An insight into the Foulkes conjecture and the Generalized Foulkes module

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I would like to dedicate this thesis to my loving parents.

Declaration

I hereby declare that the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others.

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Abstract

For integers, a, b, the Foulkes moudule $F_{(b)}^{(a)}$, is the permutation module of the symmetric group, S_{ab} acting on partitions of the ab elements into b sets of size a each. In 1950, Foulkes [1] conjectured that if $b > a$ then $F_{(a)}^{(b)}$ is a submodule of $F_{(b)}^{(a)}$. The Foulkes conjecture and the properties of the Foulkes module have been of interest for researchers in both the fields of Algebraic Combinatorics and Representation theory of Symmetric groups. Another topic of interest is the Generalized Foulkes module where we generalize $F_{(b)}^{(a)}$ with the parameter being the partitions of b.

In Chapter 3 we restrict the generalized Foulkes module to some large subgroups and in turn find a weak connection to Kronecker coefficients. We also prove a corollary that is based on this connection and obtain a result on the multiplicities of certain irreducible modules in the generalized Foulkes module.

In Chapter 4 we compare the Foulkes character values on permutations of prime power order p^k. We prove that the character value decays exponentially as k increases. We also prove that if ab is a prime power then the restriction of Foulkes conjecture to the group generated by a long cycle is true.

In Chapter 5 we compute the Foulkes character on involutions and examine if the conjecture is true when we restrict it to elementary abelian subgroups.

Table of contents

Chapter 1

Introduction

The thesis focuses on the permutation module $F^{(a)}_{(b)}$ for symmetric group S_{ab}, the Foulkes module, and its generalized form $F^{(a)}_{\nu}$ with a partition parameter $\nu \vdash b$.

The Foulkes module $F^{(a)}_{(b)}$ is defined as the transitive permutation module of S_{ab} acting on the set-partitions of $\{1, .., ab\}$ into b sets of size a each, in other words it is an induced module $F^{(a)}_{(b)} \cong 1_{S_a \wr S_b} \uparrow^{S_{ab}}$. In 1942 Thrall [2] considered the cases $F^{(2)}_{(b)}$ and $F^{(a)}_{(2)}$ and gave their decompositions into irreducible Specht modules. To describe the result we first define dominating partitions.

Definition 1. A partition λ is said to dominate γ, $\lambda \trianglerighteq \gamma$, if $\forall i$

$$\sum_{j=0}^{i} \lambda_j \geq \sum_{j=0}^{i} \gamma_j.$$

A partition is even if each part of it is even. In particular, for any λ, the double of it, $2\lambda = (2\lambda_1, \ldots, 2\lambda_k)$ is even. The following are the decompositions obtained by Thrall:

$$F^{(2)}_{(b)} = \sum_{\lambda \vdash b} S^{2\lambda};$$

$$F^{(a)}_{(2)} = \sum_{\substack{\lambda \vdash a \\ 2\lambda \trianglerighteq (a^2)}} S^{2\lambda}.$$

In 1950 Foulkes [1], while computing the multiplicities of S^λ in $F^{(a)}_{(b)}$ for certain a's and b's, hypothesised that if $a \leq b$

$$F^{(b)}_{(a)} \subseteq F^{(a)}_{(b)}. \tag{1.1}$$

The hypothesis is now known as the Foulkes Conjecture. By Thrall's result, Foulkes conjecture is true for $a = 2$. In 2000 Dent and Siemons [3] established an upper bound on multiplicities of Specht modules S^λ in $F^{(b)}_{(3)}$ and a lower bound in $F^{(3)}_{(b)}$. Consequently they proved that Foulkes conjecture holds for $a = 3$ by concluding that the multiplicity of S^λ in $F^{(b)}_{(3)}$ is less then that of $F^{(3)}_{(b)}$.

Let $n = ab = cd, a, c, d < b$. In 2003 Vessenes [4] increased the scope of conjecture. She hypothesised that there is an embedding from $F^{(b)}_{(a)}$ into $F^{(d)}_{(c)}$. She proved her hypothesis for the case $a = 2$. She also proved that every irreducible character occurring in $F^{(b)}_{(3)}$ occurs also in $F^{(d)}_{(c)}$. In her proof she employed Doran's [5] idea of tableaux constructed by him in 1998.

Another noteworthy variation is the SWS conjecture. Let $\lambda = (\lambda_1^{n_1}, \ldots, \lambda_k^{n_k}) \vdash n$. The module F^λ is defined as

$$F^\lambda = (F^{\lambda_1}_{(n_1)} \otimes .. \otimes F^{\lambda_k}_{(n_k)}) \uparrow^{S_n} .$$

Let $\overline{\lambda}$ be the conjugate partition of λ. The SWS conjecture claims that if $\overline{\lambda} \trianglerighteq \lambda$ then there is an embedding from $F^{\overline{\lambda}}$ into F^λ. In general the conjecture has been refuted. Counterexamples exist for many partitions.

For $\lambda = (a^b)$, the SWS conjecture is the Foulkes conjecture. In 2004 Mckay [6] proved that the SWS conjecture holds under certain conditions. Before we go into McKay's condition we define the sum of partitions λ, μ.

Definition 2. Let $\lambda \vdash n, \mu \vdash m$. The sum $\nu = \lambda + \mu$ is defined as the partition $(\nu_1, \ldots, \nu_k)$ such that $\nu_i = \lambda_i + \mu_i$.

Let k denote the length of λ, $l \geq k$ and $\mu = \lambda + (1^l)$. McKay defined a *standard map* and proved that if standard map is injective on $F^{\overline{\lambda}}$ then it will be injective on $F^{\overline{\mu}}$.

Muller and Neunhoffer [7] computed that the standard map is injective on $\lambda = (4^4)$, thus proving the Foulkes conjecture for $a = 4$. Although the standard map is not injective on $\lambda = (5^5)$, in 2015 Cheung, Ikenmeyer and Mkrtchyan [8] computed and proved that the standard map was injective on $\lambda = (5^6)$, thus establishing the conjecture for $a = 5$.

Foulkes conjecture claims that for all $\lambda \vdash ab$ and irreducible Specht module S^λ

$$\langle F^{(a)}_{(b)}, S^\lambda \rangle \geq \langle F^{(b)}_{(a)}, S^\lambda \rangle. \tag{1.2}$$

In 2013 Gianelli [9] proved that those partitions that are "close" to hooks do not occur in the Foulkes module thus implying that if λ is close to $(ab - r, 1^r)$ then (1.2) holds. The problem remains open in general for $a \geq 6$. The structure of the Foulkes module is in itself of interest in Algebraic Combinatorics and in the Representation Theory of the Symmetric Group. The multiplicity of S^λ in $F^{(a)}_{(b)}$ is still an open question for $a \geq 5$.

One significant result is what used to be known as Weintraub's conjecture. In 1990 Weintraub [10] claimed that for every even partition λ with at most b parts, S^λ appears with positive multiplicity in $F^{(a)}_{(b)}$. The conjecture has been independently proved by Burgeissor, Christendl and Ikenmeyer [11] in 2011 and by Manuel and Michełik [12] in 2013.

A related result proved by de Boeck [13] states that for a partition $\lambda = (ab - j^r, j^r)$, S^λ has positive multiplicity in $F^{(a)}_{(b)}$ if and only if j is even.

The Foulkes module is obtained when the trivial module of the wreath product $S_a \wr S_b$ is induced to S_{ab}. We may extend the definition to any irreducible module of $S_a \wr S_b$ induced to S_{ab}. Of particular interest are the irreducible modules S^ν of S_b inflated to $S_a \wr S_b$.

Definition 3. Let $\nu \vdash b$. The generalized Foulkes module $F^{(a)}_\nu$ is defined as

$$F^{(a)}_\nu = \mathrm{Inf}^{S_a \wr S_b}_{S_b} S^\nu \uparrow^{S_{ab}} .$$

Let $\nu' = \nu + (ab - b)$. In 1998 Agoka [14] proved that that the Specht module $S^{\nu'}$ has multiplicity 1 in $F_\nu^{(a)}$. Also, ν' is maximal with respect to the dominance order among μ such that S^μ occurs in $F_\nu^{(a)}$.

More recently, there have been further developments in the decomposition of the generalized Foulkes module. In 2019 Paget and Wildon [15] gave a description for minimal Specht modules in $F_\nu^{(a)}$. A corollary of their result is stated below.

Corollary 4. Let $\nu = (b)$ and $\overline{\nu} = (1^b)$ be its conjugate.

$$\langle F_\nu^{(a)}, \chi^{(a^b)} \rangle = 1, \quad \text{if and only if} a \text{ is even.}$$

$$\langle F_{\overline{\nu}}^{(a)}, \chi^{(a^b)} \rangle = 1, \quad \text{if and only if} a \text{ is odd.}$$

In 2015 de Boeck [13] introduced the semistandard homomorphisms and proved that if $\mu' = \mu + (1^b)$, then the multiplicity of $S^{\mu'}$ in $F_{\overline{\nu}}^{(a+1)}$ will be equal to the multiplicity of S^μ in $F_\nu^{(a)}$.

Let $\mu^* = \mu + (b)$. de Boeck [13] also proved that the multiplicity of S^{μ^*} in $F_{(b)}^{(a+1)}$ is equal to the multiplicity of S^μ in $F_{(b)}^{(a)}$ and the multiplicity of S^{μ^*} in $F_{(1^b)}^{(a+1)}$ is greater than the multiplicity of S^μ in $F_{(1^b)}^{(a)}$.

In 2018 de Boeck, Paget and Wildon [16] generalized this result for any ν.

There are many approaches to study the properties of $F_\nu^{(a)}$ but the problem of its decomposition into Specht modules still remains widely open. In this thesis we consider the properties of (Generalized) Foulkes modules restricted to some small and large subgroups H. The study of such $F_\nu^{(a)} \downarrow_H$ might give some interesting information on the properties of $F_\nu^{(a)}$ and in turn help us understand its decomposition.

The **outline** of this thesis is as follows.

In Chapter 2 we introduce the basic concepts needed to prove our main results. We start with Representation Theory of Finite groups, Character Theory and then we proceed with introducing Symmetric Groups and its representations.

In Chapter 3 we study the restriction of the generalized Foulkes module to a maximal subgroup and eventually obtain the de Boeck-Paget-Wildon Theorem as a corollary. See Corollary 44. This chapter is almost a verbatim copy of our preprint 'Some Properties of Generalized Foulkes Module.' [17]

In Chapter 4 we study the restriction of the Foulkes module to the subgroup generated by a "long cycle." We compare the character values of fixed point free elements and examine if the Foulkes conjecture is true in this particular case. We prove the restricted Foulkes conjecture when ab is a prime power.

In Chapter 5 we study the restriction of the Foulkes module to an elemenatary abelian 2-subgroup. We compute the character values of involutions and examine if the Foulkes conjecture is true when we restrict it to elementary abelian subgroups.

Chapter 2

Preliminaries

The following topics serve as a basis for this thesis. The book **Representations and Characters of Finite Groups** [18] by *Gordon James and Martin Liebeck* may be used as a reference.

- Representation
- Group Rings
- KG or G module
- Irreducible module or representation
- Maschke's Theorem
- Schur's Lemma
- Class Functions and Characters
- Orthogonality relations of Characters
- Semisimple modules

2.1 Representations and Characters of Finite Groups

Let G be a finite group. A representation ϕ of G is a homomorphism $\phi : G \to GL(V)$, where V is a vector space over a field K. The degree of the representation ϕ is the dimension of V. Equivalently, V can be considered a (left) KG module using the multiplication determined by ϕ.

In this thesis the base field $K = \mathbb{C}$ unless otherwise stated. Instead of a $\mathbb{C}G$ module we simply write it as a G module. By Maschke's theorem all finite dimensional G modules are semisimple. We recall some definitions.

Definition 5. Restricted Module: Let H be a subgroup of G and V a G module. Then H acts on V naturally thus V is also a H module. This module is called the restricted module and denoted by V_H or $V\downarrow_H$.

It is important to note that for a simple module V, V_H need not be simple.

Definition 6. Induced module: Let H be a subgroup of G with index t and the coset representatives of H over G be $\{h_i \mid 0 < i \leq t\}$. Let be V a H module. Let $h_i \otimes V \cong_K V$ for each coset of H. Let

$$V\uparrow^G = \bigoplus_{i=1}^{t} h_i \otimes V.$$

We define the action of G on $V\uparrow^G$ by

$$g(h_i \otimes v) = h_j \otimes hv, \text{ where } g \in G,\ gh_i = h_j h \text{ for some } h \in H.$$

Up to isomorphism, $V\uparrow^G$ does not depend on the coset representatives.

Definition 7. Inflated Module: Let N be a normal subgroup of G, $H = G/N$ and V a H-module. The inflated module $\mathrm{Inf}_H^G V$ of G is isomorphic to V as a vector space. The action of G on $\mathrm{Inf}_H^G V$ is defined by

$$gv = (gN)v,\ g \in G.$$

Unlike restricted modules, V is a simple $H = G/N$ module if and only if its inflated module $\mathrm{Inf}_H^G V$ is simple.

Definition 8. Representations on Tensor product: Let V and W be G modules. Then G acts on $V \bigotimes W$ by $g(v \otimes w) = gv \otimes gw$.

Similarly, let V and W be G and H modules respectively. Then $G \times H$ acts on $V \bigotimes W$ by $(g, h)(v \otimes w) = gv \otimes hw$.

The *Mackey's Decomposition Theorem* gives a formula for the restricted module of an induced module. Before we state the theorem we recall the following definition.

Definition 9. Let H be a subgroup of G and V be a H module. For each $g \in G$ H^g is the conjugate subgroup of H by g, that is $H^g = gHg^{-1}$. Similarly V^g is a H^g module defined as

$$V^g = \{g \otimes v \mid v \in V\},$$

$$ghg^{-1}(g \otimes v) = g \otimes hv.$$

Theorem 10. Mackey: Let H and K be subgroups of G and V a H module. If $G = \cup_{i=1}^{t} K g_i H$ then

$$(V \uparrow^G) \downarrow_K \cong \bigoplus_{i=1}^{t} (V^{g_i}_{K \cap H^{g_i}}) \uparrow^K$$

We recall some definitions in Character theory.

Definition 11. Let ϕ be a representation. Then character χ of ϕ is a class function and defined as

$$\chi(g) = tr(\phi(g)).$$

Inner Product: Let χ_1 and χ_2 be characters of G. The inner product $\langle ., . \rangle_G$ on characters is defined as

$$\langle \chi_1, \chi_2 \rangle_G = \frac{1}{\mid G \mid} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}.$$

Please note that the above definition of the inner product makes sense for class functions which is the subspace of complex functions on G generated by the characters. An important theorem is the following.

