nabla_Y T\big)(X_1,\dots,X_k)$.

3.2 D'Atri spaces and the total scalar curvature of hemispheres

Recall that the *volume density function* $\theta\colon \mathring{T}M \to \mathbb{R}$ is defined by the formula

$$\theta(\mathbf{v}) = \|T_{\mathbf{v}} \exp_p(\mathbf{e}_1) \wedge \cdots \wedge T_{\mathbf{v}} \exp_p(\mathbf{e}_n)\|,$$

where $\mathbf{v} \in \mathring{T}_pM$, $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ is an orthonormal basis of the Euclidean linear space $T_{\mathbf{v}}(T_pM) \cong T_pM$, and $T_{\mathbf{v}} \exp_p$ denotes the derivative map of the exponential map $\exp_p$ at $\mathbf{v}$.

For any given unit tangent vector $\mathbf{u} \in SM$, the coefficients $a_k(\mathbf{u})$ in the Taylor series $\sum_{k=0}^{\infty} a_k(\mathbf{u})r^k$ of the function $\theta(r\mathbf{u})$ can be expressed explicitly in terms of the curvature tensor of M. The initial terms are

$$\theta(r\mathbf{u}) = 1 - \frac{\rho(\mathbf{u},\mathbf{u})}{6}r^2 - \frac{\nabla_{\mathbf{u}}\rho(\mathbf{u},\mathbf{u})}{12}r^3 + O(r^4), \tag{7}$$

see ([46], Cor. 2.4).

Definition 3.3. A Riemannian manifold M is a *D'Atri space* if for any point $p \in M$, the local geodesic symmetry in p is volume-preserving, or equivalently, if for all $p \in M$, there is a ball $B_p(r) \subset \mathring{T}_pM$ such that $\theta(\mathbf{v}) = \theta(-\mathbf{v})$ for all $\mathbf{v} \in B_p(r)$.

The definition implies at once, that in a D'Atri space, all the odd coefficients $a_{2k+1}(\mathbf{u})$ in the Taylor series of the function $\theta(r\mathbf{u})$ must vanish. The identity $a_3(\mathbf{u}) \equiv 0$, also called the third Ledger condition L_3, means that the Ricci tensor of M is cyclic parallel, i.e., it satisfies the identity

$$\nabla_X\rho(Y,Z) + \nabla_Y\rho(Z,X) + \nabla_Z\rho(X,Y) = 0.$$

It was proved by Z. I. Szabó ([47], Ch. 2, Thm. 1.1), that any Riemannian manifold with cyclic parallel Ricci tensor, in particular, every D'Atri space is a real analytic Riemannian manifold, consequently in such manifolds, the function $\theta(r\mathbf{u})$ coincides with the sum of its Taylor series $\sum_{k=0}^{\infty} a_k(\mathbf{u})r^k$ when r is small. This also gives the equivalence of the D'Atri property to the vanishing of all the odd coefficients a_{2k+1}.

The following technical lemma provides a characterization of spaces with cyclic parallel Ricci tensor.

Lemma 3.4. *The following two statements are equivalent for an n-dimensional Riemannian manifold:*

(i) $\nabla_X\rho(X,X) \equiv 0$ *and* $\nabla\tau \equiv 0$.

(ii) $\nabla_X\rho(X,X) + c\nabla_X\tau\|X\|^2 \equiv 0$ *for some constant* $c \neq -2/(n+2)$.

Proof. It is clear that $(i) \Longrightarrow (ii)$, consider the converse. Polarizing (ii) we get

$$\big(\nabla_X\rho(Y,Z) + \nabla_Y\rho(Z,X) + \nabla_Z\rho(X,Y)\big) + c\big(\nabla_X\tau\langle Y,Z\rangle + \nabla_Y\tau\langle Z,X\rangle + \nabla_Z\tau\langle X,Y\rangle\big) \equiv 0.$$

To prove (i) at a particular point $p \in M$, choose an orthonormal frame $E_1, \dots, E_n$ around p, substitute $Y = Z = E_i$ into the above identity and take sum for i. Using the identity $2\operatorname{div}\rho = \nabla\tau$, this gives

$$\begin{aligned} 0 &\equiv \sum_{i=1}^{n} \Big(\nabla_X \rho(E_i, E_i) + \nabla_{E_i}\rho(X, E_i) + \nabla_{E_i}\rho(E_i, X) \Big) + c\Big(n\nabla_X\tau + 2\sum_{i=1}^{n} \nabla_{\langle X, E_i\rangle E_i}\tau \Big) \\ &= \Big(\sum_{i=1}^{n} \nabla_X\rho(E_i, E_i) \Big) + 2\operatorname{div}\rho(X) + c(n+2)\nabla_X\tau = \Big(\sum_{i=1}^{n} \nabla_X\rho(E_i, E_i) \Big) + (c(n+2)+1)\nabla_X\tau. \end{aligned}$$

Introducing the notation $\omega_i^j(X) = \langle \nabla_X E_i, E_j \rangle$ and using the skew symmetry

$$\omega_i^j(X) + \omega_j^i(X) = \nabla_X \langle E_i, E_j \rangle = 0,$$

we also have

$$\nabla_X\tau = \sum_{i=1}^{n} \nabla_X\big(\rho(E_i, E_i)\big) = \sum_{i=1}^{n} \nabla_X\rho(E_i, E_i) + 2\sum_{i,j=1}^{n} \rho(\omega_i^j(X)E_j, E_i) = \sum_{i=1}^{n} \nabla_X\rho(E_i, E_i).$$

Hence $(c(n+2)+2)\nabla_X\tau \equiv 0$, which yields $\nabla\tau = 0$ and (i). $\square$

In the special case when $c = 0$, implication $(ii) \Longrightarrow (i)$ yields an important statement.

Corollary 3.5. The scalar curvature of a connected manifold with cyclic parallel Ricci tensor is constant.

The following consequence of the Steiner-type formula of E. Abbena, A. Gray, and L. Vanhecke ([43]) will play a crucial role in the proof of the main theorem of this section.

Lemma 3.6. *The volume density function, and the scalar curvature of geodesic spheres are related by the formula*

$$\big(\rho(\gamma_{\mathbf{u}}'(r), \gamma_{\mathbf{u}}'(r)) + \tau^S(r\mathbf{u}) - \tau(\gamma_{\mathbf{u}}(r))\big)\theta(r\mathbf{u}) = \partial_r^2\theta(r\mathbf{u}) + 2(n-1)\partial_r\theta(r\mathbf{u})\frac{1}{r} + (n-1)(n-2)\theta(r\mathbf{u})\frac{1}{r^2}, \tag{8}$$

where $\mathbf{u} \in S_pM$ *is an arbitrary unit tangent vector,* $\gamma_{\mathbf{u}}$ *is the unit speed geodesic with initial velocity* $\gamma_{\mathbf{u}}'(0) = \mathbf{u}$, $0 < r < (p)$.

Proof. Choose an arbitrary open subset $\mathcal{U} = \exp_p(rU) \subset \mathcal{S}_p(r)$ of a geodesic sphere, where $U \subset S_p$ is an open subset, and compute the volume $V_{\mathcal{U}}(h)$ of the one-sided parallel domain $\bigcup_{r \le s \le r+h} \exp_p(sU)$ of height h over $\mathcal{U}$ in two different ways for $0 < r < r+h < (p)$. First, computing the volume by integrating the density function θ over the corresponding domain in the tangent space, we obtain

$$\begin{aligned} V_{\mathcal{U}}(h) &= \int_U \int_0^h \theta((r+t)\mathbf{u})(r+t)^{n-1} t\mathbf{u} \\ &= \int_U \int_0^h \Big\{ \theta(r\mathbf{u})r^{n-1} + \partial_r\big(\theta(r\mathbf{u})r^{n-1}\big)t + \partial_r^2\big(\theta(r\mathbf{u})r^{n-1}\big)\frac{t^2}{2} + O(t^3) \Big\} t\mathbf{u} \\ &= \int_U \Big\{ \theta(r\mathbf{u})r^{n-1}h + \big\{ \partial_r\theta(r\mathbf{u})r^{n-1} + (n-1)\theta(r\mathbf{u})r^{n-2} \big\}\frac{h^2}{2} \\ &\qquad + \big\{ \partial_r^2\theta(r\mathbf{u})r^{n-1} + 2(n-1)\partial_r\theta(r\mathbf{u})r^{n-2} + (n-1)(n-2)\theta(r\mathbf{u})r^{n-3} \big\}\frac{h^3}{6} + O(h^4) \Big\} \mathbf{u}. \end{aligned}$$

On the other hand, the Steiner-type formula of E. Abbena, A. Gray, and L. Vanhecke ([43], Thm. 3.5) tells us that

$$V_{\mathcal{U}}(h) = \int_U \Big\{h - H(r\mathbf{u})\frac{h^2}{2} + \big(\rho(\gamma'_{\mathbf{u}}(r), \gamma'_{\mathbf{u}}(r)) + \tau^S(r\mathbf{u}) - \tau(\gamma_{\mathbf{u}}(r))\big)\frac{h^3}{6} + O(h^4)\Big\} r^{n-1}\theta(r\mathbf{u})\mathbf{u},$$

where $H(r\mathbf{u})$ is the trace of the Weingerten map of $\mathcal{S}_p(r)$ at $\gamma_{\mathbf{u}}(r)$ relative to the normal vector $\gamma'_{\mathbf{u}}(r)$.

As the two integrals expressing $V_{\mathcal{U}}(h)$ are equal for any open subset $U \subset S_p$ and any $0 < r < (p)$, the integrands must be equal pointwise. Equating the coefficients of h^3 in the expansions of the integrands yields the desired identity. $\square$

Now we prove the main theorem of this section.

Theorem 3.7. *For a Riemannian manifold $(M, \langle\, , \rangle)$, the following statements are equivalent:*

1. *M is a D'Atri space.*
2. *The product $\tau^S\theta$ is an even function, i.e., $\tau^S(\mathbf{v})\theta(\mathbf{v}) = \tau^S(-\mathbf{v})\theta(-\mathbf{v})$ whenever both sides are defined.*
3. *The total scalar curvatures of any two geodesic hemispheres lying on an arbitrarily given geodesic sphere are equal.*

Proof. First we show the implication 1 $\Longrightarrow$ 2. Expressing the function $r \mapsto \tau^S(r\mathbf{u})$ for an arbitrary fixed unit tangent vector $\mathbf{u} \in SM$ with the help of Lemma 3.6, we obtain

$$\tau^S(r\mathbf{u}) = \frac{\partial_r^2\theta(r\mathbf{u})}{\theta(r\mathbf{u})} + 2(n-1)\frac{\partial_r\theta(r\mathbf{u})}{\theta(r\mathbf{u})}\frac{1}{r} + (n-1)(n-2)\frac{1}{r^2} + \tau(\gamma_{\mathbf{u}}(r)) - \rho(\gamma'_{\mathbf{u}}(r), \gamma'_{\mathbf{u}}(r)). \tag{9}$$

If M is a D'Atri space, then θ is an even function, M has cyclic parallel Ricci tensor, that is $\nabla_X\rho(X, X) \equiv 0$, and the scalar curvature τ of M is constant. Having cyclic parallel Ricci tensor implies that the function $\rho(\gamma'_{\mathbf{u}}, \gamma'_{\mathbf{u}})$ is constant on the domain of $\gamma_{\mathbf{u}}$. Hence the right hand side of (9), and consequently both $\tau^S(r\mathbf{u})$ and $\tau^S(r\mathbf{u})\theta(r\mathbf{u})$ are even functions of r.

To prove that 2 implies 1, we first prove that that 2 implies the Ledger condition L_3. Choose an arbitrary unit tangent vector $\mathbf{u} \in SM$ and consider the functions $\theta(r\mathbf{u})$, $\tau^S(r\mathbf{u})$ and $\tau^S(r\mathbf{u})\theta(r\mathbf{u})$. According to ([46], Thm. 4.4), we have the power expansion

$$\tau^S(r\mathbf{u}) = \frac{(n-1)(n-2)}{r^2} + \left(\tau - \frac{2(n+1)}{3}\rho(\mathbf{u}, \mathbf{u})\right) + \left(\nabla_{\mathbf{u}}\tau - \frac{n+2}{2}\nabla_{\mathbf{u}}\rho(\mathbf{u}, \mathbf{u})\right) r + O(r^2),$$

which, combined with (7), yields

$$\begin{aligned}\tau^S(r\mathbf{u})\theta(r\mathbf{u}) = \frac{(n-1)(n-2)}{r^2} &+ \left(\tau - \frac{n^2+n+6}{6}\rho(\mathbf{u}, \mathbf{u})\right)\\ &+ \left(\nabla_{\mathbf{u}}\tau - \frac{n^2+3n+14}{12}\nabla_{\mathbf{u}}\rho(\mathbf{u}, \mathbf{u})\right) r + O(r^2).\end{aligned}$$

The coefficients of odd powers of r have to vanish in the power expansion of an even function, so if $\tau^S\theta$ is even, then the coefficient $\nabla_{\mathbf{u}}\tau - \frac{n^2+3n+14}{12}\nabla_{\mathbf{u}}\rho(\mathbf{u}, \mathbf{u})$ of r in its expansion must vanish for every $\mathbf{u} \in SM$. By Lemma 3.4, this gives that M satisfies the L_3 condition and has constant scalar curvature, hence $C(\mathbf{u}) = \rho(\gamma'_{\mathbf{u}}(r), \gamma'_{\mathbf{u}}(r)) - \tau(\gamma_{\mathbf{u}}(r))$ is constant on the domain of $\gamma_{\mathbf{u}}$. Another important corollary of the third Ledger condition is that M is a real analytic Riemannian manifold, therefore the functions $\theta(r\mathbf{u})$ and $\tau^S(r\mathbf{u})\theta(r\mathbf{u})$ can be written as the sum of their Laurent series

$$\theta(r\mathbf{u}) = \sum_{k=0}^{\infty} a_k(\mathbf{u})r^k, \qquad \tau^S(r\mathbf{u})\theta(r\mathbf{u}) = \sum_{k=-2}^{\infty} b_k(\mathbf{u})r^k$$

for small values of $r \neq 0$. Substituting these Laurent series into (8) and equating the coefficients of r^k, we obtain the following recursive equation for the coefficients a_k assuming that we are given the coefficients b_k

$$a_{k+2} = \frac{1}{(k+n+1)(k+n)}(Ca_k + b_k).$$

This relation allows us to prove by an easy induction that if $\tau^S\theta$ is an even function, then θ is even as well, i.e., $a_{2k+1} = 0$ for all natural number k. The base case $a_1 = 0$ is automatically fulfilled by (7). Assume $a_{2k-1} = 0$. Then equation

$$a_{2k+1} = \frac{1}{(2k+n)(2k+n-1)}(Ca_{2k-1} + b_{2k-1}) = 0$$

completes the induction step and 1 $\iff$ 2 is proved.

Condition 2 implies 3 in an obvious way, since for any $\mathbf{v} \in S_p(r)$, the total scalar curvature of a hemisphere $\mathcal{S}^+(\mathbf{v})$ is equal to the integral $\int_{S^+(\mathbf{v})} \tau^S(\mathbf{w})\theta(\mathbf{w})\mathbf{w}$, which is exactly half the total scalar curvature of the sphere $\mathcal{S}_p(r)$ if $\tau^S\theta$ is an even function. The converse 3 $\Longrightarrow$ 2 follows from a classical result of harmonic analysis on the sphere, as 3 means that the hemispherical transformation of the restriction of the function $\tau^S\theta$ onto any sphere $S_p(r)$ of small radius r is constant and this implies by ([48], Prop. 3.4.11) that these restrictions are even functions. $\square$

Corollary 3.8. The scalar curvature function τ^S of any geodesic sphere of small radius in a D'Atri space is an even function.

Question 3.9. Assume that τ^S is an even function for a Riemannian manifold. Does it follow that the manifold is a D'Atri space?

3.3 3-dimensional D'Atri spaces and the total scalar curvature of tubes

In this section, we strengthen Theorem 3.7 in the 3-dimensional case. The distinguished role of dimension three is due to the Gauss–Bonnet theorem, controlling the total scalar curvature of surfaces.

Theorem 3.10. *For a 3-dimensional Riemannian manifold $(M, \langle\,,\rangle)$, the following conditions are equivalent:*

1. *M is a D'Atri space.*
2. *The total scalar curvature of any geodesic hemisphere is equal to 4π.*
3. *The total scalar curvature of a tube of small radius about any regular curve is 0.*
4. *The total scalar curvature of a tube of small radius about any regular curve depends only on the length of the curve and the radius of the tube.*

Proof. Theorem 3.7 implies $(ii) \implies (i)$. The total scalar curvature of a geodesic sphere of small radius in M is 8π by the Gauss–Bonnet theorem. If M is a D'Atri space, then by Theorem 3.7, the total scalar curvature of a geodesic hemisphere and its complementary hemisphere are equal, so they are both equal to 4π. Thus, $(i) \implies (ii)$ is proved.

To prove $(ii) \implies (iii)$, consider a tube $\mathcal{T}_\circ(\gamma, r)$ of small radius r about a regular parameterized curve $\gamma\colon [a,b] \to M$. We may assume without loss of generality that γ is of unit speed. The union of the tube $\mathcal{T}_\circ(\gamma, r)$ and the hemispheres $\mathcal{S}^+(-r\gamma'(a))$ and $\mathcal{S}^+(r\gamma'(b))$ is the image of a piecewise smooth $\mathcal{C}^1$-immersion of a "capsule" homeomorphic to a sphere into M so its total scalar curvature is 8π by the Gauss–Bonnet theorem. On the other hand, assumption (ii) impies that the total scalar curvature of the union $\mathcal{S}^+(-r\gamma'(a)) \cup \mathcal{S}^+(r\gamma'(b))$ is also 8π, therefore the total scalar curvature of the tube $\mathcal{T}_\circ(\gamma, r)$ must be 0. Conversely, assume that the total scalar curvature of any tube vanishes. Then computing the total scalar curvature of the immersed capsule constructed above we obtain that the sum of the total scalar curvatures of the geodesic hemisheres $\mathcal{S}^+(-r\gamma'(a))$ and $\mathcal{S}^+(r\gamma'(b))$ equals 8π. Let $\mathbf{u} \in SM$ be an arbitrary unit vector and choose the regular curve γ so that $-\gamma'(a) = \gamma'(b) = \mathbf{u}$. Then $\mathcal{S}^+(-r\gamma'(a)) = \mathcal{S}^+(r\gamma'(b)) = \mathcal{S}^+(r\mathbf{u})$, therefore $\mathcal{S}^+(r\mathbf{u})$ must have total scalar curvature 4π for any small radius r. Thus, $(iii) \implies (ii)$ is proved.

