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Elliptic Fibrations in 4-Dimensional Topology and Geometry

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In partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

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Budapest, Hungary
2022

Abstract

This thesis deals with various phenomena arising in smooth 4-dimensional topology. The connecting thread between 3 parts are *elliptic fibrations* - a structure which helps us represent some 4-manifolds as 2 dimensional families of tori, with some of the tori degenerating into interesting singular surfaces.

Part I of the thesis emphasises the difference between smooth and topological 4-dimensional worlds by constructing *exotic smooth structures* on the same underlying topological space. The construction is inspired by a similar construction which uses elliptic fibrations.

Part II is concerned with analyzing interesting 4-dimensional spaces which have very rich geometry, and come equipped with an elliptic fibration. These are moduli spaces of certain *meromorphic Higgs bundles* with an underlying *Hitchin's fibration* and we analyse a class of cases when this fibration has a special fiber called an $\tilde{E}_6$*-fiber*.

Part III deals with an interesting application of elliptic fibrations different from the first part. We use configurations of spheres to understand which *knots* in the boundary of a small 4-ball in a 4-manifold, bound a disk in the interior of the manifold. Even though we prove a general result, more specifically, using a configuration of 22 spheres, we show that many complicated knots bound disks in the $K3$*-surface*. To prove there are 22 spheres in the $K3$-surface, we can use an elliptic fibration with 3 $\tilde{E}_6$-fibers. This part is joint work with Dr. Marco Marengon.

Acknowledgments

My first thank you is to my supervisors prof. András Stipsicz and prof. Szilárd Szabó, for continued support and guidance. András put me on my mathematical feet and instilled in me what it means and what it takes to make a career out of something you love. I thank him for mathematical insights and numerous hints, including all 3 projects in this thesis, but also several others. I thank him for introducing me to the field and being patient with me, I hope I will make it worth the while. Szilárd joined us during the pandemic and kept being helpful and approachable even at my lowest points. He answered and clarified my many questions, from Higgs bundles, through Lie groups, to even physics, many times over. I am happy we both persevered. I hope I will keep interacting and learning from both of them throughout my career.

I thank Marco Marengon, the best colleague one can ask for, for both clarifying my thoughts and sharing his enlightenment, but also for the endless career advice I am privileged to have received.

For various feats of math wisdom, help, encouragement, and inclusion, I thank Kyle Larson, Antonio Alfieri, Irena Matkovič, Viktória Földvári, Arunima Ray, Peter Teichner, David Gay, Darko Milinković, Jelena Katić, Jeffrey Meier, Alex Zupan, Maggie Miller, Diego Santoro, Ludovico Battista, Andrea Bianchi, Ágnes Szilárd, Rade Živaljević and András Zempléni.

I would also wholeheartedly want to thank prof. Karoly Boroczky and Elvira Kadvany who both helped me integrate and push through the period when I started the PhD still finishing my master's. I lost a tooth in the process, and several more later on (thank you Lazo for putting them back), but it was worth it. I also thank Márton Bélik and Monika Horvath for their guidance during the pandemic and later stages of my PhD. Central European University was a truly wonderful place in many ways, I incredibly enjoyed people I met here, and if it was 'just a bubble' I don't mind to keep floating. I am thankful for the hospitality of my other mathematical home, Alfréd Rényi Institute of Mathematics. Both places I can truly call *home* - at least because I slept on their couches numerous times.

I thank my master's supervisor Marko Stošić, and my first mentor Miloš Milosavljević, they always blew wind at my back, even when we somehow didn't speak for months or years.

I thank Danica, my true friend, and soon officially collaborator, for her unfaltering support, and for being one of my harshest but most benevolent critics. I hope we do great things together one day, and in the meantime, at least you are doing pretty good on your own (greetings Verice:)!

Words of thank you to my friends and family would never end, and I would not even dare making a list. Having these humans around is my biggest treasure.