Theorem 12. Frobenius Reciprocity: Let H be a subgroup of G, χ_1 be a H character and χ_2 be a G character. Then

$$\langle \chi_1 \uparrow^G, \chi_2 \rangle_G = \langle \chi_1, \chi_2 \downarrow_H \rangle_H.$$

An important and interesting property of irreducible characters of finite groups is that the number of irreducible characters is equal to the number of conjugacy classes. Study of characters has led to a better understanding of the group structure. For further information refer to the book **Character Theory of Finite Groups** by *Martin Isaacs* [19].

2.2 Representations of the Symmetric Groups

The Symmetric group, S_n is the group of permutations of n elements. A permutation σ can be decomposed as a product of disjoint cycles. We can associate to σ a tuple that represents the sizes of its disjoint cycles. By arranging the sizes in decreasing order the tuple gives a partition of n. That partition is called the *type* of σ. Since the type of a permutation does not change under the conjugacy action of G, the number of irreducible representations is equal to the number of partitions of n.

Definition 13. Young Diagram: Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n. A *Young diagram* of λ is a 2 dimensional structure consisting of n boxes put together such that the top-most row has λ_1 boxes, just below the the top row, there are λ_2 boxes, etc. and finally the bottom-most row has λ_k boxes.

As an example let $\lambda = (5, 2, 1)$. Its Young diagram is

$$\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline & & \multicolumn{3}{c}{} \\ \cline{1-2} & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} .$$

Definition 14. Young Tableau: We can fill the boxes of a Young diagram with numbers from $\{1, \ldots, n\}$. There are $n!$ different arrangements. Each such filling of boxes is called a *Young tableau* (plural: tableaux). If the shape of the Young tableau t is λ then it is also called λ-tableau.

As an example let $\lambda = (5, 2, 1)$. Examples of λ-tableaux are

$$t_1 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & \multicolumn{3}{c}{} \\ \cline{1-2} 8 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \qquad t_2 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 6 \\ \hline 5 & 7 & \multicolumn{3}{c}{} \\ \cline{1-2} 8 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}$$

It is easy to see that S_n acts on λ-tableaux for each λ. A permutation σ is a row stabilizer of the λ-tableau t if σ only permutes elements in the same row. For example, in tableau t_1, (12) and (245) are row stabilizers but (16) is not a row stabilizer. The set of row stabilizers forms a group and is denoted by R_t. It is clear that $R_t \cong S_{\lambda_1} \times \cdots \times S_{\lambda_k}$. We can similarly define the set of column stabilizers C_t.

Definition 15. Young Tabloid: Let λ be a partition of n and t be a λ-tableau. The associated Young tabloid $\{t\}$ is a set theoretical partition of $\{1, \ldots, n\}$ with elements in each row of t constituting parts of $\{t\}$.

In a similar manner as with λ-tableaux, S_n acts on λ-tabloids and this action gives rise to permutation module M^λ. Let t be a λ-tableau, then R_t is the stabilizer of $\{t\}$ under the action of S_n. A standard way to represent tabloids is by removing the vertical lines in a Young tableau. As an example

$$t = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & \multicolumn{3}{c}{} \\ \cline{1-2} 8 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \qquad \{t\} = \begin{array}{ccccc}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & & & \\ \cline{1-2} 8 & & & & \\ \cline{1-1} \end{array} .$$

Definition 16. Polytabloids and Specht modules:

Let $k_t = \sum_{\sigma \in C_t} (-1)^{\mathrm{sgn}(\sigma)} \sigma \in KS_n$. The polytabloid e_t is defined as $e_t = k_t\{t\} \in M^\lambda$. The Specht module S^λ is the KS_n module generated by the polytabloids e_t.

Let ρ be a permutation. If $\sigma \in C_t$ then $\rho\sigma\rho^{-1} \in C_{\rho t}$. Therefore $\rho k_t = (k_{\rho t})\rho$ and $\rho e_t = e_{\rho t}$. Thus S^λ is a cyclic module, that is, it is generated by a single element.

On the set of partitions we define a partial order known as the *dominance order.*

Definition 17. Dominance Order: Let λ and μ be partitions of n of length k and r respectively. We say that λ dominates μ, $\lambda \trianglerighteq \mu$, if

$$\sum_{i=1}^{l} \lambda_i \geq \sum_{i=1}^{l} \mu_i, \text{ for each } l \in \mathbb{N}.$$

We can easily check that this is indeed a partial order. It is not a total order since partitions $(5, 2, 1)$ and $(4, 4)$ do not dominate each other.

On M^λ we can define a non-singular, symmetric and S_n-invariant bilinear form $\langle ., . \rangle_\lambda$ by

$$\langle \{t_1\}, \{t_2\} \rangle_\lambda = \delta_{ij}, \qquad \text{here } \delta \text{ is the Kronecker delta function.}$$

Theorem 18. Submodule Theorem: If $S^{\lambda^\perp}$ is the orthogonal space to S^λ with respect to the the bilinear form on M^λ. Then any submodule U of M^λ is either a submodule of $S^{\lambda^\perp}$ or it contains S^λ.

Thus $S^\lambda \cap S^{\lambda^\perp}$ is a maximal submodule of S^λ. Therefore the KS_n module $S^\lambda / S^\lambda \cap S^{\lambda^\perp}$ is irreducible. If $K = \mathbb{C}$, $S^\lambda \cap S^{\lambda^\perp} = \{0\}$. This implies that the Specht modules S^λ are irreducible modules of S_n.

Lemma 19. Let $f : M^\lambda \to M^\mu$ be a S_n homomorphism. If $S^\lambda \nsubseteq ker(f)$ then $\lambda \trianglerighteq \mu$.

Since M^λ is semisimple, any homomorphism from S^λ to M^μ can be extended to M^λ to M^μ. If $S^\lambda \cong S^\mu$ then $\lambda \trianglerighteq \mu$ and $\mu \trianglerighteq \lambda$ implying that $\lambda = \mu$. Therefore Specht modules form the complete set of irreducible moules of S_n.

Another corollary of the lemma is that if S^λ is an irreducible constituent of M^μ then $\lambda \trianglerighteq \mu$. On the set of λ tabloids we define a *total ordering.*

Definition 20. Total Ordering: Let $\{t_1\}$ and $\{t_2\}$ be λ tabloids. Then $\{t_1\} <_\lambda \{t_2\}$ if $\exists\ i \leq n$ such that

- for every $j > i$, j is in the same row of $\{t_1\}$ and $\{t_2\}$,
- i is in the higher row in $\{t_1\}$ than in $\{t_2\}$.

We describe special kinds of tablaux called "standard." They are used to determine the basis of Specht modules.

Definition 21. Standard tableau: A λ tableau t is standard if its entries are increasing from top to bottom and left to right.

Few examples of standard tableaux are

$$t_1 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \qquad t_2 = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}.$$

In the above example it is clear that the tabloids $\{t_2\} <_{(5,2,1)} \{t_1\}$ since 6 is in the higher row of t_2 and both 7 and 8 are in the same rows. In the same example consider the permutation $(1\ 6) \in C_{t_1}$. Clearly, $(1\ 6)\{t_1\} <_{(5,2,1)} \{t_1\}$. This is true for any $\sigma \in C_t$ and t, a standard tableau.

Lemma 22. Let t be a standard λ tableau and $\sigma \in C_t$ then $\sigma\{t\} <_\lambda \{t\}$.

Theorem 23. The set

$$\{e_t \mid t \text{ is a standard } \lambda \text{ tableau } \}$$

forms a basis of Specht module S^λ. Therefore, the degree of S^λ equals the number of standard λ tableau.

The proof uses Lemma 22 and *Garnir Relations*. For further information refer to **James** [20].

In general, determining the structure of the induced module is difficult. However, for Specht modules we have the following theorem.

Theorem 24. Branching Theorem: Let λ be a partition of n and $B_{\lambda,n+1}$ be the set of partitions of $n+1$ such that its Young diagram has one more box added to the Young diagram of λ. Then

$$S^\lambda \uparrow^{S_{n+1}} = \bigoplus_{\mu \in B_{\lambda,n+1}} S^\mu.$$

Let $\lambda \vdash k$ and $\mu \vdash n-k$ then $S^\lambda \otimes S^\mu$ is an $S_k \times S_{n-k}$ module. The structure of $S^\lambda \otimes S^\mu \uparrow^{S_n}$ has been determined combinatorially. It is known as the *Littlewood-Richardson rule*. First, we define *sequences*.

Definition 25. Sequences: A sequence s is a string of n integers and it is of type ν if i occurs ν_i times for each i. Whether an element in a sequence s is *good* is defined recursively:

- 1 is always good.
- An $i+1$ at position j is good if the number of good $i's$ is strictly greater then number of good $(i+1)'s$ at positions preceding j.

As an example in the following sequence 2 and 3 at positions 3 and 4 respectively are good.

$$\begin{matrix} 2 & 1 & 2 & 3 & 2 \\ \times & \checkmark & \checkmark & \checkmark & \times \end{matrix}$$

The set $s(\nu, \nu')$ consists of all sequences s of type ν such that s has at least ν'_i good $i's$. For example, the sequence **2 1 2 3 2** is in $s((1,3,2),(1^3))$. We are now ready to state the Littlewood Richardson rule.

Definition 26. Littlewood Richardson arrangement: Let ν, λ, μ be partitions of n and $\nu_i \geq \lambda_i$ for each $i \in \mathbb{N}$. The Littlewood Richardson arrangement is a setting of Young diagram of ν with parameters λ and μ such that

- The boxes in the Young diagram of ν that are not part of λ are filled with integers.

- They are non decreasing along rows from left to right and strictly decreasing along columns from top to bottom.
- When read from right to left and top to bottom the sequence is an element of $s(\mu, \mu)$.

As an example let $\lambda = (2, 1^3)$, $\mu = (2, 1^3)$ and $\nu = (4, 2^3)$. A Littlewood Richardson arrangement of ν

$$Y_\nu^{\lambda,\mu} = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & 2 \\ \cline{1-2} & 3 \\ \cline{1-2} & 4 \\ \cline{1-2} \end{array} .$$

Theorem 27. Littlewood-Richardson Rule: Let $\lambda \vdash k, \mu \vdash n-k$ and

$$S^\lambda \otimes S^\mu \uparrow^{S_n} = \bigoplus_{\nu \vdash n} a_\nu S^\nu.$$

If $\nu_i < \lambda_i$ for some i then $a_\nu = 0$. The coefficient a_ν is equal to the number of Littlewood Richardson arrangements of ν with the parameter λ and μ.

Though the Littlewood-Richardson rule helps us to get a better picture of the structure it is often computationally infeasible to compute the coefficients.

The *hook length formula* is used to determine the the dimension of the Specht module. A hook is a partition with λ_2 at most 1. We associate the hook $H(i,j)$ to the box (i,j) of a given Young diagram of shape λ such that $H(i,j)$ consists of (i,j) and the boxes to the right and bottom of it. As an example let $\lambda = (5, 4, 3, 2)$ then

$$H(2,2) = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & X & X & X \\ \cline{1-4} & X & \\ \cline{1-3} & X \\ \cline{1-2} \end{array} .$$

Theorem 28. Hook Length Formula: Let $h(i,j)$ be the number of boxes in the hook $H(i,j)$. Then

$$\dim(S^\lambda) = \frac{n!}{\prod_{(i,j)} h(i,j)}, \text{ where } (i,j) \text{ runs through the boxes of } \lambda.$$

2.3 Representations of Wreath Products

Let G be a group and $H \leq S_n$ be a permutation group. Let $X_n = \{1, \cdots, n\}$. The group

$$G^n = G_1 \times \cdots \times G_n,\ (G_i \cong G,\ 1 \leq i \leq n),$$

can also be viewed as the set of maps from X_n to G with coordinatewise multiplication, that is for 2 maps ϕ_1 and ϕ_2, $\phi_1\phi_2(i) = \phi_1(i)\phi_2(i)$.

$$G^n = \{\phi \mid \phi : X_n \to G\}.$$

Definition 29. Wreath product: The wreath product $G \wr H$ is defined as

$$G \wr H = \{(\phi, h) \mid\ \phi \in G^n,\ h \in H\}.$$

As h maps X_n to itself, the composition $\phi_h = \phi \circ h$ makes sense. Now we define the multiplication on $G \wr H$:

$$(\phi, h)(\phi^{'}, h^{'}) = (\phi\phi^{'}_h, hh^{'}).$$

The wreath product forms a group with the multiplication as defined above

- the identity element of $G \wr H$ is $(1_{G_n}, 1_H)$ and
- the inverse $(\phi, h)^{-1} = (\phi_h^- 1, h^{-1}) = ((\phi^{-1})_{h^{-1}}, h^{-1})$.

The *base group* $G^* = \{(\phi, 1) \mid\ \phi \in G^n\} \cong G^n$ is a normal subgroup of $G \wr H$ and $H^* = \{(1, h) \mid\ h \in H\} \cong H$ is its complement, that is $G \wr H$ is the semidirect product of G^* and H^*. We also employ the notation $\overline{(\phi, h)} = h$ which is a homomorphism $\bar{}: G \wr H \to H$.

Let us denote by $IR_G = \{S_1, \ldots, S_k\}$ the set of simple G modules. For each i, $1 \leq i \leq n$, take $V_i \in IR_G$. Let l_j be the number of i's such that $V_i \cong S_j$. Then the type of the simple G^n module $\mathbb{V}_\lambda = \bigotimes_i V_i$ is $(l_1, \ldots, l_k)$. The conjugate module $\mathbb{V}_\lambda^{(\phi,h)}$

is isomorphic to $\mathbb{V}_\lambda$ as a vector space and G^n acts on it by

$$\pi^{(\phi,h)}(v) = (\phi, h)(\pi, 1)(\phi, h)^{-1}(v).$$

Definition 30. Inertia Group: The inertia subgroup $G \wr H_\lambda$ of $\mathbb{V}_\lambda$ is defined as

$$G \wr H_\lambda = \{(\phi, h) \mid \mathbb{V}_\lambda^{(\phi,h)} \cong \mathbb{V}_\lambda\}.$$

Lemma 31. Let $\mathbb{V}_\lambda$ be a simple G^n module and $\lambda = (l_1, \cdots, l_k)$ be its type. If $S_\lambda = S_{l_1} \times \cdots \times S_{l_k}$ then the inertia subgroup is $G \wr H_\lambda = G \wr (H \cap S_\lambda)$.

Clifford's theory states that we can extend the G^n module $\mathbb{V}_\lambda$ to its inertia subgroup $G \wr H_\lambda$ in the following way

$$(\phi, h)(v_1 \otimes \cdots \otimes v_n) = \phi(1)v_{h^{-1}(1)} \otimes \cdots \otimes \phi(n)v_{h^{-1}(n)}.$$

We denote this module by $\mathbb{V}'_\lambda$. If M is a simple $H \cap S_\lambda$ module then it can be inflated to a simple $G \wr H_\lambda$ module denoted by M'. We obtain a $G \wr H_\lambda$ module $\mathbb{V}'_\lambda \otimes M'$.