Condition (iv) is obviously weaker than (iii). If condition (iv) holds, then there exists a function $f\colon (0, r_0) \to \mathbb{R}$ such that the total scalar curvature of a tube of small radius r about any regular curve $\gamma\colon [a,b] \to M$ of length l_γ equals $f(r)l_\gamma$. Choosing an arbitrary smoothly closed regular curve γ, the tubes of small radii about γ are immersed tori, so their total scalar curvature vanish by the Gauss–Bonnet theorem. This means that the function f must vanish around 0, hence $(iv) \implies (iii)$. ☐

Theorem 3.11. *Assume that the 3-dimensional Riemannian manifold $(M, \langle\,,\rangle)$ has the property that the total scalar curvature of a cylinder of small radius r about any geodesic segment γ depends only on the radius r and the length of γ.*

1. *Then there is a number $a \in \mathbb{R}$ and a smooth function $b\colon SM \to \mathbb{R}$ such that for any geodesic curve $\gamma_\mathbf{u}$ with initial velocity $\gamma'_\mathbf{u}(0) = \mathbf{u} \in SM$, we have*

$$K(\nu(t)) = at^2 + b(\mathbf{u})t + K(\nu(0)), \tag{10}$$

   where $K(\nu(t))$ is the sectional curvature in the direction of the normal plane $\nu(t) \subset T_{\gamma_\mathbf{u}(t)}M$ of $\gamma_\mathbf{u}$ at $\gamma_\mathbf{u}(t)$.

2. *If we assume also that M is complete and the sectional curvature of M is bounded, (e.g., if M is compact, or homogeneous), then M is a D'Atri space.*

Proof. The initial terms of the power expansion of the total scalar curvature $T_\gamma(r)$ of a tube of small radius r about a unit speed curve $\gamma\colon [a,b] \to \tilde{M}$ were computed explicitely by L. Gheysens and L. Vanhecke ([38], Thm. 5.1) in any n-dimensional Riemannian manifold $\tilde{M}$. Their formula has the form

$$T_\gamma(r) = c_{n-2} r^{n-4} \int_a^b \{(n-3)(n-2) + A(n)r^2 + B(n)r^4 + O(r^6)\}t,$$

where c_{n-2} is the volume of the unit sphere in the $(n-1)$-dimensional Euclidean space,

$$A(n) = -\frac{n-3}{6(n-1)}\{(n-4)\tau + (n+2)\rho_{11}\}(\gamma(t)),$$

$$
\begin{aligned}
B(n) = \frac{1}{n^2-1}\Big\{ & \frac{n^2-9n+2}{72}\tau^2 + \frac{n^2+3n+17}{45}\|\rho\|^2 - \frac{(n+1)(n+2)}{120}\|R\|^2 \\
& - \frac{(n-3)(n-4)}{20}\Delta\tau - \frac{(n+6)(n-3)}{40}\Delta\rho_{11} + \frac{11n^2-27n+142}{120}\nabla^2_{11}\tau \\
& + \frac{(n-4)(n+1)}{36}\tau\rho_{11} - \frac{7n^2+21n-46}{180}\sum_{i,j\geq 2}\rho_{ij}R_{1i1j} - \frac{n^2+3n-58}{120}\rho^2_{11} \\
& - \frac{7n^2+21n+194}{120}\nabla^2_{11}\rho_{11} - \frac{(n+1)(n+2)}{36}\sum_{i,j\geq 2}R^2_{1i1j} \\
& + \frac{n^2+3n+62}{180}\sum_{i\geq 2}\rho^2_{1i} - \frac{(n+1)(n+2)}{60}\sum_{i,j,k\geq 2}R^2_{1ijk} + \frac{n^2-3n+8}{6}\nabla_{\gamma''}\tau \\
& - \frac{n^2+3n+14}{6}\nabla_1\rho_{1\gamma''} - \frac{n^2+3n+14}{12}\nabla_{\gamma''}\rho_{11}\Big\}(\gamma(t)),
\end{aligned}
$$

and the tensor coordinates are taken with respect to an orthonormal frame $E_1 = \gamma', E_2, \dots, E_n$ along γ.

In particular, $c_1 = 2\pi$, $A(3) = 0$, and using the identity $\|R\|^2 = 4\|\rho\|^2 - \tau^2$, valid in any 3-dimensional Riemannian manifold, a straightforward computation shows that

$$
B(3) = \frac{1}{6}\Big\{\nabla^2_{11}\tau - 2\nabla^2_{11}\rho_{11} + \nabla_{\gamma''}\tau - 4\nabla_1\rho_{1\gamma''} - 2\nabla_{\gamma''}\rho_{11}\Big\}(\gamma(t)).
$$

In the special case when γ is a geodesic curve, all the terms containing the acceleration γ'' disappear, thus, for the total scalar curvature $T_\gamma(r)$ of a cylinder of small radius about a geodesic segment $\gamma\colon [a,b] \to M$ lying in a 3-dimensional manifold M, we have

$$
T_\gamma(r) = 2\pi\int_a^b \Big\{\big\{\nabla^2_{11}\tau - 2\nabla^2_{11}\rho_{11}\big\}(\gamma(t))\frac{r^3}{6} + O(r^5)\Big\}t.
$$

This formula implies that if the total scalar curvature of a cylinder depends only on the radius and the length of the axis of the cylinder, then the coefficient $\hat{a} = \big\{\nabla^2_{11}\tau - 2\nabla^2_{11}\rho_{11}\big\}(\gamma(t))$ of $r^3/6$ must be a constant independent of the geodesic γ and the parameter t. Set $a = \hat{a}/4$.

Now let $\gamma_{\mathbf{u}}$, ($\mathbf{u} \in SM$) be an arbitrary unit speed geodesic in M, and let $E_1 = \gamma'_{\mathbf{u}}$, E_2, E_3 be a parallel orthonormal frame along $\gamma_{\mathbf{u}}$, $\sigma_{ij}(t) \subset T_{\gamma_{\mathbf{u}}(t)}M$ the plane spanned by $E_i(t)$ and $E_j(t)$. Then the sectional curvature in the direction of the normal plane $\nu = \sigma_{23}$ can be expressed as

$$
K(\nu) = K(\sigma_{23}) = \big(K(\sigma_{12}) + K(\sigma_{23}) + K(\sigma_{31})\big) - \big(K(\sigma_{12}) + K(\sigma_{31})\big) = \frac{1}{2}\tau\circ\gamma_{\mathbf{u}} - \rho(E_1, E_1).
$$

Differentiating this equation twice with respect to the curve parameter, and using the fact that the vector field $E_1 = \gamma'_{\mathbf{u}}$ is parallel along $\gamma_{\mathbf{u}}$, we obtain

$$
K(\nu)'' = \left(\frac{1}{2}\nabla^2_{11}\tau - \nabla^2_{11}\rho(E_1, E_1)\right) = \frac{\hat{a}}{2} = 2a.
$$

Thus $K(\nu(t))$ must be a polynomial function of t of degree at most 2 with leading term at^2. In particular, (10) holds with a suitably chosen coefficient $b(\mathbf{u})$. This proves (i).

To prove (ii), assume M has bounded sectional curvature. Then for any choice of $\gamma_{\mathbf{u}}$, $K(\nu(t))$ is a bounded polynomial function defined on the whole real line, hence it is constant. Consequently, it has vanishing derivative

$$\frac{d}{dt}K(\nu(t)) = \left(\frac{1}{2}\nabla_1\tau - \nabla_1\rho(E_1, E_1)\right) = 0.$$

Evaluating this equation at $t = 0$, we obtain $\frac{1}{2}\nabla_{\mathbf{u}}\tau - \nabla_{\mathbf{u}}\rho(\mathbf{u}, \mathbf{u}) = 0$ for any $\mathbf{u} \in SM$ and by Lemma 3.4, we conclude that M satisfies the Ledger condition L_3. H. Pedersen and P. Tod (49) proved that 3-dimensional Riemannian manifolds satisfying the third Ledger condition are D'Atri spaces, so M is a D'Atri space. □

Although Theorem 3.11 does not allow us to conclude that M is D'Atri when the that M has bounded sectional curvature is not valid, however we can still infer some useful facts about M in case the sectional curvature is not assumed to be bounded as in Theorem 3.12 ,Fact 4.44.

Theorem 3.12. *Let M be a three dimensional manifold such that for any geodesic $\gamma : \mathbb{R} \to M$ and any $t \in \mathbb{R}$, we have that the sectional curvature of the plane $\{\gamma'(t)\}^{\perp}$ grows quadratic-ally as a function of t. Then $\nabla^3 R$ can be expressed using $R, \nabla R$ by a polynomial formula that holds globally over M.*

Proof. . I will write $(x_1x_2x_3x_4x_5x_6)$ as a shorthand for $\nabla^2 R(x_1, x_2, x_3, x_4, x_5, x_6)$, and will write $(x_1x_2x_3x_4x_5x_6x_7)$ as a shorthand for $\nabla^3 R(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$. If p, q, r is any orthonormal basis of tangents of manifold M , then I write $\{p\}^{\perp}$ as a shorthand for the string of characters $qrqr$. For example $(pp\{p\}^{\perp})$ will mean $\nabla^2 R(p, p, q, r, q, r)$.

Now we introduce a notation $\approx$ with a possibly ambiguous meaning, but hopefully the meaning will be become clearer as one goes through the details of the proof. If a, b are linear combination of components $\nabla^2 R$ tensor , I will write $a \approx b$ as a shorthand for this meta-statement: "We know how to express the difference $b - a$ as a polynomial function of the components of R in a way that is globally valid over M" If c, d are quantities derived from the $\nabla^3 R$ tensor, I will write $c \approx d$ as a shorthand for this meta-statement: "We know how to express the difference $d - c$ as a polynomial function of components of R, ∇R, in a way that is globally valid over M". For example

Remark 3.13. Note that using the Ricci identities we get that:$\nabla^2 R, \nabla^3 R$ are $\approx$ -symmetric in their leftmost two,three arguments respectively.

Let M be a 3 dimensional Riemannian manifold such that there exists L such that for every orthonomral frame u, v, w of tangents at any point we have

$$(uu\{u\}^{\perp}) = -L \tag{11}$$

Let $m \in M$ be arbitrary, fix an orthonomral basis x, y, z of $T_m M$.Let $a, b \in \mathbb{R}$ be arbitrary. (11) gives us that the algebraic curvature tensor defined by $(q_1, q_2, q_3, q_4) \mapsto (\frac{ax+by+z}{|ax+by+z|}, \frac{ax+by+z}{|ax+by+z|}, q_1, q_2, q_3, q_4)$ has sectional curvature equal to L on the plane $\{\frac{ax+by+z}{|ax+by+z|}\}^{\perp}$. Since $az - x, bz - y \in \{\frac{ax+by+z}{|ax+by+z|}\}^{\perp}$, thus:

$$(\frac{ax + by + z}{|ax + by + z|}, \frac{ax + by + z}{|ax + by + z|}, az - x, bz - y, bz - y, az - x) =$$
$$L(|az - x|^2 |bz - y|^2 - [(az - x) \circ (bz - y)]^2)$$

Thus:

$$(ax + by + z, ax + by + z, az - x, bz - y, bz - y, az - x) = L(a^2 + b^2 + 1)^2 \tag{12}$$

The strategy now is to try to express all components of the level $\nabla^2 R$ in terms of components of the form $(d, e, \{f\}^\perp)$, where $d, e, f \in \{x, y, z\}$. A long computation by polarization of (12) along with using Bianchi identities shows that this strategy is successful and gives the identities below:

$$(xxxzyz) \approx (xy\{x\}^\perp) \tag{13}$$

$$(yzxyzx) \approx -\frac{L}{2} - \frac{1}{4}[(yy\{z\}^\perp) + (zz\{y\}^\perp)] \tag{14}$$

$$(yzxyyz) \approx (yx\{y\}^\perp) - (yx\{x\}^\perp) \tag{15}$$

$$(xxxzyx) \approx -(zy\{x\}^\perp) \tag{16}$$

$$(yy\{z\}^\perp) + (yy\{x\}^\perp) + (zz\{y\}^\perp) + (xx\{y\}^\perp) \approx 0 \tag{17}$$

Observe that (13), (14), (15), (16), (17) will hold for any orthonomal basis and so also hold no matter how the symbols x, y, z are permuted.

Fact 3.14. *$(xyy\{x\}^\perp) \approx \frac{1}{3}(xxx\{y\}^\perp)$*

Proof. Differentiate (13) in y direction then Remark 3.13 to get:

$$(yxxxzyz) \approx (yxy\{x\}^\perp) \approx (xyy\{x\}^\perp) \tag{18}$$

By Remark 3.13 and symmetries of algebraic curvature tensors we have:

$$(yxxxzyz) \approx -(xyxzyxz) \tag{19}$$

Interchange the symbols x, z in (14), then differentiate in direction x to get:

$$(xyxzyxz) \approx -\frac{1}{4}[(xyy\{x\}^\perp + (xxx\{y\})^\perp] \tag{20}$$

Combine (18), (19), (20) to get $(xyy\{x\}^\perp) \approx \frac{1}{4}[(xyy\{x\}^\perp + (xxx\{y\})^\perp]$ which gives $(xyy\{x\}^\perp) \approx \frac{1}{3}(xxx\{y\}^\perp)$. □

Fact 3.15. *$(xyz\{x\}^\perp) \approx 0$*

Proof. Differentiate (13) in direction z then use Remark 3.13 along with symmetries of algebraic curvature tensors to get:

$$(zxy\{x\}^\perp) \approx (zxxxzyz) \approx -(xzxyzzx) \tag{21}$$

Permute the symbols x, yz in (15) cyclically (i.e. $x \mapsto y \mapsto z \mapsto x$), then differentiate in direction x to get:

$$(xzxyzzx) \approx (xzy\{z\}^\perp)) - (xzy\{y\}^\perp) \tag{22}$$

Combine (21), (22) to get

$$(zxy\{x\}^\perp) \approx (xzy\{y\}^\perp) - (xzy\{z\}^\perp) \tag{23}$$

(23) holds for all orthonormal x, y, z, in particular it still holds even when the symbols y, z are interchanged. Thus, interchange the symbols y, z in (23) and then use Remark 3.13 to get:

$$(zxy\{x\}^\perp) \approx (xzy\{z\}^\perp) - (xzy\{y\}^\perp) \tag{24}$$

Add (23), (24) to get that $(zxy\{x\}^\perp) \approx 0$. □

Fact 3.16. $(zzx\{y\}^{\perp}) \approx \frac{1}{3}(xxx\{y\}^{\perp})$

Proof. differentiate (15) in direction of y, then apply Remark 3.13 followed by symmetries of algebraic curvature tensors to get:

$$(yyx\{y\}^{\perp}) - (yyx\{x\}^{\perp}) \approx (yyzxyyz) \approx (zyyxyyz) \approx (zyyyzxy) \tag{25}$$

Interchange the symbols x, y in (16), then differentiate in direction of z to get:

$$(zyyyzxy) \approx -(zzx\{y\})^{\perp} \tag{26}$$

Set u to be y in (11), then differentiate in direction x and apply Remark 3.13 to get:

$$(yyx\{y\}^{\perp}) \approx 0 \tag{27}$$

Combine (25), (26), (27) to get

$$(yyx\{x\}^{\perp}) \approx (zzx\{y\}^{\perp}) \tag{28}$$

Combine Fact 3.14,(28), and Remark 3.13 to get $(zzx\{y\}^{\perp} \approx \frac{1}{3}(xxx\{y\}^{\perp})$ □

Fact 3.17. $(xxx\{y\}^{\perp}) \approx 0$

Proof. Differentiate (17) in direction x and apply Remark 3.13 to get

$$(yyx\{z\}^{\perp}) + (xyy\{x\})^{\perp} + (zzx\{y\}^{\perp}) + (xxx\{y\})^{\perp} \approx 0 \tag{29}$$

Since Fact 3.16 holds for all orthonormal x, y, z thus the occurrence of the symbols x, y, z can be permuted to give

$$(yyx\{z\})^{\perp} \approx \frac{1}{3}(xxx\{z\}^{\perp}) \tag{30}$$

Combine (30), Fact 3.16, Fact 3.14, (29) to get

$$(xxx\{z\})^{\perp} \approx -5(xxx\{y\}^{\perp}) \tag{31}$$

Since (31) will hold for orthonormal x, y, z, thus it will still hold even when the symbols y, z are interchanged. Hence, we get

$$(xxx\{y\})^{\perp} \approx -5(xxx\{z\}^{\perp}) \tag{32}$$

Combine (31),(32) to get $(xxx\{y\}^{\perp}) \approx 0$ □

Note that setting u to be x in (11), then differentiate in direction x to get

$$(xxx\{x\}^{\perp}) \approx 0 \tag{33}$$

Combine (33),(27)Fact 3.14,Fact 3.15, Fact 3.16, Fact 3.17 and all their permutations to get that

$$(qde\{f\}^{\perp}) \approx 0 \tag{34}$$

for all q, d, e, f

that are members of our orthonomral basis x, y, z. Since all components of $\nabla^2 R$ can be expressed using components in the form $(de\{f\}^{\perp})$ for some $d, e, f \in \{x, y, z\}$ as shown by (13), (14), (15), (16). Thus, diffrentation shows that all components of $\nabla^3 R$ can be expressed using components of the form $(qde\{f\}^{\perp})$ for some $q, d, e, f \in \{x, y, z\}$. Finally, combine the previous statment with to get that $\nabla^3 R$ may be expressed using $R, \nabla R$ by a polynomial formula. □

4 Chapter 4: Connections with Prescribed Curvature tensor and higher covariant derivatives at a point

It is well known that the pullback of an analytic Riemannian metric along normal coordinates about a point p can be expressed as a power series in terms of the covariant derivatives of the curvature tensor at p, which will satisfy some algebraic conditions (symmetry relations, Bianchi and Ricci identities, etc). It is natural to ask whether for a given sequence of multilinear maps defined on the tangent space T_pM of a manifold M at $p \in M$, which satisfy the algebraic conditions just mentioned, there exists a Riemannian metric in a neighborhood of p for which the sequence of covariant derivatives of the curvature tensor at p are equal to the sequence of multilinear maps we started with. The answer is yes (provided some not so strict inequalites are met), due to a theorem by Kowalski and Belger ([50]). This theorem can be interpreted intuitively so that all the algebraic identities one can prove for a generic analytic Riemannian metric follow from the symmetries of the curvature tensor and the Bianchi and Ricci identities.

One can repeat the same discussion for general analytic connections which are not necessarily Levi-Civitia connections of a Riemannian metric. Indeed, one can derive power series expansions for the Christoffel symbols of a connections in terms of the curvature and torsion of the connection and their derivatives at a point. The main theorem of this chapter, Theorem 4.36, is the analogue of the theorem of Kowalski and Belger for arbitrary connections.

To formulate the results of this chapter in a neat way, a new algebraic structure called Curvature-Torsion algebra, or CT algebra for short, is introduced. The CT algebra of a connection at a point encodes the local behaviour of the connection in the same way as the Lie algebra of a Lie group encodes the local behaviour of the group around its identity. This principle will be justified by the theorems of this chapter.