I would also undoubtedly forget someone at 6am in the morning so mostly from the top of my head for relevant and irrelevant events that come to mind: thank you Filip for endless conversations, math and beyond. As you would say, you taught me homology so at least I know that much before trying to defend a PhD. Thank you Luka for all the wise stuff, the laughs and the defence against the dark forces. Thank you Maja for teaching me I am a plant and for running to Budapest when I really needed it. And also thank you Johanna for bringing that same life spark even if you didn't know. Thank you Vlado for taking me skiing, it made all the difference.

Thank you Kinga for being the best and possibly ultimate version of a human being. First you ran from me, but then you promised me infinite bread, what more can a person want. Speaking of the ultimate, Anja pozz should I say more :v. Thank you Zoka and Niki for ALWAYS sticking around. Thank you Salma, Krisz, Domi, Dasha and co. for endless beautiful days in Budapest.

Thank you Tamara and Gio for literally asking me to protect your marriage from the devils, and also thank you to all the other beautiful people who made this year exciting by throwing huge parties for reasons of love. Thank you for the babies and babies to be.

Thank you Uki for thanking me in your manuscript that these thanked people would actually want to read - even though I fully read it two years later. It's still the best. Thank you Tushita for making the board game club great again. Thank you Ema for suiting me up, and thank you Aleksa and Janja for opening my mind.

Finally, thank you to my sprinkled, complicated, crazy family. Adjoint family with a few extra moms and dads included. I love you all.

Normally I wouldn't know what to do as I am not good with formalities, but this time, you made it easy nana. I dedicate this thesis to you.

Budapest, November 2022

Contents

0 Introduction

0.1 Smooth 4-manifolds

In this short introduction, we will mainly focus on an overview of elliptic fibrations, and leave the motivation to study the questions we do, to introductions to individual parts of the thesis. For additional context we refer the reader to several books on 4-manifold topology, starting with Scorpan's beautiful motivational book - *Wild World of 4-manifolds* [Sco05], Stipsicz's and Gompf's very interesting research oriented and detailed book with many excersises - *4-manifolds and Kirby calculus* [GS99], and finally Akbulut's recent short book - *4-manifolds* [Akb16] with many illustrations and great for a reader with some experience.

Essentially, in modern 4-manifold topology we are trying to understand the difference between continuous and smooth in the dimension where it is most interesting - the unique dimension where $\mathbb{R}^4$ has a non-unique smooth structure, and the unique dimension where closed manifolds have infinitely many smooth structures.

Having a different smooth structure, that we call *exotic*, means there are two smooth atlases of the topological manifold - two coverings with charts where transition functions between charts are smooth in each atlas individually, but which cannot have smooth transitions when charts are combined. Another interesting way of looking at it, is using the embedding perspective - if these 4-manifolds are exotic copies of each other, we can find their embeddings into the same higher dimensional Euclidean space, and we can isotope these embeddings into each other, but regardless of how we choose it, the isotopy can never be smooth.

In general we look for various different smooth structures on a given topological manifold. First, we need to know which 4-manifolds admit smooth structures, an unresolved but very well understood question, and second, how do their lists of smooth structures look. At the moment, we do not have ways of understanding this question completely for any given 4-manifold, but do have results which say many 4-manifolds have infinitely many smooth structures. The biggest open problem in the field, the *smooth Poincaré conjecture*, asks whether the simplest 4-manifold, the 4-dimensional sphere $\mathbb{S}^4$, has any exotic smooth structures.