Theorem 32. The induced module $\mathbb{V}'_\lambda \otimes M' \uparrow^{G \wr H}$ is simple. Moreover all simple modules of $G \wr H$ are of this kind. Let $IR^\lambda_{G^n}$ be the set of simple G^n modules of type λ and IR_{H_λ} be the set of simple $H \cap S_\lambda$ modules. The set $\{\mathbb{V}'_\lambda \otimes M' \uparrow^{G \wr H} \mid \lambda \vdash n,\ \mathbb{V}_\lambda \in IR^\lambda_{G^n},\ M \in IR_{H_\lambda}\}$ forms a complete set of simple $G \wr H$ modules.

Chapter 3

Restrictions of the generalized Foulkes module to large subgroups

3.1 Introduction

For $a, b > 1$ integers, the Foulkes module $F^{(a)}_{(b)}$ is the permutation module of S_{ab} acting on the set of partitions of $\{1, 2, \ldots, ab\}$ into b sets of size a each. Here we describe a generalization.

Let ν be a b-partition. When we inflate the Specht module S^ν to $S_a \wr S_b$ and induce it to S_{ab} we obtain the ν-generalized Foulkes module, that is, $F^{(a)}_\nu = \mathrm{Inf}^{S_a \wr S_b}_{S_b} S^\nu \uparrow^{S_{ab}}$.

As described in Chapter 1 Thrall [2] decomposed the Foulkes module into simple components for $a = 2$ and for $b = 2$:

$$F^{(2)}_{(b)} = \sum_{\lambda \vdash b} S^{2\lambda};$$

$$F^{(a)}_{(2)} = \sum_{\substack{\lambda \vdash 2a \\ \lambda_1 + \lambda_2 = 2a}} S^\lambda.$$

Foulkes [1] conjectured that if $a \leq b$ then $F^{(a)}_{(b)}$ can be embedded in $F^{(b)}_{(a)}$, for $a = 2$ this is an immediate consequence of Thrall's result.

For an integer $k < ab$ we define Ω_k as the set of partitions of k which are subpartitions of (a^b). Then the restriction of the generalized Foulkes module to $S_k \times S_{ab-k}$ has a natural decomposition (see below, Definition 40) indexed by Ω_k:

$$F_\nu^{(a)} \downarrow_{S_k \times S_{n-k}} = \bigoplus_{\lambda \in \Omega_k} V_{\nu,a}^\lambda \tag{3.1}$$

We are concerned mainly with the (1^k)-component $U_{\nu,a} = V_{\nu,a}^{(1^k)}$.

The main theorem of this paper is the following. Here, and later, $\mu^\perp$ denotes the conjugate of a partition μ. In particular, $(1^b)^\perp = (b)$.

Theorem 33. Let $k = b$ and as above, $U_{\nu,a} = V_{\nu,a}^{(1^b)}$.

Let $c_{\mu,\lambda}^\nu$ denote the Kronecker coefficient, that is the multiplicity of S^ν in the S_b module $S^\mu \otimes S^\lambda$. Then

$$U_{\nu,a} = \sum c_{\mu,\lambda}^\nu S^\mu \otimes F_\lambda^{(a-1)}. \tag{3.2}$$

In particular, for $a = 2$

$$U_{\nu,2} = \sum_{\mu,\lambda} c_{\mu,\lambda}^\nu S^\mu \otimes S^\lambda. \tag{3.3}$$

Note that $S^{(b)}$ is the trivial, while $S^{(1^b)}$ is the sign module of S_b. So $c_{(b),\nu}^\nu = 1$ and $c_{(1^b),\nu^\perp}^\nu = 1$. Hence the multiplicity of both $S^{(b)} \otimes S^\nu$ and $S^{(1^b)} \otimes S^{\nu^\perp}$ in $U_{\nu,2}$ are 1. Two important special cases of the main theorem are the following.

Corollary 34. Let $a, b \in \mathbb{N}$. The (1^b)-summand of the Foulkes module $F_{(b)}^{(a)}$ restricted to $S_b \times S_{n-b}$ is

$$U_{(b),a} = \sum_{\lambda \vdash b} S^\lambda \otimes F_\lambda^{(a-1)} \tag{3.4}$$

In particular, for $a = 2$

$$U_{(b),2} = \sum_{\lambda \vdash b} S^\lambda \otimes S^\lambda \tag{3.5}$$

Corollary 35. Let $a, b \in \mathbb{N}$. The (1^b)-summand of the generalized Foulkes module $F^a_{(1^b)}$ restricted to $S_b \times S_{n-b}$ is

$$U_{(1^b),a} = \sum_{\lambda \vdash b} S^{\lambda^\perp} \otimes F_\lambda^{(a-1)}. \tag{3.6}$$

In particular, for $a = 2$

$$U_{(1^b),2} = \sum_{\lambda \vdash b} S^{\lambda^\perp} \otimes S^\lambda \tag{3.7}$$

Paget and Wildon [15] give a description of minimal and maximal Specht modules with respect to the dominance order on partitions.

With the help of semistandard homomorphisms de Boeck [13] proved that the multiplicity of $S^{\lambda+(b)}$ in $F_{(b)}^{(a+1)}$ is at least the multiplicity of S^λ in $F_{(b)}^{(a)}$ and the multiplicity of $S^{\lambda+(1^b)}$ in $F_{(b)}^{(a+1)}$ is equal to the multiplicity of S^λ in $F_{(1^b)}^{(a)}$. This we establish as a corollary of our results. See Corollary 44.

3.2 Preliminaries

We fix a few standard notations and recall some related definitions. The definition of ν-tableau t, tabloid $\{t\}$ and polytabloid e_t appear in Section 2.2. See Definitions 14 and 15.

Let $a, b \geq 2$ fixed integers, $n = ab$. Let $H^{(a)^b}$ denote the set of ordered partitions of $\{1, 2, \ldots, n\}$ into b sets of size a each.

Definition 36. For a partition $\nu \vdash b$, a ν-tableau t, permutation $\phi \in S_b$ and $X \in H^{(a)^b}$ let $t_{X,\phi}$ be the ν shaped Young diagram with $X_{\phi(l)}$ in the (i, j)-th entry if l is the (i, j)-th entry in t. For $\phi = ()$, the identity element, we abbreviate $t_X = t_{X,()}$.

As an example let $\nu = (2, 2)$ and $\phi = (12)$. For

$$t = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad t_{X\phi} = \begin{array}{|c|c|} \hline X_2 & X_1 \\ \hline X_3 & X_4 \\ \hline \end{array}.$$

The set of ν standard polytabloids, $B_\nu = \{e_t \mid t \text{ is standard tableau}\}$ forms a basis of the Specht module S^ν. Similarly, a basis of inflation $\mathrm{Inf}_{S_b}^{S_a \wr S_b} S^\nu$ is the set

$B_{X,\nu} = \{e_{t_X} \mid t \text{ is a standard tableau}\}$. The role of X is to fix the wreath product, acting on the parts of X. The action of S_b on S^ν with respect to the basis B_ν is inflated to the action of $S_a \wr S_b$ with respect to the basis $B_{X,\nu}$. Moreover, $\mathrm{Inf}_{S_b}^{S_a \wr S_b} S^\nu$ is generated by e_{t_X} for any tableau t since S^ν is a cyclic module generated by e_t. Thus the generalized Foulkes module $F_\nu^{(a)} = \mathrm{Inf}_{S_b}^{S_a \wr S_b} S^\nu \uparrow^{S_n}$ is generated by e_{t_X} as an S_n module.

For example, let t be as in the previous example, $a = 3$, $b = 4$ and

$$X = \{(1,2,3),\ (4,5,6),\ (7,8,9),\ (10,11,12)\} \in H^{(3^4)}.$$

Then

$$e_t = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} - \begin{array}{cc}\hline 1 & 4 \\ \hline 3 & 2 \\ \hline\end{array} + \begin{array}{cc}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array} - \begin{array}{cc}\hline 3 & 2 \\ \hline 1 & 4 \\ \hline\end{array}$$

and

$$e_{t_X} = \begin{array}{cc}\hline (\ 1\ 2\ 3\ ) & (\ 4\ 5\ 6\ ) \\ \hline (\ 7\ 8\ 9\ ) & (\ 10\ 11\ 12\ ) \\ \hline\end{array} - \begin{array}{cc}\hline (\ 1\ 2\ 3\ ) & (\ 10\ 11\ 12\ ) \\ \hline (\ 7\ 8\ 9\ ) & (\ 4\ 5\ 6\ ) \\ \hline\end{array}$$

$$+ \begin{array}{cc}\hline (\ 7\ 8\ 9\ ) & (\ 10\ 11\ 12\ ) \\ \hline (\ 1\ 2\ 3\ ) & (\ 4\ 5\ 6\ ) \\ \hline\end{array} - \begin{array}{cc}\hline (\ 7\ 8\ 9\ ) & (\ 4\ 5\ 6\ ) \\ \hline (\ 1\ 2\ 3\ ) & (\ 10\ 11\ 12\ ) \\ \hline\end{array}$$

For any ν tableau t the generalized Foulkes module $F_\nu^{(a)}$ is generated by the set $\{e_{t_Y} \mid Y \in H^{(a)^b}\}$. S_b acts on $H^{(a)^b}$ by permuting the indices, that is $\sigma X = \{X_{\sigma(1)}, \ldots, X_{\sigma(b)}\}$. Let $I^{(a)^b}$ be a set of representatives of S_b orbits.

Lemma 37. Let $Z \in H^{(a)^b}$ and t be a standard ν-tableau. Suppose $Y \in I^{(a)^b}$ is in the orbit of Z, namely $Z = \sigma Y$.

If

$$e_{\sigma(t)} = \sum_{s \text{ is a standard tableau}} c_s e_s$$

then

$$e_{t_Z} = \sum_{s \text{ is a standard tableau}} c_s e_{s_Y}.$$

Proof. Observe that if $S_a \wr S_b$ acts on the parts of Y then e_{s_Y} and also e_{t_Z} belong to $\mathrm{Inf}_{S_b}^{S_a \wr S_b} S^\nu$. The claim now follows from the definition. □

Definition 38. Let t be a ν shaped Young diagram and $\tau \in S_{\{b+1,\dots,2b\}}$. We define $T(\tau) = ((1, \tau(b+1)), (2, \tau(b+2)), \dots, (b, \tau(2b))) \in H^{(2)^b}$. Then $t_{T(\tau)}$ is a ν shaped diagram such that if the (i,j)-th entry of t is l then the (i,j)-th entry of $t_{T(\tau)}$ is the pair $(l, \tau(b+l))$.

Corollary 39.

$$e_{t_{T(\tau)}} = \sum_{s \text{ is a standard tableau}} c_s e_{s_{T(\tau)}},$$

Proof. Observe that $I^{(2)^b} = \{((1,\tau(b+1)), (2,\tau(b+2)), \dots, (b, \tau(2b))) \mid \tau \in S_{\{b+1,\dots,2b\}}\}$. The rest of the proof is a direct consequence of Lemma 37. □

A basis of F_ν^a is the set $B_{\nu,(a)} = \{e_{t_Y} \mid Y \in I^{(a)^b}, t \text{ is a standard tableau}\}$. Let $1 < k < ab$. We describe a decomposition of the restriction of the generalized Foulkes module to a maximal intransitive subgroup $F_\nu^{(a)} \downarrow_{S_k \times S_{n-k}}$. As above, Ω_k is the set of those partitions of k that are subpartitions of (a^b).

Definition 40. For $\lambda \in \Omega_k$ let

$$P_\lambda = \{X \in I^{(a)^b} \mid \lambda \text{ is the partition type of } \{1, 2, \dots k\} \cap X\}$$

and let the λ-*component*, $V_{\nu,a}^\lambda$ be the module generated by the set $\{e_{t_X} \mid X \in P_\lambda\}$ under the action of $S_k \times S_{n-k}$. A basis of $V_{\nu,a}^\lambda$ is the set $B_{\nu,a}^\lambda = \{e_{t_X} \mid X \in P_\lambda, \ t \text{ is a standard tableau of shape } \nu\}$.

The mentioned decomposition is

$$F_\nu^{(a)} \downarrow_{S_k \times S_{n-k}} = \bigoplus_{\lambda \in \Omega_k} V_{\nu,a}^\lambda. \tag{3.8}$$

We provide an example of elements in $V = V^{(2,1^2)}_{(2^2),3}$. Let $a = 3$, $b = k = 4$. For $\nu = (2,2)$ we have $\Omega_4 = \{(3,1),(2^2),(2,1^2),(1^4)\}$. V is generated by the set $\{e_{t_X} \mid X \in P_{(2,1^2)}\}$. Let t be as before and let

$$X = \{(1,2,5),(3,7,6),(4,8,9),(10,11,12)\}.$$

$$e_{t_X} = \begin{array}{c}\hline (\ 1\ \ 2\ \ 5\ )\ (\ 3\ \ 6\ \ 7\ ) \\ \hline (\ 4\ \ 8\ \ 9\ )\ (\ 10\ 11\ 12\ ) \\ \hline \end{array} - \begin{array}{c}\hline (\ 1\ \ 2\ \ 5\ )\ (\ 10\ 11\ 12\ ) \\ \hline (\ 4\ \ 8\ \ 9\ )\ (\ 3\ \ 6\ \ 7\ ) \\ \hline \end{array}$$

$$+ \begin{array}{c}\hline (\ 3\ \ 8\ \ 9\ )\ (\ 10\ 11\ 12\ ) \\ \hline (\ 1\ \ 2\ \ 5\ )\ (\ 3\ \ 6\ \ 7\ ) \\ \hline \end{array} - \begin{array}{c}\hline (\ 4\ \ 8\ \ 9\ )\ (\ 3\ \ 6\ \ 7\ ) \\ \hline (\ 1\ \ 2\ \ 5\ )\ (\ 10\ 11\ 12\ ) \\ \hline \end{array}$$

For any g in $S_4 \times S_8$, $ge_{t_X} = e_{gt_X}$. Thus, ge_{t_X} is in $V^{(2,1^2)}_{\nu,3}$ If $g = (14)(567)$. Then

$$ge_{t_X} = \begin{array}{c}\hline (\ 2\ \ 4\ \ 6\ )\ (\ 3\ \ 5\ \ 7\ ) \\ \hline (\ 1\ \ 8\ \ 9\ )\ (\ 10\ 11\ 12\ ) \\ \hline \end{array} - \begin{array}{c}\hline (\ 2\ \ 4\ \ 6\ )\ (\ 10\ 11\ 12\ ) \\ \hline (\ 1\ \ 8\ \ 9\ )\ (\ 3\ \ 5\ \ 7\ ) \\ \hline \end{array}$$

$$+ \begin{array}{c}\hline (\ 1\ \ 8\ \ 9\ )\ (\ 10\ 11\ 12\ ) \\ \hline (\ 2\ \ 4\ \ 6\ )\ (\ 3\ \ 5\ \ 7\ ) \\ \hline \end{array} - \begin{array}{c}\hline (\ 1\ \ 8\ \ 9\ )\ (\ 3\ \ 5\ \ 7\ ) \\ \hline (\ 2\ \ 4\ \ 6\ )\ (\ 10\ 11\ 12\ ) \\ \hline \end{array}$$

We shall focus on $k = b$ and especially on $U_{\nu,a} = V^{(1^b)}_{\nu,a}$.