The methods of this chapter will also lead to a generalization of the Hausdorff–Campbell formula (Theorem 4.32 and Corollary 4.34), and will lead to a sufficient condition for a connection to be normal-analytic (i.e. admit an analytic atlas consisting of normal coordinates) by bounding the growth of the derivatives of its curvature and torsion tensors (Theorem 4.35).

4.1 Preliminaries

We shall consider arrays of symbols which will often have repeated entries. To reduce the length of expressions, we shall identify these finite sequences of symbols with elements of the free semigroup generated by the symbols and write them in a multiplicative form using exponents. For example, we shall write the sequence $(a, u, u, u, u, x, x, y, u)$ also as (au^4x^2yu), or (au^2u^2xxyu), or (auu^3x^2yu), etc.

Let A, B be finite dimensional $\mathbb{R}$-vector spaces, $f\colon A_0 \to B$ be a smooth function defined on an open subset A_0 of A. Then for any $p_0 \in A_0, x \in A$ we denote the derivative of f in direction of x by $\partial_x f(p_0)$, i.e., $\partial_x f(p_0) = \frac{d}{dt} f(p_0 + tx)\big|_{t=0}$. This gives us another smooth function $\partial_x f\colon A_0 \to B$. For any natural number k, we also consider a k-linear map $\partial^k f|_{p_0}\colon A^k \to B$ given by

$$\partial^k f|_{p_0}(x_1, x_2, \dots, x_k) = \partial_{x_1}\partial_{x_2}\dots\partial_{x_k} f(p_0).$$

$\partial^0 f|_{p_0}$ is interpreted to be $f(p_0)$. By Clairout's theorem, $\partial^k f|_{p_0}$ is a symmetric multilinear map. If $f\colon A_0 \to B$ happens to be analytic, then by equipping A, B with any two norms one gets that there exists $\delta > 0$ such that $B_\delta(p_0) \subseteq A_0$,

$$\frac{1}{\limsup_{k\to\infty} |\partial^k f|_{p_0}|^{1/k}} \geq \delta,$$

and for all $p \in B_\delta(p_0)$, we have

$$f(p) = \sum_{k=0}^{\infty} \frac{1}{k!} \partial^k f|_{p_0}(p - p_0, p - p_0, \ldots, p - p_0) = \sum_{k=0}^{\infty} \frac{1}{k!} \partial^k f|_{p_0}((p - p_0)^k).$$

Let $g\colon B_0 \to C$ be another smooth function, where B_0 is some open subset of B containing $f(A_0)$. Then higher order derivatives of the composition $g \circ f$ are expressed by Faà di Bruno's formula ([51])

$$\partial^k (g \circ f)|_{p_0}(x_1, x_2, \ldots, x_k) = \sum_{P \in \Pi_k} \partial^{|P|} g|_{f(p_0)} \left(\partial^{|S|} f|_{p_0}(x_S)\right)_{S \in P}, \tag{35}$$

where Π_k is the set of partitions of the set $[k] = \{1, 2, \ldots, k\}$; for a subset $S \subset [k]$ consisting of the elements $i_1 < \cdots < i_l$, x_S denotes the list $(x_{i_1}, \ldots, x_{i_l})$, and for a partition $P = \{S_1, \ldots, S_m\} \in \Pi_k$, $\left(\partial^{|S|} f|_{p_0}(x_S)\right)_{S \in P}$ unambiguously (due to symmetry of $\partial^{|S|} f|_{p_0}$) stands for the list

$$\left(\partial^{|S_1|} f|_{p_0}(x_{S_1}), \ldots, \partial^{|S_m|} f|_{p_0}(x_{S_m})\right).$$

Let (M, ∇) be a smooth boundaryless manifold equipped with a not necessarily torsion free connection. We have $C^\infty(M)$-multilinear maps $R\colon \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$, $T\colon \mathfrak{X}(M)^2 \to \mathfrak{X}(M)$ denoting the associated curvature and torsion of the tensors of ∇. These tensors are given by

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$
$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Let γ be a geodesic in (M, ∇) and J be a vector field along γ. We say that J is a Jacobi field along γ if it satisfies the differential equation

$$\nabla^2 J + R(J, \gamma', \gamma') + \nabla(T(J, \gamma')) = 0.$$

The motivation for calling such a vector field a Jacobi field is given in Proposition 4.16. By the theory of linear ordinary differential equations, there exists a unique Jacobi field J along γ with any initial conditions for $J(0)$ and $DJ(0)$.

Let m_0 be any point in M. Let ϕ be any restriction of $\exp_{m_0}$ such that ϕ is diffeomorphism between an open subset containing the origin of $T_{m_0}M$ and an open subset of M. Consider the function $F^\phi\colon dom(\phi) \times (T_{m_0}M)^2 \to T_{m_0}M$ given by

$$F^\phi(x, y, z) = (T_x \phi)^{-1}\left((\nabla_{E_z^\phi} E_y^\phi)(\phi(x))\right),$$

where for $a \in T_{m_0}M$, E_a^ϕ is the vector field over $im(\phi)$ that's given by $E_a^\phi(m) = [T_{\phi^{-1}(m)}\phi](a)$ for any $m \in Im(\phi)$. Clearly F^ϕ is smooth, and it is bilinear over $\mathbb{R}$ in its last two arguments. Throughout this chapter, we refer to F^ϕ as the Christoffel tensor.

Fact 4.1 (First Bianchi Identity). *For any $X_0, X_1, X_2 \in \mathfrak{X}(M)$, we have*

$$\sum_{i \in Z_3} [R(X_i, X_{i+1}, X_{i+2}) + \nabla T(X_i, X_{i+1}, X_{i+2}) + T(X_i, T(X_{i+1}, X_{i+2}))] = 0.$$

Fact 4.2 (Second Bianchi Identity). *For any $X_0, X_1, X_2, W \in \mathfrak{X}(M)$ we have*

$$\sum_{i \in Z_3} [\nabla R(X_i, X_{i+1}, X_{i+2}, W) + R(T(X_{i+1}, X_{i+2}), X_i, W)] = 0.$$

Fact 4.3 (Ricci Identity). *For any smooth $(k,1)$-tensor field $\omega\colon \mathfrak{X}(M)^k \to \mathfrak{X}(M)$ and for any vector fields $A, B, X_1, X_2, \dots, X_k \in \mathfrak{X}(M)$, we have*

$$\begin{aligned}\nabla^2\omega(A,B,X_1,X_2,\dots,X_k) - \nabla^2\omega(B,A,X_1,X_2,\dots,X_k)\\ = R(A,B,\omega(X_1,X_2,\dots,X_k)) - \sum_{i=1}^{k}\omega(R(A,B)^i(X_1,X_2,\dots,X_k)) -\\ - \nabla\omega(T(A,B),X_1,X_2,\dots,X_k).\end{aligned}$$

Where $R(A,B)^i(X_1,X_2,\dots,X_k)$ means $(X_1,\dots,X_{i-1},R(A,B,X_i),X_{i+1},\dots,X_k)$.

The proofs of Fact 4.1 and Fact 4.2 can be found in (52). Proof of Fact 4.3 is a straightforward application of the definitions.

4.2 Curvature-Torsion Algebras

Now we introduce a new algebraic concept that will help us formulate Theorem 4.36.

Definition 4.4 (Curvature-Torsion algebras). Let $\mathcal{A}$ be a vector space, and let $\{R^k\colon \mathcal{A}^{k+3} \to \mathcal{A}\}_{k\geq 0}$, $\{T^k\colon \mathcal{A}^{k+2} \to \mathcal{A}\}_{k\geq 0}$ be sequences of multi-linear maps satisfying the following 5 properties:
(1) For every natural number k, T^k is skew-symmetric in the rightmost two arguments, R^k is skew symmetric in 2nd and 3rd arguments counted from the right.
(2) For every natural number k and for every $x_0, x_1, x_2, u_1, u_2, \dots, u_k \in \mathcal{A}$, we have

$$\begin{aligned}\sum_{i\in\mathbb{Z}_3}[R^k(u_{[k]},x_i,x_{i+1},x_{i+2}) + T^{k+1}(u_{[k]},x_i,x_{i+1},x_{i+2})]\\ + \sum_{i\in\mathbb{Z}_3}\sum_{S\subseteq[k]} T^{|S|}(u_S,x_i,T^{|S^c|}(u_{S^c},x_{i+1},x_{i+2})) = 0.\end{aligned}$$

Where $S^c = [k]\setminus S$ is the complement of S in $[k]$.
(3) For every natural number k and for every $x_0, x_1, x_2, u_1, u_2, \dots, u_k, w \in \mathcal{A}$, we have

$$\sum_{i\in\mathbb{Z}_3}[R^{k+1}(u_{[k]},x_i,x_{i+1},x_{i+2},w) + \sum_{S\subseteq[k]}[R^{|S|}(u_S,T^{|S^c|}(u_{S^c},x_{i+1},x_{i+2}),w)] = 0.$$

(4) For any natural numbers k, r and any vectors $a, b, x, y, z,\ u_1, u_2, \dots, u_r,\ v_1, v_2, \dots, v_k \in \mathcal{A}$, we have

$$\begin{aligned}R^{k+r+2}(u_{[r]},a,b,v_{[k]},x,y,z) - R^{k+r+2}(u_{[r]},b,a,v_{[k]},x,y,z)\\ = \sum_{S\subseteq[r]}\Big(R^{|S|}(u_S,a,b,R^{|S^c|+k}(u_{S^c},v_{[k]},x,y,z))\\ - \sum_{i=1}^{k+3} R^{|S|+k}(u_S,R^{|S^c|}(u_{S^c},a,b)^i(v_{[k]},x,y,z))\\ - R^{|S|+k+1}(u_S,T^{|S^c|}(u_{S^c},a,b),v_{[k]},x,y,z)\Big).\end{aligned}$$

(5) For any natural numbers k, r and for any vectors $a, b, x, y,\ u_1, u_2, \ldots, u_r,\ v_1, v_2, \ldots, v_k \in \mathcal{A}$, we have

$$\begin{aligned} T^{k+r+2}&(u_{[r]}, a, b, v_{[k]}, x, y) - T^{k+r+2}(u_{[r]}, b, a, v_{[k]}, x, y) \\ &= \sum_{S\subseteq[r]} \Bigg(R^{|S|}(u_S, a, b, T^{|S^c|+k}(u_{S^c}, v_{[k]}, x, y)) \\ &\qquad - \sum_{i=1}^{k+2} T^{|S|+k}(u_S, R^{|S^c|}(u_{S^c}, a, b)^i(v_{[k]}, x, y)) \\ &\qquad - T^{|S|+k+1}(u_S, T^{|S^c|}(u_{S^c}, a, b), v_{[k]}, x, y) \Bigg). \end{aligned}$$

Then we call the algebraic structure $(\mathcal{A}, \left\{R^k\colon \mathcal{A}^{k+3} \to \mathcal{A}\right\}_{k\geq 0}, \left\{T^k\colon \mathcal{A}^{k+2} \to \mathcal{A}\right\}_{k\geq 0})$ a *Curvature-Torsion algebra*, or shortly a *CT algebra*.

Definition 4.5. Let $\mathcal{A}$ be the CT algebra described in Definition 4.4. A *CT monomial* in d variables is a function from $\mathcal{A}^d$ to $\mathcal{A}$ that is defined by composing a finite number of the operations of the CT algebra $\mathcal{A}$. A *CT polynomial* is a linear combination of CT monomials.

For example,

$$R^2(x, T^2(x, x, R^0(x, y, z), T^0(y, z)), R^3(y, y, y, x, y, z), y, z)$$

is a CT monomial, and

$$R^0(x, y, y) + T^0(T^0(x, y), T^0(y, z))$$

is a CT polynomial in the variables $x, y, z \in \mathcal{A}$.

We also introduce the concept of a CT tensor which will be used later on in the chapter.

Definition 4.6. Let a, c be positive integers and b, d be natural numbers. A *k-CT tensor* from $\mathcal{A}^a \times \mathbb{R}^b$ to $\mathcal{A}^c \times \mathbb{R}^d$ is a k-multilinear $\omega\colon (\mathcal{A}^a \times \mathbb{R}^b)^k \to \mathcal{A}^c \times \mathbb{R}^d$ such that the first c components of w could be expressed using the natural operations of the CT algebra $\mathcal{A}$ and the rightmost d components only depend on the real parts of the inputs.

For example, the multilinear map which sends $((x_1, r_1, s_1), (x_2, r_2, s_2), (x_3, r_3, s_3))$ to:

$$(2T^1(x_1, x_2, x_3) + 7R^0(x_3, x_1, x_2) + s_3T^0(x_1, x_2) - 5T^0(T^0(x_1, x_3), x_2), s_2r_1r_3)$$

is a 3-CT tensor from $(\mathcal{A} \times \mathbb{R}^2)^3$ to $\mathcal{A} \times \mathbb{R}$.

Remark 4.7. Let (M, ∇) be a boundaryless manifold equipped with a connection and m_0 be a point in it.

It follows that $T_{m_0}M$ equipped with the multilinear maps $\left\{\nabla^k R|_{m_0}\colon (T_{m_0}M)^{k+3} \to T_{m_0}M\right\}_{k\geq 0}$, $\left\{\nabla^k T|_{m_0}\colon (T_{m_0}M)^{k+2} \to T_{m_0}M\right\}_{k\geq 0}$ is a CT algebra. This follows from differentiating Facts 4.1, 4.2, and 4.3 as many times as necessary.

Now we prove an algebraic lemma which will be crucial to proving Theorem 4.36.

Lemma 4.8. *Let A be a vector space over the field of real numbers, and k be a positive integer such that $k \geq 2$. Let $R\colon A^{k+2} \to A, T\colon A^{k+2} \to A$ be multilinear maps satisfying the following properties:*

(L1) *$R(u_1, \ldots, u_{k-1}, x, y, z)$ is a symmetric function of the arguments $u_1, \ldots, u_{k-1}$; $T(u_1, \ldots, u_k, x, y)$ is a symmetric function of the arguments $u_1, \ldots, u_k$.*

(L2) *$R(u_1, \dots, u_{k-1}, x, y, z)$ and $T(u_1, \dots, u_k, x, y)$ are skew symmetric functions of the arguments x and y.*

(B1) *For every $x_0, x_1, x_2, u_1, u_2, \dots, u_{k-1} \in A$, we have*

$$\sum_{i \in \mathbb{Z}_3} [R(u_1, u_2, \dots, u_{k-1}, x_i, x_{i+1}, x_{i+2}) + T(u_1, u_2, \dots, u_{k-1}, x_i, x_{i+1}, x_{i+2})] = 0.$$

(B2) *For every $x_0, x_1, x_2, u_1, u_2, \dots, u_{k-2}, w \in A$, we have*

$$\sum_{i \in \mathbb{Z}_3} R(u_1, u_2, \dots, u_{k-2}, x_i, x_{i+1}, x_{i+2}, w) = 0.$$

(C1) *For every $u, v, w \in A$, we have*

$$\begin{aligned} k(k+2)R(u^{k-1}wuv) + k(k-1)R(u^{k-2}wvuu) + kR(u^{k-1}vwu) + \\ + kR(u^{k-1}vuw) + k(k+1)T(u^{k-1}wvu) + (k+1)T(u^k vw) = 0. \end{aligned}$$

Then both R and T are zero.

Proof. Plug $u = v$ into (C1) and use skew symmetry assumption (L2) to get $R(u^{k-1}wuu) = \frac{1}{k}T(u^k wu)$. Polarize this equation by replacing u by $au + bx$, where $x \in A$ and $a, b \in \mathbb{R}$, and equating the coefficients of $a^k b$. Using also the symmetry condition (L1), we obtain

$$\begin{aligned} (k-1)R(u^{k-2}xwuu) + R(u^{k-1}wxu) + R(u^{k-1}wux) \\ = T(u^{k-1}xwu) + \frac{1}{k}T(u^k wx). \end{aligned} \tag{36}$$

The last term in the right hand side and the middle term in the left hand side are skew symmetric in w, x, so symmetrization of (36) in w, x cancels them and gives

$$\begin{aligned} (k-1)R(u^{k-2}xwuu) + (k-1)R(u^{k-2}wxuu) + \\ + R(u^{k-1}wux) + R(u^{k-1}xuw) = T(u^{k-1}xwu) + T(u^{k-1}wxu). \end{aligned}$$

Rewrite the above equation noting that (B2) and (L2) give us

$$R(u^{k-2}wxuu) \overset{(B2)}{=\!=} -R(u^{k-2}xuwu) - R(u^{k-2}uwxu) \overset{(L2)}{=\!=} R(u^{k-2}xwuu) + R(u^{k-1}xwu).$$

What we get is

$$\begin{aligned} 2(k-1)R(u^{k-2}xwuu) + (R(u^{k-1}wux) + R(u^{k-1}xwu) + R(u^{k-1}uxw)) + \\ + (k-2)R(u^{k-1}xwu) + 2R(u^{k-1}xuw) = T(u^{k-1}xwu) + T(u^{k-1}wxu). \end{aligned}$$

Next add $T(u^{k-1}wux) + T(u^{k-1}xwu) + T(u^{k-1}uxw)$ and apply (B1) and (L2) to get

$$2(k-1)R(u^{k-2}xwuu) + (k-2)R(u^{k-1}xwu) + 2R(u^{k-1}xuw) = 2T(u^{k-1}xwu) + T(u^k xw).$$

Hence we have

$$(k-1)R(u^{k-2}xwuu) = \frac{(k-2)}{2}R(u^{k-1}wxu) + R(u^{k-1}uxw) + T(u^{k-1}xwu) + \frac{1}{2}T(u^k xw).$$

Substitute the last equation in (36) and use skew symmetry to get

$$\frac{k+2}{2}R(u^{k-1}wxu) + (R(u^{k-1}uxw) + R(u^{k-1}wux) + R(u^{k-1}xwu)) +$$

$$+T(u^{k-1}xwu) + \frac{1}{2}T(u^{k}xw) = T(u^{k-1}xwu) + \frac{1}{k}T(u^{k}wx).$$

Adding $T(u^{k-1}uxw) + T(u^{k-1}wux) + T(u^{k-1}xwu)$ to both sides and applying (B1), (L1) gives

$$\frac{k+2}{2}R(u^{k-1}wxu) = T(u^{k-1}xwu) + T(u^{k-1}wux) + \left(\frac{1}{k} - \frac{1}{2}\right)T(u^{k}wx).$$

From the above equation we get

$$R(u^{k-1}x_1x_2u) = \frac{2}{k+2}\left[T(u^{k-1}x_2x_1u) + T(u^{k-1}x_1ux_2) + \frac{2-k}{2k}T(u^{k}x_1x_2)\right] \tag{37}$$

for every $x_1, x_2, u \in A$. Next we polarize equation (37) by replacing u by $au + bw$ and studying coefficient of $a^{k-1}b$ of the resulting polynomials to get (using (L1))

$$\begin{aligned}(k-1)R(u^{k-2}wx_1x_2u) + R(u^{k-1}x_1x_2w) = \frac{2}{k+2}\Big[&(k-1)T(u^{k-2}wx_2x_1u) + T(u^{k-1}x_2x_1w) +\\ &+ (k-1)T(u^{k-2}wx_1ux_2) + T(u^{k-1}x_1wx_2) +\\ &+ \frac{(2-k)}{2}T(u^{k-1}wx_1x_2)\Big].\end{aligned} \tag{38}$$

Substituting v in place of x_1 and u in place of x_2 in above equation gives

$$\begin{aligned}&(k-1)R(u^{k-2}wvuu) + R(u^{k-1}vuw)\\ &\quad= \frac{2}{k+2}\Big[(k-1)T(u^{k-2}wuvu) + T(u^{k}vw) + T(u^{k-1}vwu) +\\ &\qquad\qquad + \frac{(2-k)}{2}T(u^{k-1}wvu)\Big].\end{aligned} \tag{39}$$

Substitute v for x_1, w for x_2 in equation (37) to get

$$R(u^{k-1}vwu) = \frac{2}{k+2}\Big[T(u^{k-1}wvu) + T(u^{k-1}vuw) + \frac{2-k}{2k}T(u^{k}vw)\Big]. \tag{40}$$

Multiply equation (39) by k and substitute it in LHS of (C1), substitute (40) (after multiplying it by k) in LHS of (C1) as well to get (after some algebraic manipulations, and usage of (L1), (L2))

$$R(u^{k-1}wuv) = T(u^{k-1}wuv) + \frac{1}{k}T(u^{k}wv). \tag{41}$$

Interchange v, w in (41) then combine it with (39) to get

$$R(u^{k-2}wvuu) = \frac{k-2}{k(k-1)(k+2)}T(u^{k}vw) + \frac{k}{(k-1)(k+2)}T(u^{k-1}wvu) + \frac{k+4}{(k-1)(k+2)}T(u^{k-1}vwu).$$

Replace w by v_1, and v by v_2 in above equation to get

$$R(u^{k-2}v_1v_2uu) = \alpha_k T(u^k v_2 v_1) + \beta_k T(u^{k-1}v_1v_2u) + \gamma_k T(u^{k-1}v_2v_1u), \tag{42}$$

which holds for every $u, v_1, v_2 \in A$, where $\alpha_k, \beta_k, \gamma_k$ are defined by

$$\alpha_k = \frac{k-2}{k(k-1)(k+2)}, \quad \beta_k = \frac{k}{(k-1)(k+2)}, \quad \gamma_k = \frac{k+4}{(k-1)(k+2)}.$$

Now we consider two cases, whether $k = 2$ or $k \geq 3$.