These questions motivated many results and techniques, from gauge theoretic

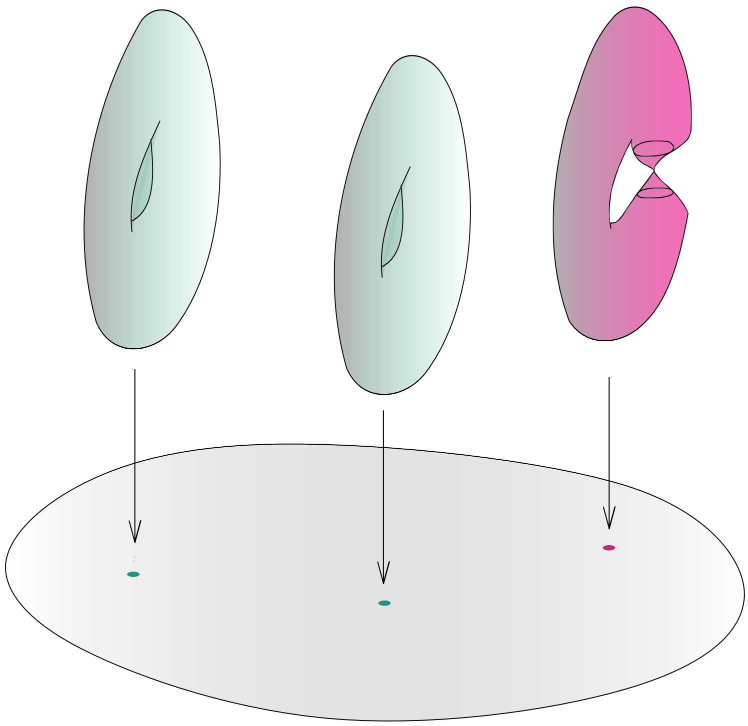


Figure 0.1: Sketch of an elliptic fibration with the simplest singular fiber, the fishtail fiber

invariants, a theory which we mention in passing in Parts I and II, to constructive techniques like the one in Part III.

In what follows, we define elliptic fibrations and give some elementary properties, but for the rest reffer to an excelent article by Stipsicz, Szabó and Szilárd [SSS07] where they give a classification of all possible combinations of singular fibers in $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$, but also survey techniques and invariants available.

0.2 Elliptic fibrations

When defining a an elliptic fibration, we can assume that both the 4-manifold X and the surface Σ that X maps to are complex, but a weaker condition also works, assuming that the map $p : X \to \Sigma$ is locally holomorphic. This means for each point $s \in \Sigma$, there is a neighborhood U_s such that the map $p_| : p^{-1}(U_s) \to U_s$ is holomorphic. This broadens the class of manifolds admiting an elliptic fibration in this sense.

Definition 0.1. *An* ellipic fibration *on a 4-manifold X is a locally holomorphic proper map $p : X \longrightarrow \Sigma$ such that there is a finite set $C = \{s_1, s_2...s_k\} \subset \Sigma$ such that if $s \in \Sigma \setminus C$, then $p^{-1}(s)$ is diffeomorphic to a 2-torus $\mathbb{T}^2$.*

The fibers $p^{-1}(s)$ for $s \in C$ are called *singular*, see Figure 0.1. In his foundational work, Kodaira [Kod63] classified all possible singular fibers in an elliptic fibration, which we list in Figure 0.3. Furthermore, taking the smallest complex elliptic surface $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$, Persson [Per90] and Miranda [Mir90] classified all the possible combinations of singular fibers that arise in this case if manifolds are complex and p is holomorphic. In [SSS07] the authors find all the possible combinations under the weaker, locally holomorphic assumption, and there are additional combinations in comparison to the globally holomorphic case. Additionally, in comparison to Persson and Miranda, they use differential topology techniques, and the article is writen in a gentle survey manner so we will use it as a further reference.

Firstly, on pages 4-6 of [SSS07] there is a full list of Kodaira's fibers with their Euler characteristics, signature and monodromy. Secondly, in the Appendix, they explain how to get various singular fibers from different *pencils of curves in* $\mathbb{CP}^2$.

Definition 0.2. *Given p_1 and p_2, two* cubic homogenious polynomials *in $\mathbb{CP}^2$, and assuming the curves defined by $p_1 = 0$ and $p_2 = 0$ intersect in only finitely many points, a pencil generated by p_1 and p_2 is the family:*

$$\{p_s = s_1p_1 + s_2p_2 | s = [s_1 : s_2] \in \mathbb{CP}^1\}$$

Curves defined by $p_s = 0$ make up a family of complex curves of degree 3, and we get a projection $x \mapsto [p_1(x) : p_2(x)]$, $\mathbb{CP}^2 \setminus \{y | p_1(y) = 0 = p_2(y)\} \rightarrow \mathbb{CP}^1$ which is almost a fibration with fibers $p_s = 0$. This family can be transformed into a fibration by *blowing-up* [GS99; Sco05] *base points* - the isolated points which are common zeros of defining polynomials. Each base point contributes to at least one section of the fibration.