3.3 Properties of $U_{\nu,a}$

In this section we trace back the arbitrary $a > 2$ case to the $a = 2$ case and prove Theorem 33.

Lemma 41. Let $a, b \in \mathbb{N}$, ν a partition of b. Then

$$U_{\nu,a} \cong \mathrm{Inf}^{S_b \times (S_{a-1} \wr S_b)}_{S_b \times S_b} U_{\nu,2} \uparrow^{S_b \times S_{n-b}} \tag{3.9}$$

Proof. For $Y = \{Y_1, \ldots, Y_b\} \in H^{(a-1)^b}$ (where the underlying set is $\{b+1,\, b+2, \ldots, n\}$) let Y_σ be the ordered set $\{(\sigma(1), Y_1), \ldots, (\sigma(b), Y_b)\}$. Similarly, we define t_{Y_σ} to be the ν diagram with (i,j)-th entry $(\sigma(l), Y_l)$, where l is the (i,j)-th entry of t. Note that t_{Y_σ} and $t_{X,\gamma}$ are different in the sense that γ permutes the indices of subpartitions of X, that is $\gamma X = \{X_{\gamma(1)}, \ldots, X_{\gamma(b)}\}$ in $t_{X,\gamma}$ but does not change the composition of $X_1, X_2, \ldots, X_b$ but σ in t_{Y_σ} permutes the first component of subsets $(1, Y_1), (2, Y_2), \ldots, (b, Y_b)$ and fixes the rest of the components. The set $\{e_{t_X} \mid X \in P_{(1^b)}\}$ can also be represented as $\{e_{t_{Y_\sigma}} \mid Y \in H^{(a-1)^b}, \sigma \in S_b\}$. As $U_{\nu,a}$ is the permutation module of the set $\{e_{t_X} \mid X \in P_{(1^b)}\}$, $U_{\nu,a}$ is also the permutation module of the set $\{e_{t_{Y_\sigma}} \mid Y \in H^{(a-1)^b}, \sigma \in S_b\}$.

Consider the action of S_b on $H^{(a-1)^b}$ which permutes the indices in the ordered set. For $\tau \in S_b$, $Y \in H^{(a-1)^b}$ we get $\tau(Y) = (Y_{\tau(1)}, Y_{\tau(2)}, .., Y_{\tau(b)})$. Fix a $Z \in H^{(a-1)^b}$. For $\sigma \in S_b$ let $(\tau, \sigma)Z = ((\sigma(1), Z_{\tau(1)}), \ldots, ((\sigma(b), Z_{\tau(b)})$. Then let W be the $S_b \times (S_{a-1} \wr S_b)$-module generated by the set $\{e_{t_{(\tau,\sigma)Z}} \mid \tau \in S_b,\, \sigma \in S_b\}$ for any tableau t. A basis of W is $\{e_{t_{(\tau,())Z}} \mid \tau \in S_b,\, t \text{ is standard}\}$.

We see that

$$U_{\nu,a} = W \uparrow^{S_b \times S_{n-b}}, \tag{3.10}$$

because the induction takes place only in the second component.

Also

$$W \cong \mathrm{Inf}_{S_b \times S_b}^{S_b \times (S_{a-1} \wr S_b)} U_{\nu,2}, \tag{3.11}$$

because the second components are preserved by the action of τ in $H^{(a-1)^b}$. These two equations prove the lemma. $\square$

Thus the study of $U = U_{\nu,2}$ might give some interesting information on the generalized Foulkes module.

Now we are ready to prove Theorem 33.

Proof of Theorem 33. Let t be a ν tableau and $\tau \in S_{\{b+1,\ldots,2b\}}$. A basis of U is $B^{(1^b)} = \{e_{t_{T(\tau)}} \mid t \text{ is a standard tableau, } \tau \in S_{(b+1,\ldots,2b)}\}$.

Now we calculate the character χ of U which will help us to identify the decomposition of U into irreducible modules.

Let χ^ν denote the irreducible character of the Specht module S^ν. We claim that for $(g_1, g_2) \in S_b \times S_b$

$$\chi(g_1, g_2) = \begin{cases} 0, & \text{if } g_1, g_2 \text{ are of different cycle structure;} \\ \mid C_{S_b}(g_1) \mid \chi^\nu(g_1), & \text{if } g_1, g_2 \text{ are of the same cycle structure.} \end{cases}$$

To prove the claim we need to find the coefficient of e_{t_τ} in $(g_1, g_2)e_{t_{T(\tau)}}$ for every (g_1, g_2) in $S_{\{1,\dots,b\}} \times S_{\{b+1,\dots,2b\}}$. Let $h = (1, b+1)(2, b+2)\cdots(b, 2b)$ and $g_3 = hg_1h$ be the shifted permutation to $\{b+1, \dots 2b\}$, that is g_3 sends $b+l$ to $b+k$ if and only if g_1 send l to k. Clearly, $(g_1, g_2)e_{t_{T(\tau)}} = e_{(g_1 t)_{T(g_2 \tau g_3^{-1})}} = e_{s_{T(\varrho)}}$ where $s = g_1 t$ and $\varrho = g_2 \tau g_3^{-1}$. Note that $s = g_1 t$ need not be standard so $e_{s_{T(\varrho)}}$ might not be a basis element!

As a consequence of Corollary 39 the coefficient of $e_{t_{T(\tau)}}$ in $e_{s_{T(\varrho)}}$ is 0 unless $\tau = \varrho$. This implies the first part of the claim, that is the coefficient of $e_{t_{T(\tau)}}$ in $(g_1, g_2)e_{t_{T(\tau)}}$ is zero if the cycle structures of g_1 and g_2 are different, since in such a case there will always exist an (i, j)-th entry in $t_{T(\tau)}$ without a corresponding entry in any box of $(g_1 t)_{T(\varrho)}$.

Let now g_1 and of g_2 be fixed and of the same cycle structure. If $g_1 e_t = \sum d_r e_r$ then

$$(g_1, g_2)e_{t_{T(\tau)}} = \sum d_r e_{r_{T(\varrho)}} \tag{3.12}$$

We need to find the number of permutations τ and standard tableau t such that $e_{t_{T(\tau)}}$ is a linear summand of $(g_1, g_2)e_{t_\tau}$. This can only happen if e_t is a linear summand of $g_1 e_t$.

Therefore $g_2 \tau g_3^{-1} = \tau$, in other words $\tau^{-1} g_2 \tau = g_3$. The number of such τ is equal to the order of the centralizer $|C_{S_b}(g_1)|$ which proves the claim.

Let $k_{g_1} = b!/|C_{S_b}(g_1)|$ denote the size of conjugacy class of g_1. Then the multiplicity $c^{\nu}_{\mu,\lambda}$ of $S^{\mu} \otimes S^{\lambda}$ in U can be expressed as

$$\begin{aligned}c^{\nu}_{\mu,\lambda} =& \frac{1}{(b!)^2} \sum k^2_{g_1} \chi^{\mu}(g_1)\chi^{\lambda}(g_2)|C_{S_b}(g_1)|\chi^{\nu}(g_1)\\ =& \frac{1}{(b!)^2} \sum k^2_{g_1} \chi^{\mu}(g_1)\chi^{\lambda}(g_1)|C_{S_b}(g_1)|\chi^{\nu}(g_1)\\ =& \frac{1}{b!} \sum k_{g_1} \chi^{\mu}(g_1)\chi^{\lambda}(g_1)\chi^{\nu}(g_1),\end{aligned}$$

which is the so called Kronecker coefficient, the multiplicity of S^{ν} in the S_b module $S^{\mu} \otimes S^{\lambda}$.

Thus,

$$U = \sum_{\mu,\lambda} c^{\nu}_{\mu,\lambda} S^{\mu} \otimes S^{\lambda}.$$

The first part of Theorem 33 follows from Lemma 41. □

Corollaries 34 and 35 follow from the observations

$$c^{(b)}_{\lambda,\lambda} = c^{(1^b)}_{\lambda^{\perp},\lambda} = 1 \tag{3.13}$$

and

$$b! = \sum_{\lambda \vdash b} \deg(S^{\lambda} \otimes S^{\lambda}) = \sum_{\lambda \vdash b} \deg(S^{\lambda^{\perp}} \otimes S^{\lambda}) \leq \deg(U_{(b),2}) = \deg(U_{(1^b),2}) = b!. \tag{3.14}$$

3.4 Implications on the Generalized Foulkes Module

Lemma 42. The multiplicity of $S^{(1^b)} \otimes S^{\mu}$ in $F^{(a)}_{\nu}$ is equal to the multiplicity of $S^{(1^b)} \otimes S^{\mu}$ in $U_{\nu,a}$.

Proof. The coefficient on the right hand side of (3.1) comes from the part $U_{\nu,a} = V^{(1^b)}_{\nu,a}$. We need prove that no other summand $V^{\lambda}_{\nu,a}$ of (3.1) contains $S^{(1^b)} \otimes S^{\mu}$. Suppose now that $\lambda \neq (1^b)$.

As we have already seen, the basis of $V^{\lambda}_{\nu,a}$ is the set

$$B^{\lambda}_{\nu,a} = \{e_{t_X} \mid X \in P_{\lambda},\ t \text{ is a standard tableau}\}.$$

Observe, that if l_1 and l_2 belong to the same part, say X_k, in $X = \{X_1, X_2, \dots, X_b\}$ then the transposition $(l_1\ l_2)$ fixes e_{t_X}, that is, $(l_1\ l_2)e_{t_X} = e_{t_X}$.

Suppose $S^{(1^b)} \otimes S^{\mu}$ can be embedded in $V^{\lambda}_{\nu,a}$. Let M be a submodule of $V^{\lambda}_{\nu,a}$ isomorphic to $S^{(1^b)} \otimes S^{\mu}$. Let m be an element of M. Then $m = \sum_{e_{t_Y} \in B^{\lambda}_{\nu,a}} c_{t_Y} e_{t_Y}$. Choose a t_Y such that $c_{t_Y} \neq 0$. As $\lambda \neq (1^b)$, there exist $l_1, l_2 \leq b$ such that l_1, l_2 belong to the same subset in Y. As a consequence of Lemma 37 for any $s_X \neq t_Y$ in $B^{\lambda}_{\nu,a}$, e_{t_Y} is not a summand of $(l_1\ l_2)e_{s_x}$. This implies that the coefficient of $e_{t_Y} = (l_1\ l_2)e_{t_Y}$ in $(l_1\ l_2)m$ is also c_{t_Y}. But, since m is an element of M, $(l_1\ l_2)m = -m$, which implies that the coefficient of e_{t_Y} is $-c_{t_Y}$, a contradiction. Thus, $S^{(1^b)} \otimes S^{\mu}$ cannot be embedded in $V^{\lambda}_{\nu,a}$ if $\lambda \neq (1^b)$. $\square$

For a partition $\mu \vdash n$ of length k and the Young subgroup $S_{\mu} = S_{\mu_1} \times S_{\mu_2} .. \times S_{\mu_k}$, we define M^{μ} as $1_{S_{\mu}} \uparrow^{S_n}$. $M^{(1^n)}$ is the regular module and $M^{(n)}$ is the trivial module. The following Lemma is specific for the regular module.

Lemma 43. Let $a, b \in \mathbb{N}$. Then

$$\mathrm{Inf}_{S_b}^{S_a \wr S_b} M^{(1^b)} \uparrow^{S_{ab}} = M^{(a^b)}.$$

Proof. A basis of M^{μ} is the set $\{\{t\} \mid t \text{ is a } \mu \text{ tableau.}\}$. Choose $X \in I^{(a)^b}$, then a basis of $\mathrm{Inf}_{S_b}^{S_a \wr S_b} M^{(1^b)}$ is the set $\{\{r_X\} \mid r \text{ is a } (1^b) \text{ tableau}\}$. Therefore a basis for $\mathrm{Inf}_{S_b}^{S_a \wr S_b} M^{(1^b)} \uparrow^{S_{ab}}$ is the set $\{\{r_X\} \mid r \text{ is a } (1^b) \text{ tableau}, X \in I^{(a)^b}\}$ which is also a basis for $M^{(a^b)}$. $\square$

Now we will prove the following corollary.

Corollary 44. Let $\mu' = \mu^{\perp} + (1^b)$. The multiplicity of $S^{\mu'}$ in $F^{(a+1)}_{\nu^{\perp}}$ is the same as the multiplicity of S^{μ} in $F^{(a)}_{\nu}$.

Proof. We know that $S^\nu \subseteq_{\mathbb{C}S_b} M^{(1^b)}$. Therefore

$$\mathrm{Inf}_{S_b}^{S_a \wr S_b} S^\nu \subseteq_{\mathbb{C}S_a \wr S_b} \mathrm{Inf}_{S_b}^{S_a \wr S_b} M^{(1^b)}.$$

From Lemma 43

$$\mathrm{Inf}_{S_b}^{S_a \wr S_b} S^\nu \uparrow^{S_{ab}} = F_\nu^{(a)} \subseteq_{\mathbb{C}S_{ab}} M^{(a^b)} = \mathrm{Inf}_{S_b}^{S_a \wr S_b} M^{(1^b)} \uparrow^{S_{ab}} . \tag{3.15}$$

The corollary follows from the Littlewood Richardson principle, (3.15) and Theorem 33. $\square$

A generalized form of Corollary 44 has been proved recently by de Boeck, Paget and Wildon [16]. Their technique is much different.

3.5 Concluding remarks and questions

Ikenmeyer, Mulmuley and Walter [21] have proved that showing that a Kronecker coefficient is non zero is NP-Hard problem. Bürgisser and Ikenmeyer [22] show the computing Kronecker coefficient is $\#P$-Hard. Theorem 33 gives a weak relationship between Kronecker coefficients and the Foulkes module which suggests that similar statements may hold for the multiplicity of S^λ in $F_\nu^{(a)}$.

Question 45. What is the computational complexity of the coefficient of S^λ in F_ν^a?