Case 1 ($k = 2$): Using (37), (42) in case $k = 2$ we get

$$R(ux_1x_2u) = \frac{1}{2}T(ux_2x_1u) + \frac{1}{2}T(ux_1ux_2), \tag{43}$$

$$R(x_1x_2uu) = \frac{1}{2}T(ux_1x_2u) + \frac{3}{2}T(ux_2x_1u). \tag{44}$$

Add equations (43),(44) and use (L1),(L2) for simplifying RHS, and use (B2) followed by (L2) for LHS to get:

$$R(x_2x_1uu) = 2T(ux_2x_1u). \tag{45}$$

Interchange x_1, x_2 in above equation, and then combine the resulting equation with (44) to get:

$$T(ux_1x_2u) = T(ux_2x_1u). \tag{46}$$

The above equation, combined with equation (43) and (L2) gives $R(ux_1x_2u) = 0$ for every $u, x_1, x_2 \in A$. Hence R is skew symmetric in the leftmost and rightmost arguments. Apply skew symmetry in leftmost,rightmost arguments of R along with (L2) on equation (45) to get:

$$R(ux_1ux_2) = 2T(ux_2ux_1). \tag{47}$$

Interchange x_1, x_2 in above equation and apply (L2) to get:

$$R(uux_2x_1) = 2T(ux_1x_2u). \tag{48}$$

By skew symmetry of R in leftmost, rightmost arguments we also have

$$R(ux_2x_1u) = 0. \tag{49}$$

Add equations (47),(48),(49) and apply (B1) on LHS to get:

$$-T(ux_1ux_2) - T(uux_2x_1) - T(ux_2x_1u) = 2T(ux_2ux_1) + 2T(ux_1x_2u).$$

Rearrange terms in above equation along with using (L2) to get:

$$T(uux_2x_1) = T(ux_2x_1u) - T(ux_1x_2u).$$

Finally combine above equation with equation (46) to get $T(uux_2x_1) = 0$ for all $x_1, x_2, u \in A$. Hence T is skew symmetric in its two leftmost arguments. We also know T is symmetric in its two leftmost arguments by (L1), hence T is the zero tensor. $T = 0$ combined with equation (45) gives that R is skew symmetric in its two rightmost arguments. The skew symmetry of R in its two rightmost argument, combined with (B1), (L2) and $T = 0$ gives that $R = 0$. So we are done if case 2 is true.

Case 2 ($k \geq 3$): Polarize equation (42) by replacing u by $au + bx$ and looking for the coefficient of $a^{k-1}b$. Using (L1) we obtain

$$\begin{aligned}(k-2)R(v_1u^{k-3}xv_2uu) + R(v_1u^{k-2}v_2xu) + R(v_1u^{k-2}v_2ux) = \\ k\alpha_k T(u^{k-1}xv_2v_1) + \beta_k[(k-1)T(u^{k-2}xv_1v_2u) + T(u^{k-1}v_1v_2x)] + \\ \gamma_k[(k-1)T(u^{k-2}xv_2v_1u) + T(u^{k-1}v_2v_1x)].\end{aligned} \tag{50}$$

Symmetrize above equation in the v_2, x arguments. The terms $R(v_1u^{k-2}v_2xu)$ and $T(u^{k-1}v_1v_2x)$ are skew symmetric in v_2, x and so vanish after symmetrization. By (B1), followed by (L2), we have $(k-2)R(v_1u^{k-3}v_2xuu) = (k-2)R(v_1u^{k-2}xv_2u) + (k-2)R(v_1u^{k-3}xv_2uu)$. Substitute RHS of previous equation in place of the $(k-2)R(v_1u^{k-3}v_2xuu)$ term in LHS of equation resulting from symmetrization to get

$$\begin{aligned}2(k-2)R(v_1u^{k-3}xv_2uu) + R(v_1u^{k-2}v_2ux) + (k-2)R(v_1u^{k-2}xv_2u) + \\ +R(v_1u^{k-2}xuv_2) = k\alpha_k(T(u^{k-1}xv_2v_1) + T(u^{k-1}v_2xv_1)] + (k-1)\beta_k[T(u^{k-2}xv_1v_2u) + T(u^{k-2}v_2v_1xu)] + \\ + \gamma_k[2(k-1)T(u^{k-2}xv_2v_1u) + T(u^{k-1}v_2v_1x) + T(u^{k-1}xv_1v_2)].\end{aligned}$$

By (B1), we can replace $R(v_1u^{k-2}v_2ux)$ in above equation by $-[R(v_1u^{k-2}xv_2u) + R(v_1u^{k-2}uxv_2) + T(v_1u^{k-2}v_2ux) + T(v_1u^{k-2}xv_2u) + T(v_1u^{k-2}uxv_2)]$. Rearrange terms in the resulting equation and simplify using (L1), (L2) applied to R to get

$$\begin{aligned}(k-2)R(v_1u^{k-3}xv_2uu) = \frac{k-3}{2}R(v_1u^{k-2}v_2xu) + R(v_1u^{k-2}uxv_2)) + \\ (\frac{(k-1)\beta_k - 1}{2})T(u^{k-2}v_1v_2xu) + (\frac{(k-1)\beta_k + 1}{2})T(u^{k-2}v_1xv_2u) + \\ (k-1)\gamma_k T(u^{k-2}v_2xv_1u) + \frac{1}{2}T(u^{k-1}v_1xv_2) + \\ (\frac{k\alpha_k - \gamma_k}{2})(T(u^{k-1}v_2xv_1) - T(u^{k-1}xv_1v_2)).\end{aligned}$$

The above equation gives a formula for $(k-2)R(v_1u^{k-3}xv_2uu)$, so substitute this formula in equation (50). In the LHS of resulting equation, apply (B1) on the terms $R(v_1u^{k-2}R(v_1u^{k-2}uxv_2) + R(v_1u^{k-2}v_2ux)$. Simplify equation (use (L1),(L2)) and definitions of $\alpha_k, \beta_k, \gamma_k$ to get

$$\begin{aligned}R(u^{k-2}v_1v_2xu) = -\frac{2}{(k-1)(k+2)}T(u^{k-1}xv_1v_2) + \frac{k-2}{(k-1)(k+2)}T(u^{k-1}v_1xv_2) + \\ -\frac{2}{(k-1)(k+2)}T(u^{k-1}v_2xv_1) + \frac{2}{k+2}T(u^{k-2}xv_1v_2u) - \frac{2}{k+2}T(u^{k-2}v_1v_2xu).\end{aligned}$$

Make the substitution $(v_1, v_2, x) \mapsto (w, x_1, x_2)$ to get

$$R(u^{k-2}wx_1x_2u) = -\frac{2}{(k-1)(k+2)}T(u^{k-1}x_2wx_1) + \frac{k-2}{(k-1)(k+2)}T(u^{k-1}wx_2x_1)$$

$$-\frac{2}{(k-1)(k+2)}T(u^{k-1}x_1x_2w)+\frac{2}{k+2}T(u^{k-2}x_2wx_1u)-\frac{2}{k+2}T(u^{k-2}wx_1x_2u).$$

Last equation gives a formula for $R(u^{k-2}wx_1x_2u)$, so substitute this formula in equation (38) and manipulate using (L1),(L2) to get that

$$R(u^{k-1}x_1x_2w)=0.$$

Finally polarize above equation by replacing u by $\sum_{i=1}^{k-1}\lambda_iu_i$ (where $\lambda_1,\lambda_2,\ldots,\lambda_{k-1}$ are arbitrary real numbers and $u_1,u_2,\ldots,u_{k-1}$ are arbitrary members of A) and look for cofficent $\lambda_1,\lambda_2,\ldots,\lambda_{k-1}$ to get using (L1) that

$$(k-1)!R(u_1,u_2,\ldots,u_{k-1},x_1,x_2,w)=0$$

for all $u_1,u_2,\ldots,u_{k-1},x_1,x_2,w\in A$. Hence R is zero tensor. Now that we know $R=0$, (C1) could be manipulated (use (L2)) to give

$$kT(u^{k-1}wuv)=T(u^{k-1}uvw). \tag{51}$$

Interchange v,w in previous equation, multiply by -1 and use (L2) to get

$$kT(u^{k-1}vwu)=T(u^{k-1}uvw). \tag{52}$$

From (B1), and the vanishing of R we know that

$$kT(u^{k-1}uvw)+kT(u^{k-1}vwu)+kT(u^{k-1}wuv)=0.$$

Substitute (51),(52) into above equation to get $(k+2)T(u^{k-1}uvw)=0$. Polarize the last equation by replacing u by $\sum_{i=1}^{k}\lambda_iu_i$ (where $\lambda_1,\lambda_2,\ldots,\lambda_k$ are arbitrary real numbers and $u_1,u_2,\ldots,u_k$ are arbitrary members of A) and look for the coefficent of $\lambda_1\lambda_2\cdots\lambda_k$ to get using (L1) that $(k+2)k!T(u_1,u_2,\ldots,u_k,v,w)=0$. As the previous equation holds for all $u_1,u_2,\ldots,u_k,v,w\in A$, we get that $T=0$, and we're done with case 2. □

Let's introduce further examples and definitions about of CT algebras.

Definition 4.9 (CT Sub-algebras)**.** : Let $(\mathcal{A},\{R^k\colon\mathcal{A}^{k+3}\to\mathcal{A}\}_{k\geq0},\{T^k\colon\mathcal{A}^{k+2}\to\mathcal{A}\}_{k\geq0})$ be a CT algebra. A CT sub-algebra S of $\mathcal{A}$ is a linear subspace of $\mathcal{A}$ such that $R^k(x_1,x_2,\ldots,x_k,y_1,y_2,y_3),T^k(x_1,x_2,\ldots,x_k,y_1,y_2)\in S$ for every $x_1,x_2,\ldots,x_k,y_1,y_2,y_3\in S$, and every $k\geq0$.

Definition 4.10 (CT Ideals)**.** : Let $(\mathcal{A},\{R^k\colon\mathcal{A}^{k+3}\to\mathcal{A}\}_{k\geq0},\{T^k\colon\mathcal{A}^{k+2}\to\mathcal{A}\}_{k\geq0})$ be a CT algebra. A CT ideal I of $\mathcal{A}$ is a linear subspace of $\mathcal{A}$ such that for every $k\geq0$ we have: $R^k(x_1,x_2,\ldots,x_k,y_1,y_2,y_3)\in I$ whenever at least one of $x_1,x_2,\ldots,x_k,y_1,y_2,y_3$ belongs to I, and we have $T^k(x_1,x_2,\ldots,x_k,y_1,y_2)\in I$ whenever at least one of $x_1,x_2,\ldots,x_k,y_1,y_2$ belongs to I.

Definition 4.11 (Quotient of CT algebras)**.** : Let $(\mathcal{A},\{R^k\colon\mathcal{A}^{k+3}\to\mathcal{A}\}_{k\geq0},\{T^k\colon\mathcal{A}^{k+2}\to\mathcal{A}\}_{k\geq0})$ be a CT algebra. Let I be a CT ideal of $\mathcal{A}$, then we get a natural CT algebra structure on $\mathcal{A}/I$ given by:

$$R^k(x_1+I,x_2+I,\ldots,x_k+I,y_1+I,y_2+I,y_3+I)=R^k(x_1,x_2,\ldots,x_k,y_1,y_2,y_3)+I$$

$$T^k(x_1+I,x_2+I,\ldots,x_k+I,y_1+I,y_2+I)=T^k(x_1,x_2,\ldots,x_k,y_1,y_2)+I$$

One can check easily that the axioms of Definition 4.4 still hold for $\mathcal{A}/I$.

Definition 4.12 (CT homomorphisms). :

Let $(\mathcal{A}, \{R^k\colon \mathcal{A}^{k+3} \to \mathcal{A}\}_{k\geq 0}, \{T^k\colon \mathcal{A}^{k+2} \to \mathcal{A}\}_{k\geq 0}), (\mathcal{A}_*, \left\{R^k_*\colon {\mathcal{A}_*}^{k+3} \to \mathcal{A}_*\right\}_{k\geq 0}, \left\{T^k_*\colon {\mathcal{A}_*}^{k+2} \to \mathcal{A}_*\right\}_{k\geq 0})$ be two CT algebras. A linear map $f : \mathcal{A} \to \mathcal{A}_*$ is said to be a CT homomorphism iff:

$$f(T^k(x_1, x_2, \dots, x_k, y_1, y_2)) = T^k_*(f(x_1), f(x_2), \dots, f(x_k), f(y_1), f(y_2))$$

$$f(R^k(x_1, x_2, \dots, x_k, y_1, y_2, y_3)) = R^k_*(f(x_1), f(x_2), \dots, f(x_k), f(y_1), f(y_2), f(y_3))$$

for every $x_1, x_2, \dots, x_k, y_1, y_2, y_3 \in \mathcal{A}$. One checks easily that $Ker(f)$ is a CT ideal of $\mathcal{A}$, and that $Im(f)$ is a CT subalgebra of $\mathcal{A}_*$. Furthermore, it is easily seen that the composition of two CT homomorphisms is again a CT homomorphism.

Definition 4.13 (Direct sum of two CT algebras). :

Let $(\mathcal{A}_1, \{R^k_1\colon \mathcal{A}_1^{k+3} \to \mathcal{A}_1\}_{k\geq 0}, \{T^k_1\colon \mathcal{A}_1^{k+2} \to \mathcal{A}_1\}_{k\geq 0}), (\mathcal{A}_2, \{R^k_2\colon \mathcal{A}_2^{k+3} \to \mathcal{A}_2\}_{k\geq 0}, \{T^k_2\colon \mathcal{A}_2^{k+2} \to \mathcal{A}_2\}_{k\geq 0})$ be two CT algebras. Then we can form the direct sum CT algebra $\mathcal{A}_1 \oplus \mathcal{A}_2$ whose operations are defined by:

$$T^k(((a_i, b_i))_{i\in[k]}, (x_1, y_1), (x_2, y_2)) = (T^k_1((a_i)_{i\in[k]}, x_1, x_2), T^k_2((b_i)_{i\in[k]}, y_1, y_2))$$

$$R^k(((a_i, b_i))_{i\in[k]}, (x_1, y_1), (x_2, y_2), (x_3, y_3)) = (R^k_1((a_i)_{i\in[k]}, x_1, x_2, x_3), R^k_2((b_i)_{i\in[k]}, y_1, y_2, y_3))$$

One easily checks that the axioms of Definition 4.4 still hold for $\mathcal{A}_1 \oplus \mathcal{A}_2$. Next we give some examples of CT algebras.

Remark 4.14. Let (M, ∇) be a manifold equipped with a connection, and let $\gamma : C \to M$ be a smooth map between manifolds. Denote the set of smooth vector fields along $\gamma : C \to M$ by $\mathfrak{X}(\gamma : C \to M)$. Then we naturally have a CT algebra structure $(\mathfrak{X}(\gamma : C \to M), \{\nabla^k R\}_{k\geq 0}, \{\nabla^k T\}_{k\geq 0})$. (Recall that if A, B, C are any smooth vector fields along γ, then $T(A, B), R(A, B, C)$ are also smooth vector fields along γ). If the map γ is taken to be the identity map $id : M \to M$, then one naturally gets a CT algebra structure on $\mathfrak{X}(M)$. If the map γ is taken to be the inclusion $i : \{m_0\} \to M$ for some point $m_0 \in M$, then one gets Remark 4.7 again.

Remark 4.15. The previous remark will give a finite dimensional CT algebra if S is the one point manifold. Here is another way to get a finite dimensional CT algebra. Let V be any finite dimensional vector space over $\mathbb{R}$. Let ∇ be any $\mathbb{R}$- bilinear map on $End_{\mathbb{R}}(V)$, next we equip $End_{\mathbb{R}}(V)$ with a CT algebra structure by stealing the formulas for curvature, torsion tensors when written using a connection on a manifold, and then replacing each occurence of the symbol of the connection by our bilinear map ∇, and replacing each occurrence of the lie bracket symbol of vector fields by the endormorphism commutator operation of $End_{\mathbb{R}}(V)$. For example, for any endormorphisms A, B, C of V, $R(A, B, C)$ will be defined as $\nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C$.