For a partition $\lambda \vdash a$, one may define F_ν^λ, a further generalization of $F_\nu^{(a)}$, see [15].

Question 46. Find the analogue of (3.1) for $F_\nu^\lambda \downarrow_{S_k \times S_{n-k}}$. Ideally, there should exist a distinguished component of $F_\nu^\lambda \downarrow_{S_k \times S_{n-k}}$ with description similar to the one in Theorem 33.

Chapter 4

Restrictions of the Foulkes module to small subgroups

In this chapter and the next we are going to prove some properties of $F^{(a)}_{(b)} \downarrow_H$ for some abelian subgroups H of S_n.

Let χ be an irreducible character of H and ψ^a_b be the character of the Foulkes module $F^{(a)}_{(b)}$. If $G = S_a \wr S_b$ with the underlying set $\{\{1, \ldots, a\}, \ldots, \{ab-a+1, \ldots, ab\}\}$ and D is the set of representatives of double cosets of G and H then according to Mackey's Theorem 10,

$$\langle \psi^b_a \downarrow_H, \chi \rangle \quad = \textstyle\sum_{\sigma \in D} \langle 1_{S_a \wr S_b{}^\sigma \cap H} \uparrow^H, \chi \rangle \tag{4.1}$$

After applying Frobenius reciprocity, Theorem 12, we get

$$\langle \psi^b_a \downarrow_H, \chi \rangle = \sum_{\sigma \in D} \langle 1_{S_a \wr S_b{}^\sigma \cap H}, \chi \downarrow_{S_a \wr S_b{}^\sigma \cap H} \rangle. \tag{4.2}$$

Since H is an abelian subgroup, χ is linear. Therefore the multiplicity in (4.2) is equal to the number of $\sigma \in D$ such that $(S_a \wr S_b)^\sigma \cap H$ is in the kernel of χ.

In this chapter we focus on subgroup H generated by a long cycle.

Before we start analysing how Foulkes module behaves if we restrict it to H, we compare the degrees of $F^{(b)}_{(a)}$ and $F^{(a)}_{(b)}$ and state the following lemma without proof.

Lemma 47. Let $1 < a < b$. The ratio of the degrees of the Foulkes characters is

$$R = \frac{\psi^a_b(1)}{\psi^b_a(1)} = \frac{(b!)^a a!}{(a!)^b b!} = \prod_{i=1}^{b-a} \frac{(a+i)^{a-1}}{a!}.$$

If $a \geq 4$ then $R > 5$. If $b \geq 4a$ then $R > 2^{ab/2}$.

4.1 Enumeration of fixed point free elements of prime order

Let $h = (1, 2, \ldots, ab)$ and $H = \langle h \rangle$. The irreducible characters of H are parametrised by $0 \leq l < ab$ such that $\chi_l(h^k) = \varepsilon^{lk}$, where $\varepsilon = e^{\frac{2\pi i}{ab}}$.

Let $m \in \mathbb{N}$ and $p \mid m$. The conjugacy class of fixed point free elements of order p in S_m has size

$$N(m,p) = \frac{m!}{p^{m/p}(m/p)!}. \tag{4.3}$$

As $\prod_{i=0}^{m/p-1}(m-pi)^p \geq 2m!$, so

$$N(m,p)^p = \frac{(m)!^p}{p^m (m/p)!^p} = (m!)^{p-1} \frac{m!}{\prod_{i=0}^{m/p-1}(m-pi)^p} \leq \frac{1}{2}(m!)^{p-1}. \tag{4.4}$$

We improve (4.4). First we consider odd primes.

Lemma 48. If $p \geq 3$ then

$$N(m,p)^p \leq \frac{1}{m}(m!)^{p-1}.$$

Proof. By $k^2-1<k^2$ we get $\prod_{\substack{j=2\\ j\neq p}}^{p+1}\frac{(k-p+j)}{k^{p-1}}<1$. We use it for $k=ip$, $(i=1,\dots,m/p-1)$ to finish the proof:

$$\begin{aligned}N(m,p)^p&=\frac{(m)!^p}{p^m(m/p)!^p}=(m!)^{p-1}\frac{a!}{\prod_{i=1}^{m/p}(ip)^p}\\&=\frac{(m!)^{p-1}}{m}\prod_{j=2}^{p-1}\frac{m-p+j}{a}\prod_{i=1}^{m/p-1}\left(\prod_{\substack{j=2\\ j\neq p}}^{p+1}\frac{(i-1)p+j}{ip}\right)<\frac{(m!)^{p-1}}{m}.\end{aligned}$$

□

In fact, as p increases, it is easy to see that the ratio $\frac{N(m,p)^p}{(m!)^{p-1}}$ decreases. If $p=2$ then Lemma 48 is not true for any m. We can still improve (4.4).

Lemma 49.

$$N(2u,2)^2\leq\frac{1}{\sqrt{2u}}(2u)!.$$

Proof. We prove it by induction on u. Clearly, $\frac{1}{2}<\frac{1}{\sqrt{2}}$. Also

$$\frac{2u-1}{2u}=\sqrt{\frac{2u-2}{2u}+\frac{1}{4u^2}}.$$

By the inductive assumption

$$\frac{N(2u-2)^2}{(2u-2)!}=\prod_{i=1}^{u-1}\frac{2i-1}{2i}<\frac{1}{\sqrt{2u-2}}.$$

Therefore

$$\prod_{i=1}^{u}\frac{2i-1}{2i}<\frac{1}{\sqrt{2u-2}}\sqrt{\frac{2u-2}{2u}+\frac{1}{4u^2}}=\sqrt{\frac{1}{2u}+\frac{1}{(4u^2)(2u-2)}}<\frac{1}{\sqrt{2u}}.$$

Thus

$$N(2u,2)^2=(2u)!\left(\prod_{i=1}^{u}\frac{2i-1}{2i}\right)<\frac{(2u)!}{\sqrt{2u}}.$$

□

Let $h_f \in H$ be a fixed point free element of order f and $N(S_a \wr S_b, f)$ be the number of fixed point free elements of order f in $S_a \wr S_b$. Then

$$\psi_a^b(h_f) = \frac{\mid C_{S_{ab}}(h_f) \mid}{\mid S_a \wr S_b \mid} N(S_a \wr S_b, f). \tag{4.5}$$

Let p be a prime factor of ab. Every element of $S_a \wr S_b$ is of the form (ϕ, h) where $\phi \in S_a^b$, $h \in S_b$, see Definition 29. If (ϕ, h) has order p then h also has order p. In the formula below we separate the elements according to k, the number of p-cycles of h. (That is, the number of fixed points of h is $b - kp$.) The components of ϕ corresponding to the moved elements of h are arbitrary, while those components corresponding to the fixed points of h are fixed-point-free elements of order p in S_a. In particular, if $p \nmid a$ then only the $k = b/p$ summand is nonzero. Lemma 50 states that even if $p|a$, the last summand dominates (for odd p).

$$N(S_a \wr S_b, p) = \sum_{k=0}^{\lfloor b/p \rfloor} \binom{b}{kp} N(a, p)^{b-kp} N(kp, p)(a!)^{kp-k}. \tag{4.6}$$

We can now prove the following lemma. We changed the parameter names not to fix which of a, b is larger. In the following we always use m, n if their order is unimportant and a, b only when $a < b$.

Lemma 50. Let $m > n$ and $p \geq 3$ be a prime factor of mn. Let $k_0 = \lfloor n/p \rfloor$.

$$N(S_m \wr S_n, p) < (1 + \frac{2}{p!}) \binom{n}{k_0 p} N(n, p)^{b-k_0 p} N(k_0 p, p)(m!)^{k_0(p-1)}$$

Proof. Since $m > n$, therefore, as $k \leq k_0$, we have $\frac{m}{k} \geq \frac{m}{k_0} > p$. Lemma 48 gives

$$N(m, p)^{n-kp} \frac{(m!)^{k(p-1)}}{k!} \geq pN(m, p)^{n-(k-1)p} \frac{(m!)^{(k-1)(p-1)}}{(k-1)!}. \tag{4.7}$$

We also have the following inequality

$$\binom{n}{kp} \frac{kp!}{p^k} = \binom{n}{kp-p} \frac{(kp-p)!}{p^{k-1}} \frac{\prod_{i=1}^{p}(n-kp+i)}{p} \geq (p-1)! \binom{n}{kp-p} \frac{(kp-p)!}{p^{k-1}}. \tag{4.8}$$

Combining (4.7) and (4.8) we obtain

$$\binom{n}{kp}\frac{kp!}{p^k}N(m,p)^{n-kp}\frac{(m!)^{kp-k}}{k!} \geq p!\binom{n}{kp-p}\frac{(kp-p)!}{p^{k-1}}N(m,p)^{n-kp+p}\frac{(m!)^{(k-1)(p-1)}}{(k-1)!}.$$

Therefore, each term in the sum (4.6) is larger than the preceding by a factor of $p!$. □

Though lemma 50 is only true for $m > n$ and $p \leq 3$ but it does not indicate that we cannot have a nice estimation for other cases. The following lemma considers cases that are not covered here.

Lemma 51. Let $p > 2$ be a prime factor of mn, $n \neq p+1$ and $k_0 = \lfloor n/p \rfloor$.

$$N(S_m \wr S_n, p) \leq \frac{1}{n}(1+\frac{2}{m})n!N(m,p)^{n-k_0p}(m!)^{k_0p-k_0}.$$

Proof. Let us assume that $n \neq p+1$ and $j \leq \lfloor n/p \rfloor$. We have the following inequality

$$p^j j!(n-jp)! \geq pj(n-pj)! \geq pj(n-jp) > n, \tag{4.9}$$

if $n - pj > 1$. If, however, $n - pj = 1$ then, by assumption $n \neq p+1$, so $j \geq 2$. Therefore

$$p^j j!(n-jp)! \geq p^2 j(n-pj) \geq p(n-1) > n.$$

If $n - pj = 0$, then $p^j j! \geq n$. We obtain, using (4.3), that

$$\binom{n}{jp}N(jp,p) = \binom{n}{jp}\frac{(jp)!}{p^j j!} = \frac{n!}{p^j j!(n-jp)!} \leq \frac{1}{n}n!. \tag{4.10}$$

By Lemma 48

$$\begin{aligned}
N(S_m \wr S_n, p) &= \sum_{k=0}^{k_0} \binom{n}{kp} N(kp,p) N(m,p)^{n-kp}(m!)^{kp-k} \\
&\leq \sum_{k=0}^{k_0} \frac{1}{n} n! N(m,p)^{n-kp}(m!)^{kp-k} \\
&\leq \frac{n!}{n} \sum_{k=0}^{k_0} \frac{1}{m^{k_0-k}} N(m,p)^{n-k_0p}(m!)^{k_0p-k_0} \\
&\leq \frac{1}{n}(1+\frac{2}{m}) n! N(m,p)^{n-k_0p}(m!)^{k_0p-k_0}.
\end{aligned}$$

□

Using Lemma 49 instead of Lemma 48 we obtain the analogous

Lemma 52. Let mn be even, $n \neq 3$ and $k_0 = \lfloor n/2 \rfloor$.

$$N(S_m \wr S_n, 2) \leq \frac{1}{n}(1+\frac{2}{\sqrt{m}}) n! N(m,p)^{n-2k_0}(m!)^{k_0}.$$

4.2 The long cycle

Since $p \mid mn$, if $n = p+1$ and $m < n$ then $m = p$. As $\frac{m}{k} \geq p$ we may use (4.7) and (4.8) we get

$$N(S_p \wr S_{p+1}) \leq (1+\frac{1}{p!}) N(p,p)(p!)^p = (1+\frac{1}{p!})(p-1)!(p!)^p. \tag{4.11}$$

Lemma 53. Let $P_{ab} = \{p_1, \ldots, p_m\}$ be the set of prime divisors of ab and L_i be the set of all possible products of i distinct prime divisors of ab. If l and ab are coprime then

$$ab\langle \psi_a^b \downarrow_H, \chi_l \rangle = \frac{ab!}{(a!)^b b!} + \sum_{i\geq 1} \sum_{l_i \in L_i} \psi_a^b (h^{\frac{ab}{l_i}})(-1)^i. \tag{4.12}$$

Proof. For any $m \mid ab$, the elements h^m and h^{nm} have the same cycle structure and thus belong to the same conjugacy class in S_{ab} if and only if $(n, ab/m) = 1$. In that case ψ_a^b has the same value on them. At the same time χ_l takes the values $\chi_l(h^{nm}) = \varepsilon^{mnl}$

on them for $1 \leq n \leq ab/m$; $(n, ab/m) = 1$. Hence

$$ab\langle \psi_a^b \downarrow_H, \chi_l \rangle = \sum_{1 \leq m \leq ab} \psi_a^b(h^m)\overline{\chi_l(h^m)} = \sum_{m|ab} \psi_a^b(h^m) \sum_{\substack{1 \leq n \leq ab/m \\ (n,ab/m)=1}} \varepsilon^{-mnl}$$
$$= \frac{(ab)!}{(a!)^b b!} + \sum_{\substack{m|ab \\ m \neq ab}} \psi_a^b(h^m)\mu(ab/m),$$

where μ is the Möbius function. Putting $m = ab/l_i$ gives the lemma. □

The Foulkes conjecture states that if $b > a$ then $F_{(b)}^{(a)}$ is a submodule of $F_{(a)}^{(b)}$. We analyse the conjecture by restricting to the subgroup H. The restriction of Foulkes conjecture is equivalent to proving

$$\langle \psi_b^a \downarrow_H, \chi \rangle \leq \langle \psi_b^a \downarrow_H, \chi \rangle$$

for any irreducible character χ of H. The first theorem of this chapter deals with the case of ab being a prime power. The following lemma will also be useful later.

Lemma 54. Let p be a divisor of mn. Then

$$\frac{p^{mn/p}(mn/p)!(m!)^{n-n/p}n!}{(mn)!} \leq c_{m,n,p} = \left(\sqrt{\frac{2\pi m}{n^m}}\right)^{n(1-1/p)} n^{n-\frac{1}{2}mn(1-1/p)}e^{-n+1+n/12m}.$$

Proof. We use Sterling's formula for estimating $d!$, that is

$$\left(\frac{d}{e}\right)^d e^{\frac{1}{12d+1}} \leq \frac{d!}{\sqrt{2\pi d}} \leq \left(\frac{d}{e}\right)^d e^{\frac{1}{12d}}. \tag{4.13}$$

Therefore

$$\begin{aligned}
p^{mn/p}(mn/p)! \leq& \sqrt{(2\pi mn/p)}(mn)^{mn/p}e^{-mn/p+p/12mn} \\
(m!)^{n-n/p} \leq& (\sqrt{2\pi m}m^m)^{n(1-p)}e^{-mn+mn/p+n/12m-n/12pm} \\
n! \leq& \sqrt{2\pi n}n^n e^{-n+1/12n} \\
(mn)! \geq& \sqrt{2\pi mn}(mn)^{mn}e^{-mn+\frac{1}{12mn+1}}.
\end{aligned}$$

Therefore

$$\frac{p^{mn/p}(mn/p)!(m!)^{n-n/p}n!}{(mn)!} \leq \left(\sqrt{\frac{2\pi m}{n^m}}\right)^{n(1-1/p)} n^{n-\frac{1}{2}mn(1-1/p)}e^{-n+1+n/12m}.$$

□

We can see that there if we can see that $p^{mn/p}(mn/p)!(m!)^{n-n/p}n!$ decays much faster as compared to $(mn)!$ as we increase p. We use Lemma 54 in the following Corollary.