4.3 Power series for a connection in normal coordinates

Let (M, ∇) be a smooth boundaryless manifold equipped with a connection (not necessarily torsion free), and let m_0 be a point in M. By inverse function theorem, one can restrict the exponential map at m_0 to get a diffeomorphism $\overline{\exp}_{m_0}$ between an open convex subset of $T_{m_0}M$ around the origin and an open subset of M around m_0. Let u be arbitrary member of $dom(\overline{\exp}_{m_0})$, and let v, w be arbitrary members of $T_{m_0}M$. Choose $\epsilon > 0$ sufficently small and consider the smooth function $\Gamma\colon]-\epsilon, \epsilon[\times]-\epsilon, \epsilon[\times[0, 1] \to M$ defined by $\Gamma(r, s, t) := \overline{\exp}_{m_0}(t(u + rv + sw))$. In this section, we denote $E_z^{\overline{exp}_{m_0}}$ simply by E_z for every $z \in T_{m_0}M$.

Clearly, Γ is a 2-parameter variation through geodesics. Consider the geodesic $\gamma\colon [0, 1] \to M$ given by $\gamma(t) := \Gamma(0, 0, t)$. Consider the smooth vector fields H, J, X along γ that are given by

$$H(t) := \partial_1\Gamma(0, 0, t), \quad J(t) := \partial_2\Gamma(0, 0, t), \quad X(t) := \nabla_2\partial_1\Gamma(0, 0, t).$$

Proposition 4.16. *The vector fields H, J satisfy the following differential equations:*

$$\nabla^2 H + R(H, \gamma', \gamma') + \nabla(T(H, \gamma')) = 0,$$
$$\nabla^2 J + R(J, \gamma', \gamma') + \nabla(T(J, \gamma')) = 0$$

subject to initial conditions $H(0) = 0, \nabla H(0) = v$, $J(0) = 0, \nabla J(0) = w$.

Proof. We prove only the differential equation and the initial conditions for H. A similar argument will work for J as well.

$$\nabla_3\nabla_3\partial_1\Gamma = \nabla_3(\nabla_1\partial_3\Gamma - T(\partial_1\Gamma, \partial_3\Gamma)) = \nabla_3\nabla_1\partial_3\Gamma - \nabla_3(T(\partial_1\Gamma, \partial_3\Gamma)) =$$
$$\nabla_1\nabla_3\partial_3\Gamma - R(\partial_1\Gamma, \partial_3\Gamma, \partial_3\Gamma) - \nabla T(\partial_3\Gamma, \partial_1\Gamma, \partial_3\Gamma) - T(\nabla_3\partial_1\Gamma, \partial_3\Gamma) - T(\partial_1\Gamma, \nabla_3\partial_3\Gamma).$$

After noting that $\nabla_3\partial_3\Gamma = 0$ (as Γ is a variation through geodesics) and rearranging the above equation, we get

$$\nabla_3\nabla_3\partial_1\Gamma + R(\partial_1\Gamma, \partial_3\Gamma, \partial_3\Gamma) + T(\nabla_3\partial_1\Gamma, \partial_3\Gamma) + \nabla T(\partial_3\Gamma, \partial_1\Gamma, \partial_3\Gamma) = 0.$$

Compose the above equation with $i\colon [0,1] \to]-\epsilon, \epsilon[\times]-\epsilon, \epsilon[\times[0,1]$ that is given by $i(t) := (0, 0, t)$ to get

$$\nabla^2 H + R(H, \gamma', \gamma') + \nabla(T(H, \gamma')) = 0.$$

$\Gamma(r, s, 0)$ is constant at m_0 for all r, s. Hence, it follows that $H(0) = 0$. Next we have

$$\nabla H(0) = \nabla_3\partial_1\Gamma(0,0,0) = \nabla_1\partial_3\Gamma(0,0,0) - T(\partial_1\Gamma(0,0,0), \partial_3\Gamma(0,0,0)).$$

As $\Gamma(r, s, 0)$ is constant at m_0, and as $\partial_3\Gamma(r, s, 0) = u + rv + sw$ for all r, s, it follows that $\nabla_1\partial_3\Gamma(0,0,0) = \frac{d}{dr}|_{r=0}(u + rv + sw) = v$. Furthermore, $\partial_1\Gamma(0,0,0) = H(0) = 0$, thus $T(\partial_1\Gamma(0,0,0), \partial_3\Gamma(0,0,0)) = 0$, hence we get

$$\nabla H(0) = v - 0 = v.$$ □

Remark 4.17. Proposition 4.16 can also be used to give us that for every $z \in T_{m_0}M$ we have $lE_z \circ \gamma$ (l is given by $t \mapsto t$) is the unique Jacobi field along γ with initial value 0 and initial derivative z.

Definition 4.18. Define the sequence of CT polynomials $\left\{h^k\colon T_{m_0}M \times T_{m_0}M \to T_{m_0}M\right\}_{k\geq 0}$ recursively as follows $h^0(p, q) = 0, h^1(p, q) = q$ for all $p, q \in T_{m_0}M$, and

$$h^{k+2}(p,q) = -\sum_{i=0}^{k}\binom{k}{i}\nabla^i R|_{m_0}(p^i, h^{k-i}(p,q), p, p)$$
$$-\sum_{i=0}^{k+1}\binom{k+1}{i}\nabla^i T|_{m_0}(p^i, h^{k+1-i}(p,q), p)$$

for every natural k, and every $p, q \in T_{m_0}M$.

Remark 4.19. For every $p, q \in T_{m_0}M$, $k \geq 1$ and real number λ, we have $h^k(\lambda p, q) = \lambda^{k-1}h^k(p, q)$.

Remark 4.20. Note that differentiating the differential equation satisfied by H in Proposition 4.16 k times, gives us a recurrence relation for computing $\nabla^{k+2}H(0)$ in terms of $\nabla^0 H(0), \nabla^1 H(0), \ldots, \nabla^{k+1}H(0)$. We leave it to the reader to check that this recurrence relation gives us that $\nabla^k H(0) = h^k(u, v)$ for every natural number k.

Proposition 4.21. *For every $x, y \in T_{m_0}M$, and every natural k, we have*

$$\nabla^k_{E_x} E_y|_{m_0} = \frac{1}{k+1} h^{k+1}(x, y).$$

Proof. From definition of H, it follows that $H = lE_v \circ \gamma$ (Where $l\colon [0,1] \to R$ is given by $l(t) = t$). Differentiate $k+1$ times to get $\nabla^{k+1}(H) = \nabla^{k+1}(lE_v \circ \gamma) = (k+1)\nabla^k(E_v \circ \gamma) + l\nabla^{k+1}(E_v \circ \gamma) = (k+1)\nabla^k_{E_u} E_v + l\nabla^{k+1}_{E_u} E_v$. Finally evaluate last equation at $t = 0$ and apply Remark 4.20 to get that $\nabla^k_{E_u} E_w|_{m_0} = \frac{h^{k+1}(u,v)}{k+1}$. The previous equality will hold for any $u \in dom(\overline{exp_{m_0}}), v \in T_{m_0}M$. However, by Remark 4.20 we could get that equality holds for all $u, v \in T_{m_0}M$ by scaling the $u-$ argument of the equality appropriately. $\square$

Proposition 4.22. *The vector field X satisfies the following differential equation:*

$$\nabla^2 X + R(X, \gamma', \gamma') + \nabla(T(X, \gamma'))$$

$$\begin{gathered}\nabla R(\gamma', J, \gamma', H) + R(\nabla J, \gamma', H) + 2R(J, \gamma', \nabla H) + \nabla R(J, H, \gamma', \gamma') \\ +R(H, \nabla J + T(J, \gamma'), \gamma') + R(H, \gamma', \nabla J + T(J, \gamma'))+ \\ \nabla^2 T(J, \gamma', H, \gamma') + \nabla T(\nabla J + T(J, \gamma'), H, \gamma')+ \\ \nabla T(\gamma', H, \nabla J + T(J, \gamma'))+ \\ \nabla T(J, \nabla H, \gamma') + T(R(J, \gamma', H), \gamma') + T(\nabla H, \nabla J + T(J, \gamma')) = 0\end{gathered}$$

subject to the inital conditions $X(0) = 0, \nabla X(0) = 0$

Proof. From the proof of Proposition 4.16, we have the following equation:

$$\nabla_3\nabla_3\partial_1\Gamma + R(\partial_1\Gamma, \partial_3\Gamma, \partial_3\Gamma) + T(\nabla_3\partial_1\Gamma, \partial_3\Gamma) + \nabla T(\partial_3\Gamma, \partial_1\Gamma, \partial_3\Gamma) = 0.$$

Apply the differentiation operator ∇_2 to the previous equation to get

$$\begin{aligned}\nabla_2\nabla_3\nabla_3\partial_1\Gamma + \nabla R(\partial_2\Gamma, \partial_1\Gamma, \partial_3\Gamma, \partial_3\Gamma) + R(\nabla_2\partial_1\Gamma, \partial_3\Gamma, \partial_3\Gamma)+ \\ +R(\partial_1\Gamma, \nabla_2\partial_3\Gamma, \partial_3\Gamma) + R(\partial_1\Gamma, \partial_3\Gamma, \nabla_2\partial_3\Gamma) + \nabla T(\partial_2\Gamma, \nabla_3\partial_1\Gamma, \partial_3\Gamma)+ \\ +T(\nabla_2\nabla_3\partial_1\Gamma, \partial_3\Gamma) + T(\nabla_3\partial_1\Gamma, \nabla_2\partial_3\Gamma) + \nabla^2 T(\partial_2\Gamma, \partial_3\Gamma, \partial_1\Gamma, \partial_3\Gamma)+ \\ +\nabla T(\nabla_2\partial_3\Gamma, \partial_1\Gamma, \partial_3\Gamma) + \nabla T(\partial_3\Gamma, \nabla_2\partial_1\Gamma, \partial_3\Gamma)+ \\ +\nabla T(\partial_3\Gamma, \partial_1\Gamma, \nabla_2\partial_3\Gamma) = 0.\end{aligned} \tag{53}$$

We also have the following formulae for $\nabla_2\nabla_3\partial_1\Gamma, \nabla_2\nabla_3\nabla_3\partial_1\Gamma, \nabla_2\partial_3\Gamma$ respectively:

$$\nabla_2\nabla_3\partial_1\Gamma = \nabla_3\nabla_2\partial_1\Gamma + R(\partial_2\Gamma, \partial_3\Gamma, \partial_1\Gamma), \tag{54}$$

$$\begin{aligned}\nabla_2\nabla_3\nabla_3\partial_1\Gamma = \nabla_3\nabla_2\nabla_3\partial_1\Gamma + R(\partial_2\Gamma, \partial_3\Gamma, \nabla_3\partial_1\Gamma) = \\ \nabla_3(\nabla_3\nabla_2\partial_1\Gamma + R(\partial_2\Gamma, \partial_3\Gamma, \partial_1\Gamma)) + R(\partial_2\Gamma, \partial_3\Gamma, \nabla_3\partial_1\Gamma) = \\ \nabla_3\nabla_3\nabla_2\partial_1\Gamma + \nabla R(\partial_3\Gamma, \partial_2\Gamma, \partial_3\Gamma, \partial_1\Gamma) + R(\nabla_3\partial_2\Gamma, \partial_3\Gamma, \partial_1\Gamma) \\ +2R(\partial_2\Gamma, \partial_3\Gamma, \nabla_3\partial_1\Gamma),\end{aligned} \tag{55}$$

$$\nabla_2\partial_3\Gamma = \nabla_3\partial_2\Gamma + T(\partial_2\Gamma, \partial_3\Gamma). \tag{56}$$

Substitute the formulae from equations (54),(55),(56) above for $\nabla_2\nabla_3\partial_1\Gamma, \nabla_2\nabla_3\nabla_3\partial_1\Gamma, \nabla_2\partial_3\Gamma$ into LHS of equation (53), evaluate the resulting equation at $(0,0,t)$ for any $t \in [0,1]$ and the differential equation follows. By definition of Γ, one sees easily that $\partial_1\Gamma = \pi_3 E_v \circ \Gamma$, (Where $\pi_3 \in C^\infty(]-\epsilon,\epsilon[\times],\epsilon[\times[0,1])$ is given by $\pi_3(r,s,t) = t$). Apply ∇_2 to get that

$$\nabla_2\partial_1\Gamma = \pi_3\nabla_2(E_v \circ \Gamma). \tag{57}$$

Evaluate previous equation at $(0,0,0)$ to get that $X(0) = 0$. Apply ∇_3 to (57) to get that

$$\nabla_3\nabla_2\partial_1\Gamma = \nabla_2(E_v \circ \Gamma) + \pi_3\nabla_3(\nabla_2(E_v \circ \Gamma)).$$

Evaluating the above equation at $(0,0,0)$ gives that $\nabla X(0) = 0$ as

$$\nabla_2(E_v \circ \Gamma)|_{(0,0,0)} = \nabla_{\partial_2\Gamma(0,0,0)}E_v = \nabla_0 E_v = 0.$$

$\square$

Note that by differentiating the differential equation of Proposition 4.22 k times, one gets a recurrence relation for computing $\nabla^{k+2}X(0)$ from $\nabla^0 X(0), \nabla^1 X(0), \dots, \nabla^{k+1}X(0)$. This recursion gives that $\nabla^k X(0)$ is a CT polynomial in u, v, w for every $k \geq 0$.

As $\overline{exp}_{m_0}$ is a diffeomorphism, one can get a smooth function $\alpha : [0,1] \to T_{m_0}M$ such that $E_{\alpha(t)}(\gamma(t)) = [\nabla_{E_w}E_v] \circ \gamma(t)$, for every $t \in [0,1]$. We wish to show that $\alpha(0), \alpha'(0), \alpha''(0), \dots$ are all CT polynomials in the variables u, v, w.

By (57), and by $\partial_2\Gamma = \pi_3 E_w \circ \Gamma$ we get

$$\nabla_2\partial_1\Gamma = \pi_3^2[\nabla_{E_w}E_v] \circ \Gamma.$$

Evaluate above equation at $(0,0,t)$ for any $t \in [0,1]$ to get

$$X = l^2[\nabla_{E_w}E_v] \circ \gamma = l^2 E_\alpha \gamma.$$

Where $l \in C^\infty([0,1])$ is given by $l(t) := t$. Differentiate $X = l^2 E_\alpha\gamma$ for $k+2$ times (Where k is any natural number), then evalute at $t = 0$ to get

$$\nabla^{k+2}X(0) = (k+2)(k+1)D^k(E_\alpha\gamma)|_0. \tag{58}$$

By induction, we have that for every natural number k we have

$$D^k(E_\alpha\gamma) = \sum_{i=0}^{k}\binom{k}{i}[\nabla^i_{E_u}E_{\alpha^{(k-i)}}]\gamma.$$

Where $[\nabla^i_{E_u}E_{\alpha^{(k-i)}}]\gamma$ is defined as the vector field along γ given by

$$t \mapsto [\nabla^i_{E_u}E_{\alpha^{(k-i)}(t)}]\gamma(t).$$

Evaluate previous equation at $t = 0$ and rearrange terms to get

$$\alpha^{(k)}(0) = \nabla^k(E_\alpha\gamma)|_0 - \sum_{i=1}^{k}\binom{k}{i}[\nabla^i_{E_u}E_{\alpha^{(k-i)}(0)}](m_0).$$

Hence, by (58) and Proposition 4.21 applied to above equation, we get

$$\alpha^{(k)}(0) = \frac{\nabla^{k+2}X(0)}{(k+1)(k+2)} - \sum_{i=1}^{k} \binom{k}{i} \frac{h^{i+1}(u, a^{(k-i)}(0))}{i+1}.$$

The recursion above gives that all of $\alpha^{(k)}(0)$ is a CT polynomial in u, v, w for all natural numbers k. A moment of thought gives that $\alpha^{(k)}(0)$ is a degree $k+2$ homogeneous polynomial in the variable (u, v, w) as well. So get a sequence of homogeneous CT polynomials $\left\{q_k \colon (T_{m_0}M)^3 \to T_{m_0}M\right\}_{k \geq 0}$ such that q_k is homogeneous with degree k, q_0, q_1 are zero polynomials, and for every $k \geq 2$ we have $q_k(u, v, w)$ matches $\alpha^{(k-2)}(0)$ as a CT polynomial in u, v, w. A moment of thought shows that one can get a sequence $\{\eta_k\}_{k \geq 0}$ such that for every k we have that η_k is a k-CT tensor from $(T_{m_0}M)^3$ to $T_{m_0}M$ and $\eta_k(x, x, \ldots, x) = q_k(x)$ for every $x \in (T_{m_0}M)^3$. Replacing η_k by it's symmetrization if necessary, we may assume without loss of generality that η_k is symmetric for every k as well.

4.4 The Analytic Case

Definition 4.23. Let (M, ∇) be a manifold equipped with a connection. (M, ∇) is said to be tame iff any of the following two equivalent conditions holds:

1) For every Riemannian metric g on M and for every $m_1 \in M$ there exists an open set U around m_1 and $C, L \geq 0$ such that for every $m \in U$ we have $|\nabla^k T|_m|, |\nabla^k R|_m| \leq Ck!L^k$ for every natural k.

2) There exists a Riemannian metric g on M such that for every $m_1 \in M$ there exists an open set U around m_1 and $C, L \geq 0$ such that for every $m \in U$ we have $|\nabla^k T|_m|, |\nabla^k R|_m| \leq Ck!L^k$ for every natural k.

Theorem 4.24. *Let (M, ∇, m_0) be a pointed tame manifold with connection, then one can choose normal coordinates ϕ (i.e. a diffeomorphism between an open subset containing the origin of $T_{m_0}M$ and an open subset of M containing m_0) such that F^{ϕ} is analytic, and such that $\partial^k F^{\phi}|_0$ is a k-CT tensor for every $k \geq 0$.*

Proof. In this proof, we only care to verify convergence and so do not care to prove the sharpest bounds in our inequalities. We break the proof into several steps.

Step 1: By tameness, get some Riemannian metric g on M, some open set U containing m_0, some A, λ_0 such that for every m in U, and every natural k we have $|\nabla^k T(m)|, |\nabla^k R(m)| \leq Ak!\lambda_0^k$. Choose $r_1 > 0$ small enough so that $B_{2r_1}(0) \subseteq dom(exp_{m_0})$, $exp_{m_0}(B_{2r_1}(0)) \subseteq U$, and $exp_{m_0}|B_{2r_1}(0) \colon B_{2r_1}(0) \to exp_{m_0}(B_{2r_1}(0))$ is diffeomorphism. Define $\widehat{exp}_{m_0}$ to be $exp_{m_0}|B_{2r_1}(0)$. I will denote $E_a^{\widehat{exp}_{m_0}}$ simply by F_a for any $a \in T_{m_0}M$.

Step 2: Define T to be $exp_{m_0}(\overline{B_{r_1}(0)})$. By compactness of T, we may get positive real numbers L_1, L_2, L_3 such that for every $m \in T$, $y_1, y_2 \in T_{m_0}M$, we have

$$|\nabla_{F_{y_2}} F_{y_1}(m)| \leq L_2 |y_1||y_2|,$$
$$L_3 |y_1| \leq |F_{y_1}(m)| \leq L_1 |y_1|.$$

Set L_4, L_5 to be

$$L_4 = \max\{|\nabla_{F_{y_3}} \nabla_{F_{y_2}} F_{y_1}(m)| : m \in T, y_1, y_2, y_3 \in T_{m_0}M, |y_1|, |y_2|, |y_3| \leq r_1\},$$
$$L_5 = \max\{|\nabla_{F_{y_2}} F_{y_1}(m)| : m \in T, y_1, y_2 \in T_{m_0}M, |y_1|, |y_2| \leq r_1\}.$$

Step 3: Fix a basis $z_1, z_2, \ldots, z_n$ of $T_{m_0}M$. For all $x \in dom(\widehat{exp}_{m_0})$, let $\gamma_x \colon [0, 1] \to M$ be the geodesic given by $\gamma_x(t) = \widehat{exp}_{m_0}(tx)$. Let $Q_1, Q_2, \ldots, Q_n$ be vector fields locally defined over $\mathrm{Im}(\widehat{exp}_{m_0})$ such that for every $i \in [n]$ we have

a) $Q_i(m_0) = z_i$,

b) for every $x \in \operatorname{dom}(\widehat{exp}_{m_0})$, we have $Q_i \circ \gamma_x$ is parallel along γ_x.

Let $m \in \operatorname{Im}(\widehat{exp}_{m_0})$ we have that $\{Q_i(m)\}_{i\in[n]}$ is a basis for $T_m M$. Hence the matrix $\left[g(Q_i(m), Q_j(m))\right]_{i,j\in[n]}$ has an inverse which we denote by $\left[d_{i,j}(m)\right]_{i,j\in[n]}$.

By compactness, we define Θ as the maximum of $|Q_i(m)|, |d_{i,j}(m)|$ as m ranges over T, and i, j range over $[n]$.

Step 4: Choose $\lambda > 0$ large enough so that $\lambda_0 \le \lambda$ and $\frac{A}{2\lambda^2} + \frac{A}{\lambda} \le \frac{1}{2}$. Define σ by

$$\sigma = 1 + 2\lambda L_1 n^2 \Theta^3 \max\left\{L_1, \frac{L_1 + L_2}{\lambda L_1}\right\}.$$

Choose $r > 0$ small enough so that

$$r < \max\{r_1, 1\}, \quad \lambda L_1 r < 2^{-100}, \text{ and } \quad 2r\lambda L_1 \max\{16, \frac{2\sigma}{L_3} + 1\} < 2^{-10}.$$

Set $\overline{exp}_{m_0}$ to be $\widehat{exp}_{m_0}|B_r(0)$. Let $u, v, w \in B_r(0)$ be arbitrary. Now we are in a setting which is a special case of that of Section 4, so retain all notation, definitions, propositions of Section 4.

Step 5: We prove the following

Claim 4.25. *Let θ be any Jacobi field along γ such that $\theta(0) = 0$, then for every $t \in [0, 1]$ and natural number k we have*

$$|\nabla^k \theta(t)| \le |D\theta(0)| \max\left\{L_1, \frac{L_1 + L_2 r}{\lambda L_1 r}\right\} k!(\lambda L_1 r)^k.$$

The proof is by induction on k. Use Remark 4.17 of section 4 to prove the base case. Prove the induction step by differentiating the Jacobi equation $\nabla^2\theta + R(\theta, \gamma', \gamma') + \nabla(T(\theta, \gamma')) = 0$ k times. Make $\nabla^{k+2}\theta$ term the subject, then apply the triangle inequality followed by induction hypothesis. The inequality below will be helpful in proving the induction step

$$|\gamma'| \le L_1|u| \le L_1 r. \tag{59}$$

Step 6: Introduce the following definition.

Definition 4.26. Let $\eta\colon \mathfrak{X}(M)^l \to \mathfrak{X}(M)$ be $\mathcal{C}^\infty(M)$-multilinear, we say η *is of type L with constant C*, where C, L are non-negative reals iff for every natural k, and every $m \in U$ we have $|\nabla^k \eta(m)| \le Ck!L^k$. We adopt similar definitions for smooth vector fields along γ and for smooth scalar fields on $[0, 1]$.

Claim 4.27. *Let $a \ge 0$. Let $B\colon \mathfrak{X}(M)^l \to \mathfrak{X}(M)$ be $\mathcal{C}^\infty(M)$-multilinear of type $a\lambda$ with constant C, and $S_1, S_2, \dots, S_l$ be smooth vector fields along γ be of type $a\lambda L_1 r$ with constants $C_1, C_2, \dots, C_l$ respectively. Then $B(S_1, S_2, \dots, S_l)$ is of type $2a\lambda L_1 r$ with constant $2^l CC_1C_2\cdots C_l$.*

Proof. Apply the triangle inequality on the equation

$$\nabla^k(B(S_1, S_2, \dots, S_l)) = \sum_{\substack{(b_0, b_1, \dots, b_l)\in\mathbb{N}^{l+1},\\ b_0+b_1+\cdots+b_l=k}} \binom{k}{b_0, b_1, \dots, b_l} \nabla^{b_0} B((\gamma')^{b_0}, \nabla^{b_1}S_1, \nabla^{b_2}S_2, \dots, \nabla^{b_l}S_l).$$

Hint: Use inequality (59) along with the combinatorial inequality

$$\#\{(b_0, b_1, \dots, b_l) \in \mathbb{N}^{l+1} : b_0 + b_1 + \cdots + b_l = k\} = \binom{k+l}{l} \le 2^{k+l}.$$ □

Step 7: Using Proposition 4.22, we know that X satisfies the differential equation of the form

$$\nabla^2 X + R(X, \gamma', \gamma') + \nabla(T(X, \gamma')) + S = 0,$$

where S is $\nabla R(\gamma', J, \gamma', H) + R(\nabla J, \gamma', H) + \ldots$. Note that H, J are type $\lambda L_1 r$ by step 5. Applying findings of step 6, one can see after some computations that S is type $8\lambda L_1 r$. So one can get some constant $C_{00} > 0$ such that $|D^k S| \leq C_{00} k!(8\lambda L_1 r)^k$. Tedious computations on the formula of S show that one can get C_{00} so that it is independent of u, v, w. (To get independency, we recall that $|u|, |v|, |w| < r$). Set A_2 to be $\max\{L_5, \frac{L_4+2L_5}{8\lambda L_1 r}, \frac{C_{00}}{(8\lambda L_1 r)^2}\}$. We bound the growth rate of the derivatives of X by arguing inductively as in step 5 to get that X is of type $8\lambda L_1 r$ with constant A_2. The base of the induction step will be proven using the equation $X = l^2[\nabla_{E_w} E_v] \circ \gamma$ of Section 4.

Step 8: We begin with proving the following calculus fact:

Fact 4.28. *Let $g_1, g_2 \in \mathcal{C}^\infty([0,1])$ be such that $g_1 = lg_2$, where l is the linear function given by $l(t) = t$). It is also given that g_1 is of type L with constant C for some $L \leq \frac{1}{2}$. Then g_2 is of type $2L$ with constant $2LC$.*

Proof. The bound we have on the growth of derivatives of g_1 allows us to represent g_1 by its Taylor series, and so by $g_1 = lg_2$ we get a convergent power series expansion for g_2 in terms of $\{g_1^{(i)}(0)\}_{i \geq 0}$. Differentiating this power series as many times as we wish and bounding it appropriately we find the rate of growth of the derivatives of g_2. ∎

The next thing to do is to prove the "manifold-analouge" of the above fact:

Claim 4.29. *Let V, W be any two vector fields along γ such that $V = lW$, and such that V is type L_0 with constant C_0 for some $L_0 \leq \frac{1}{2}$, then W is type $2L_0$ with constant $2L_0 n^2 \Theta^3 C_0$*

Proof. Use the vector fields $Q_1, Q_2, \ldots, Q_n$ introduced in step 3, to get a parallel frame $Q_1 \circ \gamma, Q_2 \circ \gamma, \ldots, Q_n \circ \gamma$ along γ. Hence $V = \sum_{i=1}^n a_i Q_i \circ \gamma, W = \sum_{i=1}^n b_i Q_i \circ \gamma$ for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathcal{C}^\infty([0,1])$. It follows from $V = lW$ that $a_i = lb_i$. By hypothesis of the claim we have a bound on the growth rate of the derivatives of V, which gives us a bound on the growth rates of the derivatives of a_i's. Now use the above fact to get a bound on the growth rate of the derivatives of the b_i's, which in turn will give a bound on the growth rate of the derivatives of W. ∎

Step 9: We prove the following claim:

Claim 4.30. *For every $z \in T_{m_0}M$, we have that $\|\nabla^k_{E_u} E_z|_{\gamma(t)}\| = \|\nabla^k(E_z \circ \gamma)(t)\| \leq \sigma |z| k!(2\lambda L_1 r)^k$ for every $t \in [0,1]$ and every natural k.*

Proof. Remark 4.17 of Section 4 tells us that for any tangent $z \in T_{m_0}M$, we have that $lE_z \circ \gamma$ is a Jacobi field along γ with initial value 0 and initial derivative z. Hence, by step 5 we have that $lE_z \circ \gamma$ is of type $\lambda L_1 r$ with constant $|z| \max\left\{L_1, \frac{L_1+L_2 r}{\lambda L_1 r}\right\}$. As $\lambda L_1 r < \frac{1}{2}$, we may apply claim of step 8 to bound the growth rate of the derivatives of $E_z \circ \gamma$ and we will be done. (Hint: $r < 1$.) ∎

Step 10: Next we know that $X = l^2 E_\alpha \gamma$ (see section 4). By step 7, we know that X is type $8L_1 r$ with constant A_2. As $\lambda L_1 r < 2^{-100}$, we may apply claim of step 8 twice to get that $E_\alpha \gamma$ is type $32\lambda L_1 r$ with constant $2(16\lambda L_1 r n^2 \Theta^3)^2 A_2$. To reduce length of expressions, I will refer to $2(16\lambda L_1 r n^2 \Theta^3)^2 A_2$ by C_{15} from now on.

Step 11: Now we are in a position to bound the derivatives of α. We prove the claim below:

Claim 4.31. *For every natural k, and every $t \in [0,1]$, we have*

$$|\alpha^{(k)}(t)| \leq A_3 k!(2\lambda L_1 r \max\{16, \frac{2\sigma}{L_3}+1\})^k,$$

where A_3 is defined to be $\frac{2C_{15}}{L_3}$.

Proof. By induction on k. Base ($k=0$):then we have by choice of L_3 and by step 10 that

$$L_3|\alpha(t)| \leq |E_{\alpha(t)}(\gamma(t))| \leq c_{15}.$$

This gives us that $|\alpha(t)| \leq \frac{c_{15}}{L_3} \leq A_3$

Induction step: For any $t \in [0,1[$ and for any natrual k, we have

$$\nabla^k(E_\alpha\gamma)|_t = \sum_{i=0}^{k}\binom{k}{i}[\nabla^i_{E_u}E_{\alpha^{(k-i)}}(t)]|_{\gamma(t)} = E_{\alpha^{(k)}(t)}(\gamma(t)) + \sum_{i=1}^{k}\binom{k}{i}[\nabla^i_{E_u}E_{\alpha^{(k-i)}(t)}]|_{\gamma(t)}.$$

Hence

$$\begin{aligned}
|E_{\alpha^{(k)}(t)}(\gamma(t))| &= |\nabla^k(E_\alpha\gamma)|_t - \sum_{i=1}^{k}\binom{k}{i}[\nabla^i_{E_u}E_{\alpha^{(k-i)}(t)}]|_{\gamma(t)}|,\\
|E_{\alpha^{(k)}(t)}(\gamma(t))| &\leq |\nabla^k(E_\alpha\gamma)|_t| + \sum_{i=1}^{k}\binom{k}{i}|[\nabla^i_{E_u}E_{\alpha^{(k-i)}(t)}]|_{\gamma(t)}|,\\
{}_3|\alpha^{(k)}(t)| &\leq |E_{\alpha^{(k)}(t)}(\gamma(t))| \leq c_{15}k!(32\lambda L_1 r)^k + \sum_{i=1}^{k}\binom{k}{i}\sigma|\alpha^{(k-i)}(t)|i!(2\lambda L_1 r)^i,\\
|\alpha^{(k)}(t)| &\leq \frac{c_{15}}{L_3}k!(32\lambda L_1 r)^k + \sum_{i=1}^{k}\binom{k}{i}\frac{\sigma}{L_3}|\alpha^{(k-i)}(t)|i!(2\lambda L_1 r)^i.
\end{aligned}$$

The last inequality gives an upper bound on $|\alpha^{(k)}(t)|$ using $|\alpha^{(0)}(t)|, |\alpha^{(1)}(t)|, \ldots, |\alpha^{(k-1)}(t)|$. This will allow us to use the induction hypothesis to get a bound on $|\alpha^{(k)}(t)|$ and complete the induction step. □

Step 12: By step 11 and choice of r, we know that for every natural k and $t \in [0,1]$ we have $|\frac{\alpha^{(k)}(t)}{k!}| \leq A_3(2\lambda L_1 r max\{16, \frac{2\sigma}{L_3}+1\})^k \leq A_3 2^{-10k}$. By Taylor's theorem, this gives that for all

$$(T_u\overline{exp}_{m_0})^{-1}[\nabla_{E_w}E_v|_{\overline{exp}_{m_0}(u)}] = \alpha(1) = \sum_{k=0}^{\infty}\frac{\alpha^{(k)}(0)}{k!} = \sum_{k=0}^{\infty}\frac{\eta_k((u,v,w)^k)}{k!},$$

$$\left|\frac{\eta_k((u,v,w)^k)}{k!}\right| = \left|\frac{\alpha^{(k)}(0)}{k!}\right| \leq A_3 2^{-10k}.$$

Hence, for all $u, v, w \in B_r(0)$, we have

$$(T_u\overline{exp}_{m_0})^{-1}[\nabla_{E_w}E_v|_{\overline{exp}_{m_0}(u)}] = \sum_{k=0}^{\infty}\frac{\eta_k((u,v,w)^k}{k!},$$

$$\left|\frac{\eta_k((u,v,w)^k)}{k!}\right| \leq A_3 2^{-10k}.$$

Set ϕ to be $\overline{exp}_{m_0}|B_{\frac{r}{100}}(0)$ for example, and we are done because by previous equation we have $\partial^k F^{\phi}|_0$ equals η_k which is a k-CT tensor. ☐

Now let (M, ∇, m_0) be a pointed tame manifold equipped with connection. Using Theorem 4.24, choose a restriction ϕ of exp_{m_0} such that $dom(\phi)$ is an open ball $B_{\delta_0}(0)$ in $T_{m_0}M$, such that $\phi : dom(\phi) \to Im(\phi)$ is diffeomorphism, F^{ϕ} is analytic, and $\partial^k F^{\phi}|_0$ is k-CT tensor for every $k \geq 0$. Consider the operation $*$ defined on some suitably small open set of $(T_{m_0}M)^2$ containing the origin given by

$$x * y \to \phi^{-1}(exp_{\phi(x)}(\hat{y})), \tag{60}$$

where $\hat{y}$ denotes the parallel transport of y along $\gamma_x \colon [0,1] \to M$ given by $\gamma_x(t) = \phi(tx)$.

Theorem 4.32. *The operation $*$ in (60) is analytic, and for every $k \geq 0$, $\partial^k(*)|_{(0,0)}$ is a k-CT tensor.*

Proof. We break the proof in to several steps. Let's introduce a useful definition: Let f be a smooth map from some open subset of $(T_{m_0}M)^a \times \mathbb{R}^b$ containing the origin to $(T_{m_0}M)^c \times \mathbb{R}^d$. We say f is a "CT-map" iff $\partial^k f|_0$ is k-CT tensor for every $k \geq 0$.

Step 1: Use (35) to show that for any two CT maps f, g such that $f(0) = 0$ we have that gf is a CT map.

Step 2: We prove the following claim: Let $g \colon (T_{m_0}M)^a \to (T_{m_0}M)^a$ be a CT map and let G be its flow, then G is CT map

Proof. As G is the flow of g, one can

Step 3: We prove the following claim:

Claim 4.33. *Let H be a CT map from an open subset of $(T_{m_0}M)^2 \times \mathbb{R}$ that contains the origin to $T_{m_0}M$. It is also given that:*

1) for every $t \in [0,1]$: $(0_{m_0}, 0_{m_0}, t) \in dom(H), H(0_{m_0}, 0_{m_0}, t) = 0_{m_0}$

2)For any $x, y \in T_{m_0}M$,$s, t \in R$ such that $(x, y, st), (x, sy, t) \in dom(H)$ we have $H(x, y, st) = H(x, sy, t)$

For any $c \in]0,1]$, let $\Lambda_c \colon (T_{m_0}M)^2 \to (T_{m_0}M)^2 \times \mathbb{R}$ be the smooth map given by $(x,y) \mapsto (x, \frac{y}{c}, c)$. Then $H\Lambda_1$ is a CT map.