Corollary 55. Let h_p be a fixed point free element of prime order p in H. For any $m, n \in \mathbb{N}$,

$$\psi_m^n(h_p) \leq \frac{c_{m,n,p}}{2n}\frac{(mn)!}{(m!)^n n!}.$$

Proof. Let $k_0 = \lfloor n/p \rfloor$. By (4.5) and Lemmas 51 and 52, we get

$$\begin{aligned}
\mid S_m \wr S_n \mid \psi_m^n(h_p) =& \mid C_{S_{mn}}(h_p) \mid N(S_m \wr S_n, p) \\
&\leq \frac{1}{n}(1+\frac{2}{\sqrt{m}}) \mid C_{S_{mn}}(h_p) \mid n!(m!)^{n-n/p}.
\end{aligned}$$

Lemma 54 implies

$$\mid C_{S_{mn}}(h_p) \mid n!(m!)^{n-n/p} = p^{mn/p}(mn/p)!n!(m!)^{n-n/p} \leq c_{m,n,p}(mn)!$$

therefore we conclude

$$\psi_m^n(h_p) \leq \frac{c_{m,n,p}}{2n}\frac{(mn)!}{(m!)^n n!}. \tag{4.14}$$

thus proving the corollary. □

Theorem 56. Let p be a prime, ab be a power of p, $a < b$ and $l \leq ab$. If $p \nmid l$ then

$$\langle \psi_b^a \downarrow_H, \chi_l \rangle \leq \langle \psi_a^b \downarrow_H, \chi_l \rangle.$$

Proof. Let h_p be a fixed-point-free element of order p. As there are only two terms in (4.12) of Lemma 53, we need to show that for $\chi = \chi_l$,

$$ab\langle \psi_a^b \downarrow_H, \chi \rangle = \frac{ab!}{(a!)^b b!} - \psi_a^b(h_p) \geq \frac{ab!}{(b!)^a a!} - \psi_b^a(h_p) = ab\langle \psi_b^a \downarrow_H, \chi \rangle.$$

Since $ap \leq b$, we have $2(a!)^b b! < (b!)^a a!$. By Corollary 55, $\psi_a^b(h_p) \leq \frac{1}{b}\frac{(ab)!}{(a!)^b b!}$. Then we get the required inequality:

$$\begin{aligned} \frac{(ab)!}{(b!)^a a!} - \psi_b^a(h_p) < \frac{(ab)!}{(b!)^a a!} < \frac{1}{2}\frac{(ab)!}{(a!)^b b!} \\ \leq (1 - \frac{1}{b})\frac{(ab)!}{(a!)^b b!} \leq \frac{(ab)!}{(a!)^b b!} - \psi_a^b(h_p). \end{aligned}$$

□

From Corollary 55 we get that the character value of an element of prime order in H is so small that if $\gcd(l, ab) = 1$ the multiplicity of χ_l in $\psi_a^b \downarrow_H$ is approximately equal to the degree of ψ_a^b divided by ab if it is a prime power. Can the same approximation hold true if l and ab are not co-prime?

For any l, there could be many terms in the multiplicity of χ_l. We show that irrespective of whether $\gcd(l, ab) = 1$ a result analogous to Theorem 56 is true.

First we compare the character values, $\psi_m^n(h_{\frac{k}{p}})$ and $\psi_m^n(h_{p^{k+1}})$, p being a prime divisor of mn. We show the following inequalities.

Lemma 57. Let $p \mid n$ and $q \mid m$ be are arbitrary. Then

$$\frac{p^{mn/p}(mn/p)!}{(pq)^{mn/pq}(mn/pq)!} \geq p^{\frac{mn}{p} - \frac{mn}{pq}} \left(\frac{m!}{q^{m/q}(m/q)!} \right)^{n/p}.$$

Proof. Observe that

$$\prod_{\substack{i=1\\ q\nmid i}}^{mn/p}(ip) \geq \prod_{\substack{j=1\\ q\nmid j}}^{m}(jp)^{n/p},$$

because the number of factors are $mn/p - mn/pq$ on both sides, but on the right hand side they increase at a slower pace. But

$$\prod_{\substack{i=1\\ q\nmid i}}^{mn/p}(ip) = \frac{p^{mn/p}(mn/p)!}{(pq)^{mn/pq}(mn/pq)!}$$

and

$$\prod_{\substack{j=1\\ q\nmid j}}^{m}(jp)^{n/p} = p^{\frac{mn}{p}-\frac{mn}{pq}}\prod_{\substack{j=1\\ q\nmid j}}^{m} j^{n/p} = p^{\frac{mn}{p}-\frac{mn}{pq}}\left(\frac{m!}{q^{m/q}(m/q)!}\right)^{n/p},$$

so we are done.

□

Lemma 58. Let p, p_1, q be arbitrary and $p \mid n,\ p_1 q \mid m$. Then

$$p^{mn/pp_1}\frac{(mn/pp_1)!}{((m/p_1)!)^{n/p}} \geq p^{\frac{mn}{pp_1}-\frac{mn}{pqp_1}}p^{mn/pqp_1}\frac{(mn/pqp_1)!}{((m/qp_1)!)^{n/p}}$$

Proof. The p-powers cancel so it is enough to show that $\frac{(mn/pp_1)!}{((m/p_1)!)^{n/p}} \geq \frac{(mn/pqp_1)!}{((m/qp_1)!)^{n/p}}$. Both these are polynomial coefficients (with n/p equal parts):

$$\binom{\frac{mn}{pp_1}}{\frac{m}{p_1};\ldots;\frac{m}{p_1}} = \frac{(mn/pp_1)!}{((m/p_1)!)^{n/p}} \quad \text{and} \quad \binom{\frac{mn}{pqp_1}}{\frac{m}{qp_1};\ldots;\frac{m}{qp_1}} = \frac{(mn/pqp_1)!}{((m/qp_1)!)^{n/p}}.$$

Clearly, the first is larger, for there are more ways to choose t equal subsets of a set of size lq than t equal subsets of a set of size l. □

Lemma 59. Let p, q be arbitrary and $pq|n$. Then

$$\frac{p^{mn/p}(mn/p)!}{(pq)^{mn/pq}(mn/pq)!}(m!)^{\frac{n}{pq}-\frac{n}{p}} \geq p^{(m-1)(\frac{n}{p}-\frac{n}{pq})}\frac{p^{n/p}(n/p)!}{p^{n/pq}(n/pq)!}.$$

Proof. Observe that

$$\frac{p^{mn/p}(mn/p)!}{(pq)^{mn/pq}(mn/pq)!} = \frac{(p)^{n/p}(n/p)!}{(pq)^{n/pq}(n/pq)!}(\prod_{\substack{j>n/p \\ q\nmid j}}^{mn/p} jp)$$

and

$$\prod_{\substack{j>n/p \\ q\nmid j}}^{mn/p} jp = p^{(m-1)(\frac{n}{p}-\frac{n}{pq})}(\prod_{\substack{j>n/p \\ q\nmid j}}^{mn/p} j).$$

Since

$$(m!)^{\frac{n}{p}-\frac{n}{pq}} = \prod_{i=2}^{m} i^{\frac{n}{p}-\frac{n}{pq}} \leq \prod_{\substack{j=2 \\ q\nmid j}}^{\frac{mn}{p}-\frac{n}{p}+1} j \leq \prod_{\substack{j>n/p \\ q\nmid j}}^{mn/p} j,$$

the inequality is true.

□

Let ab be a p-prime power and H be the group generated by a long cycle. We compare $\psi_n^m(h_{p^k})$ with $\psi_n^m(h_{p^{k-1}})$. We show that $\psi_n^m(h_{p^k})$ decays exponentially and therefore the multiplicity of the H module χ_l, with $\gcd(l, ab) > 1$, is close to $\psi_n^m(1) + (p-1)\psi_n^m(h_p)$.

Before we prove the Theorem we need another inequality.

Lemma 60. Let p, q be arbitrary and $pq|m$. Then

$$\frac{((m/pq)!)^n(mn/p)!}{((m/p)!)^n(mn/pq)!} \geq (p/q)^{\frac{n-1}{2}} n^{\frac{mn}{p}-\frac{mn}{pq}}.$$

Proof. The proof directly follows from Sterling's formula (4.13). □

Now we state and prove Theorem 61.

Theorem 61. Let p be a prime divisor of mn. Then

$$\frac{\psi_n^m(h_{p^{k+1}})}{\psi_n^m(h_{p^k})} \leq p^{-m(1-1/p)}(1 + p^{3-p-m/p}p^{-(m-2)(p-2)} + p^{-(m-1)(p-1)}).$$

Proof. By (4.5), to calculate the character value of $h_{p^{k+1}}$ we need to calculate the number of fixed point free elements $g \in S_m \wr S_n$ of order p^{k+1}. Depending on the cycle structure of $\overline{g} \in S_n$ we separate two categories. Let $k' = \min(k, \log_p n)$. Then

$$\begin{aligned} F_1 =&\{g \in S_m \wr S_n \text{ fixed point free} \mid o(\overline{g}) \leq p^{k'}\}; \\ F_2 =&\{g \in S_m \wr S_n \text{ fixed point free} \mid o(\overline{g}) = p^{k'+1}\}. \end{aligned}$$

Let $p_1 = p^k$ and $n = a_{k'}p^{k'} + a_{k'-1}p^{k'-1} \dots a_1 p + a_0$ be a partition of n such that $a_l = 0$ if $mp^l < p^{k+1}$. Then in the products

$$\prod_{\substack{i=1 \\ p \nmid i}}^{mn/p_1} i; \qquad (\prod_{\substack{i=1 \\ p \nmid i}}^{a_{k'}mp^{k'-k}} i) \cdots (\prod_{\substack{i=1 \\ p \nmid i}}^{a_1 mp^{1-k}} i)(\prod_{\substack{i=1 \\ p \nmid i}}^{a_0 mp^{-k}} i)$$

the number of factors is the same, $nm(p-1)/pp_1$, but on the right hand side these are not larger than in the left hand side. So

$$\frac{p_1^{mn/p_1}(mn/p_1)!}{(pp_1)^{mn/pp_1}(mn/pp_1)!} = \prod_{\substack{i=1 \\ p \nmid i}}^{mn/p_1}(ip_1) \geq (\prod_{\substack{i=1 \\ p \nmid i}}^{a_{k'}mp^{k'-k}}(ip_1)) \cdots (\prod_{\substack{i=1 \\ p \nmid i}}^{a_0 mp^{-k}}(ip_1)). \tag{4.15}$$

Let the number of fixed point free elements $g \in S_m \wr S_n$ of order p^k and cycle type $\overline{g} = \{a_{k'}, \dots, a_0\} = (a_{k'}p^{k'}, a_{k'-1}p^{k'-1} \cdots a_1 p, a_0)$ be $N^k_{\{a_{k'}, \dots, a_0\}}$. Let $l_i = k - k_i$ and

$l' = k - k'$. Then

$$\begin{aligned} N^k_{\{a_{k'},\ldots,a_0\}} = & \binom{n}{a_{k'}p^{k'}} N(a_{k'}p^{k'}, p^{k'})(m!)^{a_{k'}(p^{k'}-1)} N(m, p^{l'})^{a_{k'}} \\ & \binom{n - a_{k'}p^{k'}}{a_{k'-1}p^{k'-1}} N(a_{k'-1}p^{k'-1}, p^{k'-1})(m!)^{a_{k'-1}(p^{k'-1}-1)} N(m, p^{l'-1})^{a_{k'-1}} \\ & \ldots \\ & \ldots \\ & \binom{a_0 + a_1 p}{a_1 p} N(a_1 p, p)(m!)^{a_1(p-1)} N(m, p^{l_1})^{a_1} \\ & N(m, p^{l_0})^{a_0}. \end{aligned}$$

Similarly, the number of elements $g \in F_1$ and cycle type of $\overline{g} = \{a_{k'}, \ldots, a_0\}$ is

$$\begin{aligned} N^{k+1}_{\{a_{k'},\ldots,a_0\}} = & \binom{n}{a_{k'}p^{k'}} N(a_{k'}p^{k'}, p^{k'})(m!)^{a_{k'}(p^{k'}-1)} N(m, p^{l'+1})^{a_{k'}} \\ & \binom{n - a_{k'}p^{k'}}{a_{k'-1}p^{k'-1}} N(a_{k'-1}p^{k'-1}, p^{k'-1})(m!)^{a_{k'-1}(p^{k'-1}-1)} N(m, p^{l'})^{a_{k'}} \\ & \ldots \\ & \ldots \\ & \binom{a_0 + a_1 p}{a_1 p} N(a_1 p, p)(m!)^{a_1(p-1)} N(m, p^{l_1+1})^{a_1} \\ & N(m, p^{l_0+1})^{a_0}. \end{aligned}$$

It follows that

$$\frac{|C_{S_{mn}}(h_{p^{k+1}})| N^{k+1}_{\{a_{k'},\ldots,a_0\}}}{|C_{S_{mn}}(h_{p^k})| N^{k}_{\{a_{k'},\ldots,a_0\}}} = \frac{|C_{S_{mn}}(h_p^{k+1})| N(m, p^{l'+1})^{a_{k'}} \cdots N(m, p^{l_0+1})^{a_0}}{|C_{S_{mn}}(h_p^{k})| N(m, p^{l'})^{a_{k'}} \cdots N(m, p^{l_0})^{a_0}}.$$

By using (4.15) and Lemmas 57, 58 and Lemma 60 we get

$$\frac{|C_{S_{mn}}(h_{p^{k+1}})| N^{k+1}_{\{a_{k'},\ldots,a_0\}}}{|C_{S_{mn}}(h_{p^k})| N^{k}_{\{a_{k'},\ldots,a_0\}}} \leq \prod p^{-a_i m(1-1/p)} = p^{-(\sum a_i) m(1-1/p)} \leq p^{-m(1-1/p)},$$

since $\sum a_i \geq 1$. Therefore

$$|C_{S_{mn}}(h_{p^{k+1}})||F_1| \leq p^{-m(1-1/p)}\psi_n^m(h_{p^k}).$$

We now consider the second category, F_2. If F_2 is non-empty then $a_{k'+1} \neq 0$ for at least one element of $S_m \wr S_n$. This can only happen if $k' = \min(k, \log_p n) < \log_p n$. Therefore $k' = k$. First we consider a subcategory F_3 of F_2.

$$F_3 = \{g \in F_2 \mid \text{ cycle type of } \overline{g} = \{a_{k+1}, a_k, \dots a_0\} \text{ and } a_k = 0\}.$$

Let $\{b_k, b_{k-1} \dots b_0\} = \{pa_{k+1}, a_{k-1} \dots, a_0\}$. For F_3, we compare $N^{k+1}_{\{a_{k+1},\dots,a_0\}}$ with $N^k_{\{b_k,b_{k-1}\dots b_0\}}$.