Proof. Let $c \in]0,1]$ be arbitrary, by condition 2 of the hypothesis of the claim we have that $H\Lambda_1, H\Lambda_c$ locally agree at the origin. Hence for any natural number k, we have

$$\partial^k(H\Lambda_1)|_0 = \partial^k(H\Lambda_c)|_0.$$

Evaluate previous equation at $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ that are some arbitrary members of $(T_{m_0}M)^2$ then expand RHS using (35) to get

$$\partial^k(H\Lambda_1)|_0((x_1,y_1),(x_2,y_2),\ldots,(x_k,y_k)) = \partial^k H|_{(0,0,c)}[(x_i, \frac{y_i}{c}, 0)]_{i\in[k]}.$$

Set $\overline{1}$ to be $(0,0,1)$, $\overline{x_i}$ to be $(x_i, 0, 0)$ and $\overline{y_i}$ to be $(0, y_i, 0)$. Expand RHS of above equation to get

$$\partial^k(H\Lambda_1)|_0((x_1,y_1),(x_2,y_2),\ldots,(x_k,y_k)) = \sum_{S\subseteq[k]} \frac{1}{c^{|S|}} \partial^k H|_{(0,0,c)}[(\overline{x_i})_{i\in S^c}, (\overline{y_i})_{i\in S}].$$

Apply $\lim_{c\to 0^+}$ to above equation and use the observation (which will be explained later) that

$$\lim_{c\to 0^+}\frac{1}{c^{|S|}}\partial^k H|_{(0,0,c)}[(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}]=\frac{1}{|S|!}\frac{d^{|S|}}{dc^{|S|}}|_{c=0}\partial^k H|_{(0,0,c)}[(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}],$$

to get:

$$\partial^k(H\Lambda_1)|_0((x_1,y_1),(x_2,y_2),\dots,(x_k,y_k))=\sum_{S\subseteq[k]}\frac{1}{|S|!}\frac{d^{|S|}}{dc^{|S|}}|_{c=0}\partial^k H|_{(0,0,c)}[(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}]$$

$$\partial^k(H\Lambda_1)|_0((x_1,y_1),(x_2,y_2),\dots,(x_k,y_k))=\sum_{S\subseteq[k]}\frac{1}{|S|!}\partial^{k+|S|}H|_0[(\overline{1})_{i\in S},(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}]$$

The last equation shows how to express $\partial^k(H\Lambda_1)|_0$ using the derivatives of H at the origin which are CT tensors as H is a CT map. So all derivatives of $H\Lambda_1$ at the origin are CT tensors. Now we are done except that we have to show that

$$\lim_{c\to 0^+}\frac{1}{c^{|S|}}\partial^k H|_{(0,0,c)}[(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}]=\frac{1}{|S|!}\frac{d^{|S|}}{dc^{|S|}}|_{c=0}\partial^k H|_{(0,0,c)}[(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}].$$

as promised earlier. The above equation would follow from Taylor's theorem mean value theorem if we can show that for every $j<|S|$ we have

$$\frac{d^j}{dc^j}|_{c=0}[\partial^k H|_{(0,0,c)}[(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}]]=0.$$

Equivalently, we need to see that for every $j<|S|$ we have

$$\partial^{k+j}H|_0[(\overline{1})_{i\in[j]},(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}]=0. \tag{61}$$

Consider $L:(T_{m_0})^2\times\mathbb{R}\to(T_{m_0})^2$ given by $L(x,y,t)=(x,ty)$. Notice that by condition 2 of the claim we have that $H,H\Lambda_1L$ locally agree at the origin, hence

$$\partial^{k+j}H|_0[(\overline{1})_{i\in[j]},(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}]=\partial^{k+j}(H\Lambda_1L)|_0[(\overline{1})_{i\in[j]},(\overline{x_i})_{i\in S^c},(\overline{y_i})_{i\in S}].$$

RHS of the aobve equation can be seen to be 0 by applying (35) on $H\Lambda_1L$ as the composition of $H\Lambda_1$ and L. ☐

Step 4: Consider CT mpas $L_1\colon(T_{m_0}M)^3\to(T_{m_0}M)^3, L_2\colon T_{m_0}M\to(T_{m_0}M)^3$ given by $L_1(x,y,z)=(y,0,0), L_2(x)=(0,-x,-x)$ respectively. Clearly $L_1+L_2F^\phi$ will be an analytic CT map from an open subset of $(T_{m_0}M)^3$ to $(T_{m_0}M)^3$.Let F_* be flow of $L_1+L_2F^\phi$. Hence by step 2, F_* is an analytic CT map from an open subset of $(T_{m_0})^3\times\mathbb{R}$ to $(T_{m_0}M)^3$. Consider an analytic CT map Geo from a suitable open subset of $(T_{m_0}M)^2\times\mathbb{R}$ to $T_{m_0}M$ given by $Geo(x,y,t)=\pi_1F_*(x,y,y,t)$ (LHS is defined whenever RHS is, and $\pi_1\colon(T_{m_0}M)^3\to T_{m_0}M$ is 1st natural projection). Geo arose from F_* which is a flow to some other map, and so a solution to a certain ODE. One can verify that Geo satisfies the same ODE as the ODE of the geodesic equation ,in the coordinates ϕ, starting at $\phi(x)$ with velocity $D\phi|_xy$, and then it will follow that for every $(x,y,t)\in dom(Geo)$ we have

$$Geo(x,y,t)=\phi^{-1}[exp_{\phi(x)}(tD\phi|_xy)].$$

Set geo to be $Geo\circ\Lambda_1$ (Λ_1 is defined like in claim of step 3). Apply claim of step 3, to get that geo is an analytic CT map.

Step 5: The idea of step 5 is similar to that of step 4, so I write less details. By solving the parallel transport ODE in coordinates of ϕ (like how the geodesic equation ODE in coordinates of ϕ was considered in step 4), one can get an analytic CT map $Para$ from an open subset of $(T_{m_0}M)^2 \times \mathbb{R}$ to $T_{m_0}M$ such that:

1) $dom(Para) = \{(x, y, t) | x, tx \in dom(\phi)\}$

2) For any $x \in dom(\phi)$, if $\gamma_x :]-\frac{\delta_0}{|x|}, \frac{\delta_0}{|x|}[\to M$ denotes the geodesic given by $\gamma_x(\tau) = \phi(\tau x)$ and W denotes the unique parallel vector field along γ_x such that $W(0) = y$, then $W(t) = D\phi|_{tx} Para(x, y, t)$. By using claim of step 3, one can get that we have a CT map $para$ from an open subset of $(T_{m_0}M)^2 \to T_{m_0}M$ gievn by $para(x, y) = Para(x, y, 1)$.

Step 6: Consider a CT map $\overline{para}$ from an open subset of $(T_{m_0}M)^2$ to $T_{m_0}M)^2$ given by $\overline{para}(x, y) = (x, para(x, y))$. Now we get that $*$ is a CT map as it's the composition of CT maps ($* = geo \circ \overline{para}$), and we're done. □

Now, we use the previous theorems as machinery to prove a century old result([53])([54])([55])([56]).

Corollary 4.34 (Hausdorff–Campbell–Baker Formula)**.** *In a Lie group, pullback of multiplication using sufficiently small normal coordinates around the origin is analytic and could be expressed as an infinite series using the group's Lie bracket.*

Proof. Let G be a Lie group. Equip it with the unique connection ∇ that makes left invariant vector fields as parallel vector fields. One verifies that for this choice of connection we have $\nabla^k R, \nabla^{k+1} T$ vanish for all $k \geq 0$. This gives us that (G, ∇) is tame. Take m_0 of Theorem 4.32 to be the identity of G. One verifies that $*$ described in Theorem 4.32 will coincide with the pull back of group operation along sufficiently small normal coordinates (which will be ϕ as in proof of Theorem 4.32). Hence, by Taylor's theorem $*$ could be expressed as an infinite sum of CT polynomials. However, CT polynomials in the setting of this corollary are nothing but Lie polynomials because $T|_e$ is the additive inverse of the group's Lie bracket and $\nabla^k R|_e, \nabla^{k+1} T|_e$ vanish for all $k \geq 0$. □

Theorem 4.35. *If (M, ∇) is tame, then it is normal-analytic.*

Proof. Our goal is to equip M with an analytic atlas consisting of normal coordinates, such that the Cristoffel symbols are analytic. Retain the setting of the proof of the previous theorem. Then ϕ is a restriciton of exp_{m_0} to an open set around the origin such that $\phi\colon dom\phi) \to Im(\phi)$ is diffeomorphism and such that the Cristoffel tensor $F^\phi\colon dom(\phi) \times (T_{m_0}M)^2 \to T_{m_0}M$ is analytic. Let m be arbitrary in $Im(\phi)$. On some sufficiently small open $U_m^{m_0}$ set around the origin 0_m of T_mM, we have for all $y \in U_m^{m_0}$ that

$$\phi^{-1} exp_m(y) = geo(\phi^{-1}(m), [D\phi|_{\phi^{-1}(m)}]^{-1}(y)).$$

geo is analytic as shown in proof of previous theorem, hence RHS depends analytically on y. Thus, $\phi^{-1} exp_m | U_m^{m_0} \colon U_m^{m_0} \to T_{m_0}M$ is an analytic map. To summarize what we have so far: For every $p \in M$, we get a normal coordinates diffeomorphism ϕ_p between an open set of T_pM around the origin, and an open subset of M around p such that $F^{\phi_p}\colon dom(\phi_p) \times (T_pM)^2 \to T_pM$ is analytic, and for every $q \in Im(\phi_p)$ we get an open set U_q^p around the origin of T_qM such that $\phi_p^{-1} exp_q | U_q^p \colon U_q^p \to T_pM$ is analytic.By inverse function theorem and restricting U_q^p to a smaller open subset if necessary, we may also assume WLOG that $exp_q | U_q^p$ is a diffeomorphism between U_q^p and an open subset of M around q.

We take our atlas to be the family $\{\phi_p\colon dom(\phi_p) \to Im(\phi_p)\}_{p \in M}$. Clearly, the cristoffel symbols are analytic with respect to our member of our atlas. Next we show that the transition maps are analytic by showing that they are locally analytic. Let x be any member of $dom(\phi_{p_2}^{-1}\phi_{p_1})$. Set m to be $\phi_{p_1}(x)$. By definition we have that the maps below are analytic:

$$\phi_{p_1}^{-1}exp_m|U_m^{p_1} : U_m^{p_1} \to T_{p_1}M$$
$$\phi_{p_2}^{-1}exp_m|U_m^{p_2} : U_m^{p_2} \to T_{p_2}M$$

Now note that on a sufficiently small open subset around x, we have that $\phi_{p_2}^{-1}\phi_{p_1}$ equals $(\phi_{p_2}^{-1}exp_m|U_m^{p_2}) \circ (\phi_{p_1}^{-1}exp_m|U_m^{p_1})^{-1}$, where the latter is a composition of analytic functions so it is analytic. □

4.5 Connection with a prescribed CT Algebra

We prove our main theorem in this section

Theorem 4.36. *Let $(\mathcal{A}, \{R^k : \mathcal{A}^{k+3} \to \mathcal{A}\}_{k\geq 0}, \{T^k : \mathcal{A}^{k+2} \to \mathcal{A}\}_{k\geq 0})$ be a finite dimensional CT algebra. Assume that there exists a norm on $\mathcal{A}$ and $C, L \geq 0$ such that $|T^k|, |R^k| \leq Ck!L^k$ for every natural number k. Then there exists a pointed manifold (M, m_0) equipped with a connection ∇ such that $T_{m_0}M = \mathcal{A}$.*

Proof. Section 4 gives us a formula for the Christoffel symbols in normal coordinates centered at m_0 using $\{\nabla^k R(m_0)\}_{k\geq 0}$ and $\{\nabla^k T(m_0)\}_{k\geq 0}$. We "steal" this formula and replace each occurrence of $\nabla^k R(m_0)$ in the formula by R^k, and each occurence of $\nabla^k T(m_0)$ by T^k. This will give us a connection $\overline{\nabla}$ on a small ball $B_\delta(0)$ centered around the origin of $\mathcal{A}$ (small ball to ensure that the infinite sum in the formula converges). More precisely, let $\overline{q}_k$ be the CT polynomial produced from taking the formula of q_k of section 4 and replacing each occurrence of $\nabla^k R(m_0)$ by R^k and each occurence of $\nabla^k T(m_0)$ by T^k. Finally, we define $\overline{\nabla}$ by requiring that for any $u \in B_\delta(0), v, w \in \mathcal{A}$ we have : $\overline{\nabla}_{\overline{w}}\overline{v}|_u = \sum_{k=2}^{\infty} \frac{\overline{q}_k(u,v,w)}{k!}$ (Where for any $a \in \mathcal{A}$, $\overline{a}$ denotes the smooth vector field over $B_\delta(0)$ that sends each point to a.)

Set the triplet (M, ∇, m_0) to be $(B_\delta(0), \overline{\nabla}, 0)$ The only remaining thing is to verify that $T_{m_0}M = \mathcal{A}$. First note that for any $u \in B_\delta(0)$ we have that $\gamma_u : [0,1] \to B_\delta(0)$ given by $\gamma_u(t) = tu$ is geodesic in $(B_\delta(0), \overline{\nabla})$. That's because for any $t \in [0,1]$ we have

$$\overline{\nabla}\gamma_u'|_t = \overline{\nabla}_{\overline{u}}\overline{u}|_{tu} = 0.$$

In the above equation, we have that $\overline{\nabla}_{\overline{u}}\overline{u}|_{tu} = 0$ because by def of $\overline{\nabla}$ we have that $\overline{\nabla}_{\overline{u}}\overline{u}|_{tu}$ is an infinite sum of CT polynomials evaluated at (tu, u, u). By skew symmetry, a CT polynomial vanishes when evaluated at (tu, u, u). This observation about geodesics of $(B_\delta(0), \overline{\nabla})$ gives that $\exp_0 : B_\delta(0) \to B_\delta(0)$ is just the identity map.

As exp_0 is diffeomorphism with convex domain, so we are in a similar position as that of section 4. Retain all notation, definitions, propositions of section 4. For any $t \in [0,1]$ we have

$$\begin{aligned}\alpha(t) &= (T_{tu}exp_0)^{-1}[\overline{\nabla}_{E_w}E_v|_{exp_0(tu)}] = \overline{\nabla}_{\overline{w}}\overline{v}|_{tu} \\ &= \sum_{k=2}^{\infty} \frac{\overline{q}_k(tu,v,w)}{k!} = \sum_{k=2}^{\infty} t^{k-2}\frac{\overline{q}_k(u,v,w)}{k!}.\end{aligned}$$

Differentiating the above power series for α we get that $\alpha^{(k)}(0) = \overline{q}_{k+2}(u,v,w)$ for every natural k. However, by section 4 we also know that $\alpha^{(k)}(0) = q_{k+2}(u,v,w)$ for every natural k. Hence,

$$\overline{q}_{k+2}(u,v,w) = q_{k+2}(u,v,w)$$

for every natural k and every $u \in B_\delta(0), v, w \in \mathcal{A}$. By scaling the u-argument of the previous equation appropriately, we get that $\overline{q}_k = q_k$ for every natural k. Finally an induction argument on k with the help of

Lemma 4.8 applied on previous statement will give that $\nabla^k R(0) = R^k, \nabla^k T(0) = T^k$ for all natural numbers k. $\square$

4.6 CT Algebras in the Riemannian Case

Definition 4.37. Let $(\mathcal{A}, \{R^k : \mathcal{A}^{k+3} \to \mathcal{A}\}_{k\geq 0}, \{T^k : \mathcal{A}^{k+2} \to \mathcal{A}\}_{k\geq 0})$ be a CT algebra such that T^k is zero for all k, $\mathcal{A}$ is equipped with an inner product $\circ$ such that for every natural k, and every $u_1, u_2, \dots, u_k, x, y \in \mathcal{A}$ we have that the endomorphism of the inner product space $(\mathcal{A}, \circ)$ given by $z \to R^k(u_{[k]}, x, y, z)$ is skew symmetric. Then the structure $\left(\mathcal{A}, \{R^k : \mathcal{A}^{k+3} \to \mathcal{A}\}_{k\geq 0}, \{T^k : \mathcal{A}^{k+2} \to \mathcal{A}_{k\geq 0}, \circ\right)$ is said to be a *Riemannian curvature torsion algebra*, abbreviated as "RCT algebra".

Fact 4.38. *Let (M, g, m_0) be a pointed Riemannian manifold and let ∇ be it's Levi-Civita connection. Then $T_{m_0}M$ is an RCT algebra.*

Proof. Easy verification by differentiating the symmetries of the Riemann curvature tensor sufficiently many times.

Fact 4.39 (Semisimplicity of RCT algebras)**.** *Let $(\mathcal{A}, \{R^k : \mathcal{A}^{k+3} \to \mathcal{A}\}_{k\geq 0}, \circ)$ be an RCT algebra and let I be an ideal of $\mathcal{A}$, then $I^\perp$ is also an ideal of $\mathcal{A}$. Thus, $\mathcal{A} = I \oplus I^\perp$ splits as a direct sum of two RCT algebras.*

Proof. We prove by induction on k that for all $k \geq 0$ we have $R^k(x_k, \dots, x_2, x_1, p, q, r) \in I^\perp$ whenever at least one of $x_k, \dots, x_2, x_1, p, q, r$ belongs to $I^\perp$. Equivalently, it suffices to show that for all $k \geq 0$, we have $R^k(x_k, \dots, x_2, x_1, p, q, r) \circ s = 0$ whenever $s \in I$ and whenever at least one of $x_k, \dots, x_2, x_1, p, q, r$ belongs to $I^\perp$. We consider five cases below.

Case 1 ($p \in I^\perp$) : By symmetries of algebraic curvature tensors we have $R^k(x_k, \dots, x_2, x_1, p, q, r) \circ s$ equals $-R^k(x_k, \dots, x_2, x_1, r, s, q) \circ p$ which equals 0 because $p \in I^\perp, R^k(x_k, \dots, x_2, x_1, r, s, q) \in I$ (as $s \in I$ and I is an ideal of $\mathcal{A}$).

Case 2 ($q \in I^\perp$) : By skew symmetry we have $R^k(x_k, \dots, x_2, x_1, p, q, r) \circ s$ equals $-R^k(x_k, \dots, x_2, x_1, q, p, r) \circ s$ which equals 0 by case 1.

Case 3 ($r \in I^\perp$) : By Second Bianchi symmetry (i.e. axiom (3) of Definition 4.4) of algebraic curvature tensors we have:

$$R^k(x_k, \dots, x_2, x_1, p, q, r) \circ s = -R^k(x_k, \dots, x_2, x_1, r, p, q) \circ s - R^k(x_k, \dots, x_2, x_1, q, r, p) \circ s \tag{62}$$

The RHS of (62) vanishes by cases 1,2, and so its LHS vanishes as well.

Case 4 ($x_1 \in I^\perp$) : By first Bianchi symmetry (i.e. axiom (2) of Definition 4.4) of algebraic curvature tensors we have:

$$R^k(x_k, \dots, x_2, x_1, p, q, r) \circ s = -R^k(x_k, \dots, x_2, p, q, x_1, r) \circ s - R^k(x_k, \dots, x_2, q, x_1, p, r) \circ s \tag{63}$$

The RHS of (63) vanishes by cases 1,2, thus LHS of (63) vanishesas well.

Case 5 ($x_i \in I^\perp$ for some $i \geq 2$): This is the case where we need to use the induction hypothesis. By axiom (4) of Definition 4.4 we may keep interchanging the position of x_i step by step until it becomes the fourth rightmost argument of R^k, then we may apply case 4 and the induction hypothesis to finish the argument. We illustrate this idea explicitly below in case $i = 2$ as an example. In the equation below, $(u_1, u_2, ..., u_{k-2})$ is defined to be $(x_k, \dots, x_4, x_3)$.

$$\begin{aligned} R^k(x_k, \dots, x_3, x_2, x_1, p, q, r) &= +R^k(x_k, \dots, x_3, x_1, x_2, p, q, r) \\ &= \sum_{S \subseteq [k-2]} \Bigg(R^{|S|}(u_S, x_2, x_1, R^{|S^c|}(u_{S^c}, p, q, r)) \\ &- \sum_{i=1}^{3} R^{|S|}(u_S, R^{|S^c|}(u_{S^c}, x_2, x_1)^i(p, q, r)) \Bigg) \end{aligned}$$

.

The R^k term in above equation belongs to $I^\perp$ by case 4, and the sum that ranges over all subsets of $[k-2]$ in above equation also belongs to $I^\perp$ by induction hypothesis. Thus, $R^k(x_k, \dots, x_3, x_2, x_1, p, q, r) \in I^\perp$ and we are done.

□

A kind of a converse of to Fact 4.38 is stated in the theorem below:

Theorem 4.40 (Kowalski–Belger)**.** *Let $(\mathcal{A}, \{R^k\}_{k\geq 0}, \{T^k\}_{k\geq 0}, \circ)$ be a finite dimensional RCT Algebra of a finite radius of convergence (i.e. satisfies hypothesis of Theorem 4.36) then there exists a pointed Riemannian manifold (M, g, m_0) such that:*

$$(T_{m_0}M, \{\nabla^k R|_{m_0}\}_{k\geq 0}, \{\nabla^k T|_{m_0}\}_{k\geq 0}, g(m_0)) = (\mathcal{A}, \{R^k\}_{k\geq 0}, \{T^k\}_{k\geq 0}, \circ)$$

Proof. This was proven by Kowalski and Belger (50). □

Now we wish to answer a natural question: Given an algebraic curvature tensor, can one extend it to an RCT algebra? Theorem 4.42 of this sections says yes, and proves a stronger statement. First, let's recall a lemma due to Gilkey (57).

Lemma 4.41 (Gilkey)**.** *Let $\mathcal{A}$ be a finite dimensional vector space over the real numbers, and let T be a $(k+4, 0)$ tensor on V $(k \geq 0)$ such that*

1) T is an algebraic curvature tensor in its rightmost four arguments.

2) T is symmetric in its leftmost k arguments

3) $\sum_{i \in Z_3} T(u_1 u_2 \dots u_{k-1} x_i x_{i+1} x_{i+2} y z) = 0$, for every $u_1, u_2, \dots, u_{k-1} \in \mathcal{A}$, and for every $x_0, x_1, x_2, y, z \in \mathcal{A}$

Then T is the zero tensor if and only if $T(u^k, v, u, u, v) = 0$ for all $u, v \in \mathcal{A}$.

Proof. Similar to the proof of lemma 2 of (57). □

Theorem 4.42. *Let $(\mathcal{A}, \circ)$ be any finite dimensional inner product space and let $\{M^k : \mathcal{A}^{k+4} \to \mathcal{A}\}_{k\geq 0}$ be a sequence of multilinear maps that are skew symmetric in the rightmost first and second argument, and skew symmetric in the rightmost third and fourth argument. It is also given that for every $k \geq 0, x \in \mathcal{A}$ we have that $M^k(x^k y_1 x x y_2)$ is symmetric in arguments y_1, y_2. Then there exists a unique RCT algebra structure $(\mathcal{A}, \{R^k\}_{k\geq 0}, \{T^k\}_{k\geq 0}, \circ)$ on $(\mathcal{A}, \circ)$ such that for every*

$$k \geq 0, x, y \in \mathcal{A} \text{ we have:} R^k(x^k y x x y) = M^k(x^k y x x y) \tag{64}$$

Proof. Lemma 4.41 combined with induction on k will give us uniqueness, so we focus on existence. Since we wish to have an RCT algebra structure, set all the T^k(s) to be zero. Next, we construct the sequence $\{R_k\}_{k\geq 0}$ recursively.Let $k \geq 0$ be arbitrary, suppose we already constructed $R^0, R^1, \ldots, R^{k-1}$ and wish to construct R^k so that the symmetries of the RCT algebra (i.e. the Ricci identities), and so that (64) holds. Put a Riemannian metric on a sufficiently small (to ensure positive definiteness) open ball centered at origin of $(\mathcal{A}, \circ)$) defined by

$$g|_v(w,w) = \sum_{i=2}^{k+4}\sum_{a=1}^{i-1} \frac{p^a(v,w) \circ p^{i-a}(v,w)}{a!(i-a)!}, \tag{65}$$

where $p^l(v,w)$ is defined recursively as follows: $p^0(v,w) = 0, p^1(v,w) = w$, and for every $l \in \{0, 1, \ldots, k+1\}$, we set

$$p^{l+2}(v,w) = -\sum_{j=0}^{l-1} \binom{l}{j} M^j(v^j, p^{l-j}(v,w), v, v).$$

(In the above equation, $M^j(v^j xvv)$ is the unique member of $\mathcal{A}$ such that $M^j(v^j xvv) \circ y = M^j(v^j xvvy)$ for all $y \in \mathcal{A}$) . The next step is to note that the exponential map, exp_0, of the metric g is just the identity map. This can be proven by showing (using the skew symmetries of the tensors $\{M^k\}_{k\geq 0}$) that linear curves of the form $t \mapsto tu$, for any $u \in \mathcal{A}$, are critical points of the energy functional under proper variations and so they are geodesics.

It is well known that the power series expansion of $exp_0^* g|_{tv}(w,w)$ has the form below

$$exp_0^* g|_{tv}(w,w) = \sum_{i=2}^{k+4}(\sum_{a=1}^{i-1} \frac{g|_0(q^a(v,w), q^{i-a}(v,w))}{a!(i-a)!})t^{i-2} + O(t^{k+3}). \tag{66}$$

Where $q^l(v,w)$ is defined recursively as follows: $q^0(v,w) = 0, q^1(v,w) = w$, and for every $l \in \{0, 1, \ldots, k+1\}$:

$$q^{l+2}(v,w) = -\sum_{j=0}^{l-1} \binom{l}{j} \nabla^j R|_0(v^j, p^{l-j}(v,w), v, v)$$

Replace v by tv in (65), then compare the resulting power series to that of (66) along with using the fact that the $exp_0 = id$ to get that for every $i \in \{2, 3, \ldots, k+4\}$ and for every $v, w \in \mathcal{A}$ we have

$$\sum_{a=1}^{i-1} \frac{p^a(v,w) \circ p^{i-a}(v,w)}{a!(i-a)!} = \sum_{a=1}^{i-1} \frac{g|_0(q^a(v,w), q^{i-a}(v,w))}{a!(i-a)!}.$$

The above equation (ranging over all $i \in \{2, 3, \ldots, k+4\}$) along with Lemma 4.41 gives $g|_0 = \circ$ and $\nabla^j R|_0 = R^j$ for all $j \in \{0, 1, 2, \ldots, k-1\}$ by induction on j. Set R^k to be $\nabla^k R|_0$ and we're done.

□

As an application of the methods of this chapter we prove the three dimensional case of a result due to DeTurck and Kazdan(58).

Theorem 4.43. *Let (M, g) be a Riemannian 3-manifold with the property that $\nabla\rho(X, X, X) = 0$ for all X (Where ρ is the Ricci tensor of (M, g)), then (M, g) is tame and sp admits an analytic atlas consisting of normal coordinates by Theorem 4.35.*

Proof. To reduce the length of notation, we write (A_1, A_2, A_3, P, Q) as a short hand for $\nabla^3\rho(A_1, A_2, A_3, P, Q)$. For any vector fields A_1, A_2, A_3, P, Q and B_1, B_2, B_3, X, Y we write $(A_1, A_2, A_3, P, Q) \approx (B_1, B_2, B_3, X, Y)$ iff the difference:

$$(A_1, A_2, A_3, P, Q) - (B_1, B_2, B_3, X, Y)$$

can be expressed naturally using the vector fields $A_1, A_2, A_3, P, Q, B_1, B_2, B_3, X, Y$ and using a finitely many applications of $R, \nabla R, \nabla^2 R$. It is clear that $\approx$ gives an equivalence relation that respects addition and multiplication by scalars. Differentiating the identity $\nabla\rho(X, X, X) = 0$ twice gives the identity $(A, B, X, X, X) = 0$ for all A, B, X. Polarizing the X variable of the previous equation gives us the cyclic identity below:

$$(A, B, X, Y, Z) + (A, B, Y, Z, X) + (A, B, Z, X, Y) \approx 0$$

The Ricci identity also gives us (A, B, C, X, Y) is symmetric in the A, B, C arguments with respect to $\approx$. By symmetries of the Ricci tensor, we also know that (A, B, C, X, Y) is symmetric in its two rightmost arguments. Next we use these identities to make the manipulations below:

$$(A, B, C, X, Y) \approx -(A, B, X, Y, C) - (A, B, Y, C, X) \approx -(A, X, B, Y, C) - (A, Y, B, C, X) \approx$$

$$[(A, X, Y, C, B)] + (A, X, C, B, Y)] + [(A, Y, C, X, B) + (A, Y, X, B, C)] \approx$$

$$[(A, Y, X, B, C) + (A, C, X, B, Y)] + [(A, C, Y, X, B) + (A, Y, X, B, C)] \approx$$

$$2(A, Y, X, B, C) + (A, C, X, B, Y) + (A, C, Y, X, B) \approx 2(A, Y, X, B, C) - (A, C, B, Y, X)) \approx$$

$$2(A, Y, X, B, C) - (A, B, C, X, Y)$$

Thus, $(A, B, C, X, Y) \approx 2(A, X, Y, B, C) - (A, B, C, X, Y))$. This gives that

$$(A, B, C, X, Y) \approx (A, X, Y, B, C) \tag{67}$$

(67) gives that $(U, U, U, X, Y) \approx (U, X, Y, U, U) \approx (X, Y, U, U, U) \approx 0$. Thus, $(U, U, U, X, Y) \approx 0$. Polarize this identity in the U variable and use the symmetry of $(-, -, -, -, -)$ in the leftmost three variables to get that $(U_1, U_2, U_3, X, Y) \approx 0$. Thus $\nabla^3\rho$ can be expressed by a formula using $R, \nabla R$ that holds on all the manifold. Since in dimension three R can be recovered from the Ricci tensor ρ, thus it follows that $\nabla^3 R$ can be expressed polynomially using $R, \nabla R$ that holds on all the manifolds. Differentiating this polynomial sufficiently many times will give an exponential bound on the growth rate of $\nabla^k R$ as k grows that will allow to prove (M, g) is tame. $\square$

Fact 4.44. *Let (M, g) be a Riemannian manifold that satisfies hypothesis of Theorem 3.12, then M is tame and so admits an analytic atlas consisting of normal coordinates by Theorem 4.35.*

Proof. Theorem 3.12 shows us that $\nabla^3 R$ can be expressed polynomially using $R, \nabla R$. Differentiating this polynomial sufficiently many times will give an exponential bound on the growth rate of $\nabla^k R$ as k grows that will allow to prove (M, g) is tame . $\square$

Motivated by the proofs of Theorem 3.12 and Theorem 4.43, we introduce a special class called of CT algebras called the "Algebraic CT algebras". The definition might not state all details explicitly, but this okay for two reasons. Firstly, this defintion will not be used in any of the results of this thesis but will only be used to discuss some conjectures in the last section "Future Research". Secondly, I am not sure yet if this is the "right" notion of what counts as an algebraic CT algebra.

Definition 4.45. Let $(\mathcal{A}, \{R^k : \mathcal{A}^{k+3} \to \mathcal{A}\}_{k\geq 0}, \{T^k : \mathcal{A}^{k+2} \to \mathcal{A}\}_{k\geq 0})$ be a CT algebra. We say it is algebraic iff there exists a sufficiently large natural number N such that R^N, T^N can be "expressed" using $R^0, R^1, R^2, \ldots, R^{N-1}, T^0, T^1, \ldots, T^{N-1}$. Furthermore, all formal derivatives (see example below to know what I mean by all formal derivatives) of this "expression" remain valid.

For example, let's say $(\mathcal{A}, \{R^k : \mathcal{A}^{k+3} \to \mathcal{A}\}_{k\geq 0}$ is a CT algebra with the property that $T^2 = 0$, and

$$R^2(x_1, x_2, x_3, x_4, x_5) = 7R^0(x_1, x_2, R^0(x_3, x_4, x_5)) + 2T^0(R^1(x_1, x_2, x_5, x_4), x_3) \tag{68}$$

for all $x_1, x_2, x_3, x_4, x_5 \in \mathcal{A}$. For this setup to yield an algebraic CT algebra, we also need to require that all formal derivatives of the previous two equations remain valid in $\mathcal{A}$. This will give us that we need $\mathcal{A}$ to have the property that $T^k = 0$ for all $k \geq 2$. Furthermore, we will need $\mathcal{A}$ to satisfy the relations resulting from formally differentiating (68) with respect to u_1, and formally (68) with respect to u_1, u_2, and formally (68) with respect to u_1, u_2, u_3 etc etc. Thus we will need $\mathcal{A}$ to satisfy the relation below for all $u_1, u_2, \ldots, u_r, x_1, x_2, x_3, x_4, x_5 \in \mathcal{A}$:

$$R^{r+2}(u_{[r]}, x_1, x_2, x_3, x_4, x_5) = 7\sum_{S\subseteq[r]} R^{|S|}(u_S, x_1, x_2, R^{|S^c|+1}(u_{S^c}, x_3, x_4, x_5))+$$

$$+2\sum_{S\subseteq[r]} T^{|S|}(u_S, R^{1+|S^c|}(u_{S^c}, x_1, x_2, x_5, x_4), x_3)$$

Thus, a CT algebra $\mathcal{A}$ that satisfies the above equation for all $u_1, u_2, \ldots, u_r, x_1, x_2, x_3, x_4, x_5 \in \mathcal{A}$ and satisfies $T^k = 0$ for all $k \geq 2$ would be an example of an algebraic CT algebra.

One sees easily that Lie algebras, Lie Triple systems are algebraic CT algebras. Theorem 3.12 and Theorem 4.43 give us that their respective manifolds posses algebraic CT algebras.

5 Future Research

In this section we pause some problems motivated by the results of the previous chapters:

1. Theorem 4.36 is a local analogue of Theorem 1.6. Thus, it's natural to ask if there exists a global analogue of Theorem 4.36. We introduce some definitions to formulate a conjectural global analouge of Theorem 4.36.

   Definition 5.1 (Maximal Manifold)**.** . Let (M, ∇) be a connected tame n-dimensional boundaryless manifold equipped with a connection (Recall Definition 4.23). We say (M, ∇) is maximal iff for every tame connected n-dimensional boundaryless manifold $(\overline{M}, \overline{\nabla})$ such that M is an open submanifold of $\overline{M}$ and such that the inclusion map $i : (M, \nabla) \to (\overline{M}, \overline{\nabla})$ is a connection preserving map, we must have $M = \overline{M}$

   Now we are are ready to formulate a conjectural global analogue of Theorem 4.36

   Conjecture 5.2. Let $(\mathcal{A}, \left\{R^k \colon \mathcal{A}^{k+3} \to \mathcal{A}\right\}_{k\geq 0}, \left\{T^k \colon \mathcal{A}^{k+2} \to \mathcal{A}\right\}_{k\geq 0})$ be a finite dimensional CT algebra. Assume that there exists a norm on $\mathcal{A}$ and $C, L \geq 0$ such that $\left|T^k\right|, \left|R^k\right| \leq Ck!L^k$ for every natural number k. Then there exists a pointed simply connected manifold (M, m_0) equipped with a connection ∇ such that (M, ∇) is tame and maximal, and such that $T_{m_0}(M, \nabla) = \mathcal{A}$. Furthermore, the triplet (M, ∇, m_0) is unique up to isomorphisms.

   I already made considerable progress to prove the above conjecture, but a complete proof is not ready yet. I would like to discuss below some of the consequences of Truth of the above conjecture. Given a CT algebra $\mathcal{A}$, I use the notation $Max(\mathcal{A})$ to denote the manifold whose existence is guranteed by the truth of the above conjecture.

   Conjecture 5.3. Assume the truth of conjecture 1 above. Let $\mathcal{A}$ be an algebraic CT algebra. Then $Max(\mathcal{A})$ is geodesic-ally complete.

   The truths of Conjecture 5.3 along with Conjecture 5.2 and its Riemannian analogue can be used to deduce the following well known facts as special cases:

   (a) For every positive integer n, there exists a simply connected geodesically complete riemannian manifold H^n of constant curvature -1. This can proven assuming the truths of Conjecture 5.3 combined with Conjecture 5.2 and its Riemannian analogue along with taking our RCT algebra to be $\mathbb{R}^n$ equipped with euclidean dot product, and setting R^0 to be given by $R^0(x, y, z) = -[(y \circ z)x - (x \circ z)y]$ and setting R^k to be zero for every $k \geq 1$.

   (b) Theorem 1.6 follows taking our CT algebra to be $\mathfrak{g}$ to get a simply connected geodesically manifold (M, ∇) whose connection has a parallel Torsion tensor and everywhere vanishing curvature. Next use Theorem 1.7 to get tat (M, ∇) is actually the Lie group we are looking for.

   (c) By assuming the truths of Conjecture 5.3 along with Conjecture 5.2 and its Riemannian analogue then applying them to the case where we take our CT algebra is any Lie triple system, we get Theorem 1.10.

   The truths of Conjecture 5.3 and Conjecture 5.2 may also be used to give a version of De Rham decomposition theorem formulated using CT algebras as follows: Let $\mathcal{A}_1, \mathcal{A}_2$ be two finite dimensional CT algebras with finite radii of convergence. It follows $\mathcal{A}_1 \oplus \mathcal{A}_2$ will also be a finite dimensional CT algebra with a finite radius of convergence. It might be possible to use the CT correspondence theorems to get that $Max(\mathcal{A}_1 \oplus \mathcal{A}_2) \cong Max(\mathcal{A}_1) \times Max(\mathcal{A}_2)$.

2. Theorem 4.43 gives us that in dimension 3, manifolds satisfying condition (2) must have algebraic CT algebras. I used a computer search to check whether four dimensional RCT algerbas satisfying (2) are algebraic or not, and the answer seems to be not. A very long computation by hand, assuming no errors were made, also gives that the (2) condition and all of its formal derivatives do not force the RCT algebra to be algebraic when the dimension is at least 4.

   Question 5.4. Do the conditions (2),(4) and all of its formal derivatives force the RCT algebra to be algebraic in the four dimensional case ?

3. Theorem 3.12 tells us that the geodesic tube property forces the RCT algebra to be algebraic by showing that $\nabla^3 R$ can be expressed using $R, \nabla R$. This was first noticed via a computer search. Interestingly, the computer search also revealed that there will be some nonlinear constraints relating $\nabla R, R$ together, however the number of constraints is not enough to force ∇R to be expressible using R. Differentiating these constraints will give further constraints relating $R, \nabla R, \nabla^2 R$. This time the number of constraints might be sufficiently large to force $\nabla^2 R$ to be expressible using $R, \nabla R$ which will be a stronger statement than Theorem 3.12. This might allow for a classification of the RCT algebras of *Theorem* 3.12, and will probably give a classification of simply connected complete 3-manifolds satisfying hypothesis of Theorem 3.12. The challenge with this approach is to find a way to shorten the long computations.

4. *Theorem* 3.11 deduces that M has to be a D'Atri space under the additional assumption that M has bounded sectional curvature. The proof uses the power series expansion of the scalar curvature of a tube of small radius r truncated until the r^4 term. One might go further in the power expansion until the r^6 term to deduce further constraints on the RCT algebra T_pM that can aid in the classification of the possible isomorphism classes of the RCT algebra T_pM, which will probably lead to the classifcation of the possible isomorphism classes of M. Once again, the challenge is to deal with the long formulas obtained from the r^6 term.

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