The first factor of $N^{k+1}_{\{a_{k+1},\dots,a_0\}}$

$$\binom{n}{a_{k+1}p^{k+1}} N(a_{k+1}p^{k+1}, p^{k+1})(m!)^{a_{k+1}(p^{k+1}-1)}.$$

Similarly, the first factor of $N^k_{\{b_k,b_{k-1}\dots b_0\}}$ equals

$$\binom{n}{pa_{k+1}p^k} N(pa_{k+1}p^k, p^k)(m!)^{pa_{k+1}(p^k-1)}.$$

Therefore

$$\frac{|C_{S_{mn}}(h_{p^{k+1}})|N^{k+1}_{\{a_{k+1},\dots,a_0\}}}{|C_{S_{mn}}(h_{p^k})|N^k_{\{b_k,\dots,b_0\}}} = \frac{|C_{S_{mn}}(h_{p^{k+1}})|(m!)^{a_{k+1}(p-1)}(p^k)^{pa_{k+1}}(pa_{k+1})!}{|C_{S_{mn}}(h_{p^k})|(p^{k+1})^{a_{k+1}}(a_{k+1}!)} \frac{N(m,p^l)^{a_{k-1}}\cdots N(m,p^{l_0+1})^{a_0}}{N(m,p^{l-1})^{b_{k-1}}\cdots N(m,p^{l_0})^{b_0}}.$$

From (4.15) and Lemmas 57-60 we get

$$\frac{|C_{S_{mn}}(h_{p^{k+1}})|N^{k+1}_{\{a_{k+1},\dots,a_0\}}}{|C_{S_{mn}}(h_{p^k})|N^k_{\{b_k,\dots,b_0\}}} \leq p^{-a_{k+1}(m-1)(p-1)}p^{-(\sum_{i<k} a_i)m(1-1/p)} \leq p^{-(m-1)(p-1)}.$$

Now consider the remaining $F_2 \setminus F_3$, that is, when $a_k \neq 0$. Let $\{b_k, b_{k-1}, \ldots, b_0\} = \{pa_{k+1} + a_k, a_{k-1}, \ldots, a_0\}$. The first two factors of $N^{k+1}_{\{a_{k+1},\ldots,a_0\}}$ are

$$\begin{aligned} N_1 =& \binom{n}{a_{k+1}p^{k+1}} N(a_{k+1}p^{k+1}, p^{k+1})(m!)^{a_{k+1}(p^{k+1}-1)} \text{ and} \\ N_2 =& \binom{n - a_{k+1}p^{k+1}}{a_k p^k} N(a_k p^k, p^k)(m!)^{a_k(p^k-1)} N(m,p)^{a_k}. \end{aligned}$$

The first factor of $N_{\{b_k, b_{k-1}\ldots b_0\}}$ equals

$$N_3 = \binom{n}{(pa_{k+1} + a_k)p^k} N(b_k p^k, p^k)(m!)^{b_k(p^k-1)}.$$

Note that $b_k = pa_{k+1} + a_k$. For convenience let us denote a_{k+1} as f and a_k as e. Then

$$\frac{N_1 N_2}{N_3} = \binom{pf+e}{pf} \frac{(pf)! p^{kpf} (m!)^{f(p-1)}}{p^{(k+1)} f!} N(m,p)^e.$$

By Lemmas 57 and 59 we get

$$\begin{aligned} \frac{(p^{k+1})^{(pf+e)m/p}((pf+e)m/p)!}{p^{k(pf+e)m}((pf+e)m)!} \frac{N_1 N_2}{N_3} \leq & \frac{\binom{pf+e}{pf}}{\binom{(pf+e)m}{pfm}} \Big(\frac{(p^{k+1})^{fm}(fm)!}{(p^k)^{pfm}(pfm)!}\Big) \frac{(pf)! p^{kpf}(m!)^{f(p-1)}}{p^{(k+1)} f!} \\ & \frac{(p^{k+1})^{em/p}(em/p)!}{p^{k(em)}(em)!} N(m/p)^e \\ \leq & p^{-f(m-1)(p-1)} p^{-em(1-1/p)}. \end{aligned} \tag{4.16}$$

By using (4.16), (4.15) and Lemmas 57-60 we can compute the ratio to obtain

$$\frac{|C_{S_{mn}}(h_{p^{k+1}})| N^{k+1}_{\{a_{k+1},\ldots,a_0\}}}{|C_{S_{mn}}(h_{p^k})| N^k_{\{b_k,\ldots,b_0\}}} \leq p^{-(m-1)(p-1)} p^{-m(1-1/p)}.$$

Therefore

$$\frac{|C_{S_{mn}}(h_{p^{k+1}})||F_2|}{\psi^a_b(h_{p^k})} \leq p^{-(m-1)(p-1)} + p^{-(m-1)(p-1)} p^{-m(1-1/p)}.$$

Since $p^{-(m-1)(p-1)} = p^{3-p-m/p}p^{-(m-2)(p-2)}p^{-m(1-1/p)}$ and

$$\psi_n^m(h_{p^{k+1}}) = |C_{S_{mn}}(h_{p^{k+1}})|(|F_1| + |F_2|),$$

we get

$$\frac{\psi_n^m(h_{p^{k+1}})}{\psi_n^m(h_{p^k})} \leq p^{-m(1-1/p)}(1 + p^{3-p-m/p}p^{-(m-2)(p-2)} + p^{-(m-1)(p-1)}).$$

□

For a prime power mn, we have to determine the multiplicities $\langle \psi_n^m \downarrow_H, \chi_l \rangle$. In the next lemma we prove that $\langle \psi_n^m \downarrow_H, \chi_{l_1} \rangle > \langle \psi_n^m \downarrow_H, \chi_{l_2} \rangle$ if $\gcd(l_1, mn) < \gcd(l_2, mn)$. In particular, it is the largest for $\chi_l = 1_H$ the trivial character. Then we prove Corollary 63 which is the restricted Foulkes Conjecture in a special case.

Lemma 62. Let p be a prime, mn a power of p. Let $H = \langle h \rangle \leq S_{mn}$ be a subgroup generated by a long cycle. Let $\chi_l \in \mathrm{Irr}(H), \chi_l(h^j) = \varepsilon^{lj}$ where $\varepsilon = e^{\frac{2\pi i}{mn}}$. Let $l' = \gcd(l, mn)$. If $l' < mn$ then

$$mn\langle \psi_m^n \downarrow_H, \chi_l \rangle = \frac{(mn)!}{(m!)^n n!} + \sum_{k=1}^{\log_p(l')} \psi_m^n(h_{p^k})\phi(p^k) - \psi_m^n(h_{p^{l'+1}})$$

where ϕ is Euler's totient function. While if $l' = mn$ then

$$mn\langle \psi_m^n \downarrow_H, \, \chi_l \rangle = \frac{(mn)!}{(m!)^n n!} + \sum_{k=1}^{\log_p mn} \psi_m^n(h_{p^k})\phi(p^k).$$

Proof. Note that for every k there are $\phi(p^k)$ elements of order p^k in H. The order of h^j divides p^k exactly when $mn \mid jp^k$. If $k \leq \log_p(l')$ then it also follows that $mn|lj$, so

$\varepsilon^{lj} = 1$.

$$\begin{aligned} mn\langle \psi^n_m \downarrow_H, \chi_l \rangle &= \sum_{1\le j\le mn} \psi^n_m(h^j)\overline{\chi_l(h^j)} = \sum_{k=0}^{\log_p mn} \psi^n_m(h_{p^k}) \sum_{o(h^j)=p^k} \varepsilon^{lj} \\ &= \sum_{k=0}^{\log_p(l')} \psi^n_m(h_{p^k}) \sum_{o(h^j)=p^k} \varepsilon^{lj} + \sum_{k=\log_p(l')+1}^{\log_p mn} \psi^n_m(h_{p^k}) \sum_{o(h^j)=p^k} \varepsilon^{lj} \\ &= \sum_{k=0}^{\log_p(l')} \psi^n_m(h_{p^k})\phi(p^k) + \sum_{k=\log_p(l')+1}^{\log_p mn} \psi^n_m(h_{p^k})\mu(p^k/l') \end{aligned}$$

If $k > \log_p(l') + 1$ then $\mu(p^k/l') = 0$. □

Corollary 63. Let p be a prime, $b > a$ be powers of p. Let H be the group generated by a long cycle of S_{ab}. If $\chi \in \mathrm{Irr}(H)$ then

$$\langle \psi^b_a \downarrow_H, \chi \rangle \le \langle \psi^a_b \downarrow_H, \chi \rangle.$$

Proof. We assume $a \ge 6$ as Foulkes Conjecture holds for $a \le 5$. Since $b \ge 7$, $p^{-b(1-1/p)}(1 + p^{3-p-b/p}p^{-(b-2)(p-2)} + p^{-(b-1)(p-1)}) \le p^{-b/2+1}$. By using Theorem 61 and Lemma 62 we get

$$\langle \psi^b_a \downarrow_H, \chi \rangle \le \langle \psi^b_a \downarrow_H, 1_H \rangle \le \frac{(ab)!}{(a!)^b b!} + \frac{1}{1-p^{-b/2+1}}\psi^a_b(h_p)$$

By using Corollary 55 we get

$$\begin{aligned} \frac{(ab)!}{(b!)^a a!} + \frac{1}{1-p^{-b/2+1}}\psi^b_a(h_p) < (1 + \frac{c_{b,a,p}}{2(1-p^{-b/2+1})})(\frac{(ab)!}{(b!)^a a!}) < \frac{1}{2}\frac{(ab)!}{(a!)^b b!} \\ \le (1 - \frac{c_{a,b,p}}{2})\frac{(ab)!}{(a!)^b b!} \le \frac{(ab)!}{(a!)^b b!} - \psi^a_b(h_p). \end{aligned}$$

Let $\gcd(l, ab) = 1$. By Lemma 62, we have $\langle \psi^a_b \downarrow_H, \chi_l \rangle = \frac{(ab)!}{(a!)^b b!} - \psi^a_b(h_p)$. Putting these together, using Lemma 62 again:

$$\langle \psi^a_b \downarrow_H, \chi \rangle \ge \langle \psi^a_b \downarrow_H, \chi_l \rangle \ge \langle \psi^b_a \downarrow_H, 1_H \rangle.$$

□

4.3 Concluding remarks

Theorem 61 proves that the Foulkes character at h_{p^k}, for a prime p, decays exponentially as k increases. We conjecture that for any arbitrary m and n, there is an exponential decay of the character values at $h_{n^k m}$ as k increases.

Conjecture 64. Let $m, n \mid ab$ be arbitrary. Then $\psi_a^b(h_{n^k m})$ decays exponentially as we increase k.

Corollary 63 proves that if ab is a prime power then the restricted Foulkes conjecture is true on the group generated by a long cycle. Conjecture 64 would enable to extend it for arbitrary ab.

Chapter 5

Restrictions of the Foulkes module to elemantary abelian subgroups

In this chapter we focus on the properties of the Foulkes module $F_{(b)}^{(a)}$ restricted to the elementary abelian 2-subgroups. In particular, we consider the case of the subgroups, H_k, generated by k transpositions.

In Section 5.1 we analyse if the Foulkes conjecture is true when we restrict it H_1. Then we generalize the result to some larger elementary abelian subgroups H_k. In particular we prove that the multiplicity of the trivial character in $F_{(a)}^{(b)}$ is smaller than that of $F_{(b)}^{(a)}$.

In Section 5.2 we compare the Foulkes character value of the involutions formed by k transpositions. Then we look at the multiplicities of irreducible modules of H_k in $F_{(b)}^{(a)} \downarrow_{H_K}$.

5.1 Foulkes Character value on Involutions

Let $1 < a < b$ and ψ_b^a denote the Foulkes character. Let further g_k be a product of k disjoint transpositions and $G_k = \langle g_k \rangle$, $\mathrm{Irr}(G_k) = \{1, \varepsilon\}$. Let $s(a, b, k)$ denote the number of S_{ab}-conjugates of g_k in $S_a \wr S_b$. Let

$$e = \psi_b^a(1) = \frac{(ab)!}{(a!)^b b!} \quad \text{and} \quad f = \psi_b^a(g_k) = \frac{s(a,b,k)2^k(ab-2k)!k!}{(a!)^b b!}. \tag{5.1}$$

Further, let $m = \langle \psi_b^a \downarrow_{G_k}, 1\rangle_{G_k} = \frac{e+f}{2}$ and $n = \langle \psi_b^a \downarrow_{G_k}, \varepsilon\rangle_{G_k} = \frac{e-f}{2}$ be the two multiplicities. Finally, let e', f', m' and n' denote the respective notions for changing the roles of a and b.

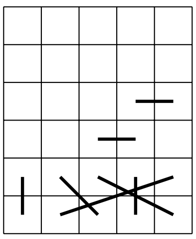
Foulkes conjecture restricted to G_k claims that $m \geq m'$ and $n \geq n'$. This is what we confirm:

Theorem 65. For $1 < a < b$, $k > 0$ we have

1. $e \geq \alpha f$, with $\alpha = \frac{b^2}{b+1} \geq 2.25$;
2. $e \geq \beta e'$, with $\beta \geq 2$;
3. $m > m'$ and $n > n'$.

Proof. In the proof of claim 1, we do not assume $a < b$ as we need this for e' and f' too!

The first claim is equivalent to $\frac{1}{\alpha} \geq \frac{f}{e} = \frac{s(a,b,k)2^k k!}{ab(ab-1)\cdots(ab-2k+1)}$. This last fraction can be reinterpreted as the probability of consecutively choosing $2k$ boxes (paired) from an $a \times b$ grid such that the pairs come out are all horizontal or completely connecting two rows. Here is and example of a good pairing with $a = 5$, $b = 6$, $k = 7$:



Suppose $k < a$, then the choice of the second box should be in the same row as the first so the probability of a correct choice is $\frac{a-1}{ab-1} < \frac{1}{b} < \frac{b+1}{b^2}$.

If, on the other hand, $k \geq a$ then possibly the second box is in another row. Suppose the first $l \leq b/2$ pairs of boxes are all from distinct rowsand the $l+1$-st is not. Then they are either in

- a) the same row,

- b) two rows that already have one "connecting" pair.

The first case, case a), can only happen if there is still an unused row left, that is $2l < b$. In that scenario the probability of choosing the second box for the $l+1$-st pair correctly is $\frac{a-1}{ab-2l-1} \leq \frac{1}{b}$.

In the second case, case b), the probability of choosing the second box correctly is $\frac{a-1}{ab-2l-1} \leq \frac{1}{b} + \frac{1}{b^2} = \frac{b+1}{b^2}$, as required.

Claim 2. is clear since the ratio of $|S_b \wr S_a|$ with $|S_a \wr S_b|$ is quite large and increases exponentially as b increases. See Lemma 47 in Section 4.1.

To prove claim 3. we observe that $\alpha(\beta - 1) \geq 2.25 > 1$ and $\frac{(\alpha-1)\beta}{\alpha} = \beta(1 - \frac{1}{\alpha}) \geq \frac{10}{9} > 1$.

Now,

$$m = \frac{e+f}{2} > \frac{e}{2} \geq \beta\frac{e'}{2} = \frac{e'}{2} + \frac{(\beta-1)}{2}e' \geq \frac{e'}{2} + \frac{(\beta-1)\alpha}{2}f' \geq m'; \text{ and}$$

$$n = \frac{e-f}{2} = \frac{\alpha e - \alpha f}{2\alpha} \geq \frac{\alpha - 1}{2\alpha}e \geq \frac{\beta(\alpha-1)}{2\alpha}e' \geq \frac{e'-f'}{2} = n'.$$

□

The claim 1 of the Theorem 65 can be extended to the order p elements of the group for a prime p.

Lemma 66. Let p be a prime and $k \leq ab/p$. Let $g_{p,k}$ be an element that is a disjoint product of k p-cycles. Then $\psi_b^a(1) \geq \alpha\psi_b^a(g_k)$ with $\alpha = \frac{b^2}{b+1}$.

Proof. The proof is along similar lines to that of Claim 1 of the Theorem 65. □

The following corollary generalizes the first part of the third claim in Theorem 65 for a sufficiently large b.

Corollary 67. Let $b > 3a, a > 1$ and $k \leq ab/2$. Let H_k be the elementary abelian 2-subgroup generated by k transpositions. Then

$$\langle \psi_a^b \downarrow_{H_k}, 1\rangle_{H_k} \leq \langle \psi_b^a \downarrow_{H_k}, 1\rangle_{H_k}.$$

Proof. Let $f_r = \psi_b^a(g_r)$ and $f'_r = \psi_a^b(g_r)$ then

$$2^k \langle \psi_a^b \downarrow_{H_k}, 1\rangle_{H_k} = e' + \sum_{l=1}^{k} \binom{k}{l} f'_l.$$

By Theorem 65, $\frac{e'}{f'_l} \geq \frac{b^2}{b+1} > b-1$. Therefore

$$e' + \sum_{l=1}^{k} \binom{k}{l} f'_l \leq e' + \sum_{l=1}^{k} \binom{k}{l} \frac{e'}{b-1} \leq (1 + \frac{2^k - 1}{b-1})e'.$$

By Lemma 47, if $b \geq 4a$, then $\frac{e}{e'} \geq 2^{ab/2} \geq 1 + \frac{2^{ab/2}-1}{b-1}$. Therefore

$$\begin{aligned} 2^k \langle \psi_a^b \downarrow_{H_k}, 1\rangle_{H_k} &\leq (1 + \frac{2^k - 1}{b-1})e' \leq (1 + \frac{2^{ab/2} - 1}{b-1})e' \\ &\leq e < 2^k \langle \psi_b^a \downarrow_{H_k}, 1\rangle_{H_k}. \end{aligned}$$

□

It can be easily verified that for $p \geq 4$, $p^{ab/p} \leq 2^{ab/2}$. So Corollary 67 can be extended to the elementary abelian p-subgroups that are generated by p-cycles.

Corollary 68. Let $b > 3a, a > 1$, $p \geq 5$ be a prime and $k \leq ab/p$. Let $H_{p,k}$ be the elementary abelian p-subgroup generated by k p-cycles Then

$$\langle \psi_a^b \downarrow_{H_{p,k}}, 1_{H_{p,k}}\rangle \leq \langle \psi_b^a \downarrow_{H_k}, 1_{H_{p,k}}\rangle.$$

Proof. Since $p^{ab/p} \leq 2^{ab/2}$, this follows from Lemma 66 by a proof analogous to that of Corollary 67. □

5.2 Elementary

Let $E_k = S_{\{1,2\}} \oplus \ldots \oplus S_{\{2k-1,2k\}} \subset S_{ab}$. We focus on the restriction of the Foulkes character, $\psi_b^a \downarrow_{E_k}$. The irreducible characters of E_k are of the form $\eta_1 \times \cdots \times \eta_k$, where $\eta_i \in \mathrm{Irr}(S_{\{2i-1,2i\}})$. The group S_k acts on E_k and on $\mathrm{Irr}(E_k)$ by permutations:

$$\sigma(2i-1, 2i) = (2_{\sigma(i)} - 1, 2_{\sigma(i)}) \tag{5.2}$$

$$\sigma(\eta_1 \times \cdots \times \eta_k) = \eta_{\sigma(1)} \times \cdots \times \eta_{\sigma(k)}. \tag{5.3}$$

Lemma 69. Let $\eta_1 \times \cdots \times \eta_k \in \mathrm{Irr}(E_k)$ and $\sigma \in S_k$. Then

$$\langle \psi_b^a \downarrow_{E_k}, \eta_1 \times \cdots \times \eta_k \rangle = \langle \psi_b^a \downarrow_{E_k}, \sigma(\eta_1 \times \cdots \times \eta_k) \rangle \tag{5.4}$$

Proof. As $\psi_b^a(g) = \psi_b^a(h)$ whenever g, h have the same cycle structure so $\sigma(\psi_b^a \downarrow_{E_k}) = \psi_b^a \downarrow_{E_k}$. Also, the two actions, (5.2) and (5.3), of σ are compatible, so

$$\langle \psi_b^a \downarrow_{E_k}, \eta_1 \times \cdots \times \eta_k \rangle = \langle \sigma(\psi_b^a \downarrow_{E_k}), \sigma(\eta_1 \times \cdots \times \eta_k) \rangle = \langle \psi_b^a \downarrow_{E_k}, \sigma(\eta_1 \times \cdots \times \eta_k) \rangle.$$

$\square$

By Lemma 69, the position of trivial or sign characters in $\eta_1 \times \cdots \times \eta_k$ does not have an effect on $\psi_b^a \downarrow_{E_k}$. Therefore, from now on let $\eta_{j,k}$ be the irreducible character of E_k where first j $\eta_i' s$ are trivial and rest are sign characters and $N_{j,k} = N_{j,k}^{a,b} = \langle \psi_b^a \downarrow_{E_k}, \eta_{j,k} \rangle$. One of the interesting properties of $N_{j,k}$ is

$$N_{j,k} = N_{j+1,k+1} + N_{j,k+1} \tag{5.5}$$

Note that since $\eta_{j,k}$ is a linear character it can also be seen as an E_k module. In fact, the following holds

$$N_{k-i,k}^{a,b} = \frac{1}{|E_k|} \sum_{j \le k} \Big(\sum \binom{i}{w} \binom{k-i}{j-w} \cdot (-1)^w \Big) \psi_b^a(a_j).$$

However, we do not use this formula later.

Let a and b be even. The restriction of the Foulkes conjecture in this particular case is equivalent to proving that $\forall j \leq ab/2$

$$N^{a,b}_{j,\frac{ab}{2}} = N^{a,b}_{j} \geq N^{b,a}_{j} = N^{b,a}_{j,\frac{ab}{2}}.$$

We abbreviate $N^{a,b}_{j,\frac{ab}{2}}$ and $N^{b,a}_{j,\frac{ab}{2}}$ as $N^{a,b}_{j}$ and $N^{b,a}_{j}$ respectively. Let $j = ab/2$ then by (5.2)

$$N^{a,b}_{0} = \frac{1}{|E_{\frac{ab}{2}}|} \sum_{i=0}^{\frac{ab}{2}} \binom{ab/2}{i} \psi^{a}_{b}(a_j). \tag{5.6}$$

Therefore proving $\psi^{a}_{b}(a_j) - \psi^{b}_{a}(a_j) \geq 0$ for each j is sufficient to prove that $N^{a,b}_{0} - N^{b,a}_{0} \geq 0$.

Let $n(a, b, r)$ be the number of involutions that are product of r transpositions in $\bigoplus^{b} S_a$. Then

$$s(a, b, r) = \sum_{i=0}^{b/2} \binom{b}{2i} (a!)^{i} \frac{(2i)!}{2^{i} i!} n(a, b - 2i, r - ai). \tag{5.7}$$

By (5.1), $\psi^{a}_{b}(a_j) - \psi^{b}_{a}(a_j) \geq 0$ is equivalent to proving that $\forall j \leq ab/2$

$$\frac{s(b, a, j)}{s(a, b, j)} \leq \frac{| S_b wr S_a |}{| S_a wr S_b |} = \frac{(b!)^{a-1}}{(a!)^{b-1}}. \tag{5.8}$$

By (5.7)

$$\frac{s(b, a, j)}{s(a, b, j)} = \frac{\sum_{i_1} \binom{a}{2i_1} (b!)^{i_1} \frac{(2i_1)!}{2^{i_1} i_1!} n(b, a - 2i_1, j - bi_1)}{\sum_{i_2} \binom{b}{2i_2} (a!)^{i_2} \frac{(2i_2)!}{2^{i_2} (i_2)!} n(a, b - 2i_2, j - ai_2)} \tag{5.9}$$

in turn implying

$$\frac{s(b, a, j)}{s(a, b, j)} \leq \max_{0 \leq i \leq a/2} \left\{ \frac{\binom{a}{2i} (b!)^{i} n(b, a - 2i, j - bi)}{\binom{b}{2i} (a!)^{i} n(a, b - 2i, j - ai)} \right\} \tag{5.10}$$

We show the following lemma for the case $j = ab/2$.

Lemma 70. Let a and b be even, $a \leq b$ and $j' = ab/2$. Then

$$\frac{s(b,a,j')}{s(a,b,j')} \leq \frac{n(b,a,j')}{n(a,b,j')}.$$

Proof. Since $n(b,1,b/2) = \prod_{i_1=0}^{b/2-1}(2i_1+1)$, a simple comparison shows that

$$(a-2)! \prod_{i_1=0}^{b/2-1}(2i_1+1)^2 \geq (b-2)! \prod_{i_2=0}^{a/2-1}(2i_2+1)^2.$$

Therefore

$$\frac{n(b,1,b/2)^2}{n(a,1,a/2)^2} = \frac{\prod_{i_1=0}^{b/2-1}(2i_1+1)^2}{\prod_{i_2=0}^{a/2-1}(2i_2+1)^2} \geq \frac{(b-2)!}{(a-2)!}. \tag{5.11}$$

Since $i \leq a/2 \leq b/2$ then $\frac{a-2i}{b-2i} < \frac{a}{b}$.Therefore

$$\frac{a(a-1)}{b(b-1)} > \frac{(a-2i)(a-2i-1)}{(b-2i)(b-2i-1)}$$

and

$$\frac{(b-2)!}{(a-2)!} > \frac{b!(a-2i)(a-2i-1)}{a!(b-2i)(b-2i-1)} = \frac{\binom{b}{2i_1}\binom{a}{2i_1+2}b!}{\binom{a}{2i_1}\binom{b}{2i_1+2}a!}.$$

Since $j' = ab/2$, we are considering the number of involutions that are fixed point free. Therefore, for any $i' \leq a/2$, $n(b, a-2i', j'-bi') = n(b,1,b/2)^{a-2i'}$. Similarly we can deduce it if we change a with b. By the above inequalities and (5.11)

$$\frac{\binom{a}{2i}(b!)^i n(b,1,b/2)^{a-2}}{\binom{b}{2i}(a!)^i n(a,1,a/2)^{a-2}} \geq \frac{\binom{a}{2i+2}(b!)^{i+1} n(b,1,b/2)^{a-2i-2}}{\binom{b}{2i+2}(a!)^{i+2} n(a,1,a/2)^{b-2i-2}}.$$

Therefore, by (5.10) and the above inequality

$$\frac{s(b,a,j')}{s(a,b,j')} \leq \frac{n(b,a,j')}{n(a,b,j')}.$$

□

If a and b and even, if we show that

$$\prod_{i=1}^{a/2}(2i)^{b-a} \leq \frac{a!}{b!}\prod_{j=a/2+1}^{b/2}(2j)^{a}. \tag{5.12}$$

by the Lemma 70 and (5.12) we can prove that

$$\frac{s(b,a,ab/2)}{s(a,b,ab/2)} \leq \frac{n(b,a,ab/2)}{n(a,b,ab/2)} \leq \frac{(b!)^{a-1}}{(a!)^{b-1}} \tag{5.13}$$

For any general j and $i \leq a/2$ showing that

$$\frac{n(b,a-2i,j-bi)}{n(a,b-2i,j-ai)} \geq \frac{(b-2)!}{(a-2)!}\cdot\frac{n(b,a-2(i+1),j-b\cdot(i+1)}{n(a,b-2(i+1),j-a\cdot(i+1)}$$

and

$$\frac{n(b,a,j)}{n(a,b,j)} \leq \frac{(b!)^{a-1}}{(a!)^{b-1}}$$

can show that the inequality in (5.8) is true and hence $N_0^{a,b} - N_0^{b,a} \geq 0$. Though these computations can help us in understanding the Foulkes conjecture but it can become a bit cumbersome.

5.3 Concluding remarks

In the proof of Corollary 63 we compare the smallest multiplicity of ψ_b^a with ψ_a^b and that the restricted Foulkes conjcture is true in the “long cycle” case. The following questions can be of interest.

Question 71. Let $N_{j,k}^{a,b} = \langle \psi_b^a \downarrow_{E_k}, \eta_{j,K}\rangle$. Can we find a pattern on $N_{j,k}$ with the parameter j?

Question 72. Let $b > a$. If $N_{s,k}^{a,b}$ is the smallest multiplicity in $\psi_b^a \downarrow_{E_k}$ and $N_{l,k}^{b,a}$ be the largest multiplicity in $psi_a^b \downarrow_{E_k}$. What are the necessary conditions on a, b so that $N_{s,k}^{a,b} \geq N_{l,k}^{b,a}$?

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