Here, we present one example of a pencil made out of a curve $p_1 = 0$ consisting of three complex lines in $\mathbb{CP}^2$ and a curve $p_2 = 0$ consisting of one complex line of multiplicity 3. Figure 0.2 represents an example of a blow-up procedure leading to a special combination of singular fibers that we will encounter in Part II of the thesis.

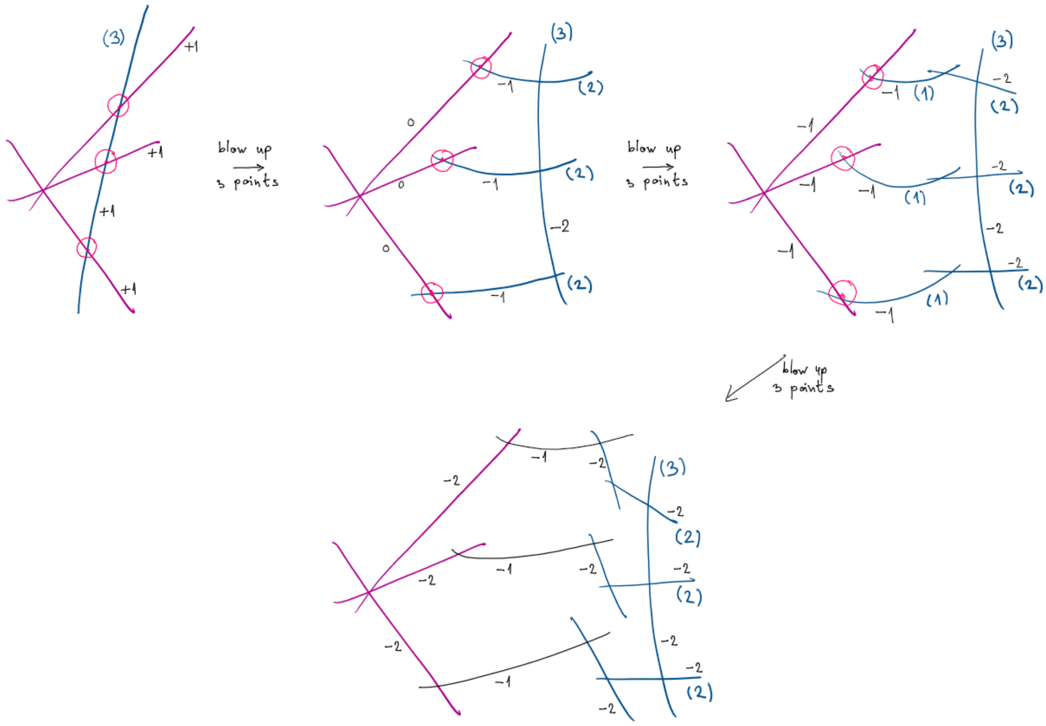


Figure 0.2: Example of a pencil and the blow-up procedure leading to an elliptic fibration with 2 singular fibers, one of type IV and the other of type $\tilde{E}_6$: The flowchart starts in the upper left corner where we have two curves, purple and blue, that define the pencil, as well as 3 marked intersection points that we blow up. The numbers in brackets describe the multiplicity of that component of the curve. In the second subfigure the 3 new curves come with self-intersection -1 and become a part of the blue curve with multiplicity 2 (notice that for the blue and purple curve to become fibers in the elliptic fibration, they have to be in the same homology class, and their self-intersections have to drop to 0 in the end). After 6 more blow ups in the next two subfigures, we manage to separate them, and have a fibration with 2 singular fibers and three -1-sections depicted by black curves.

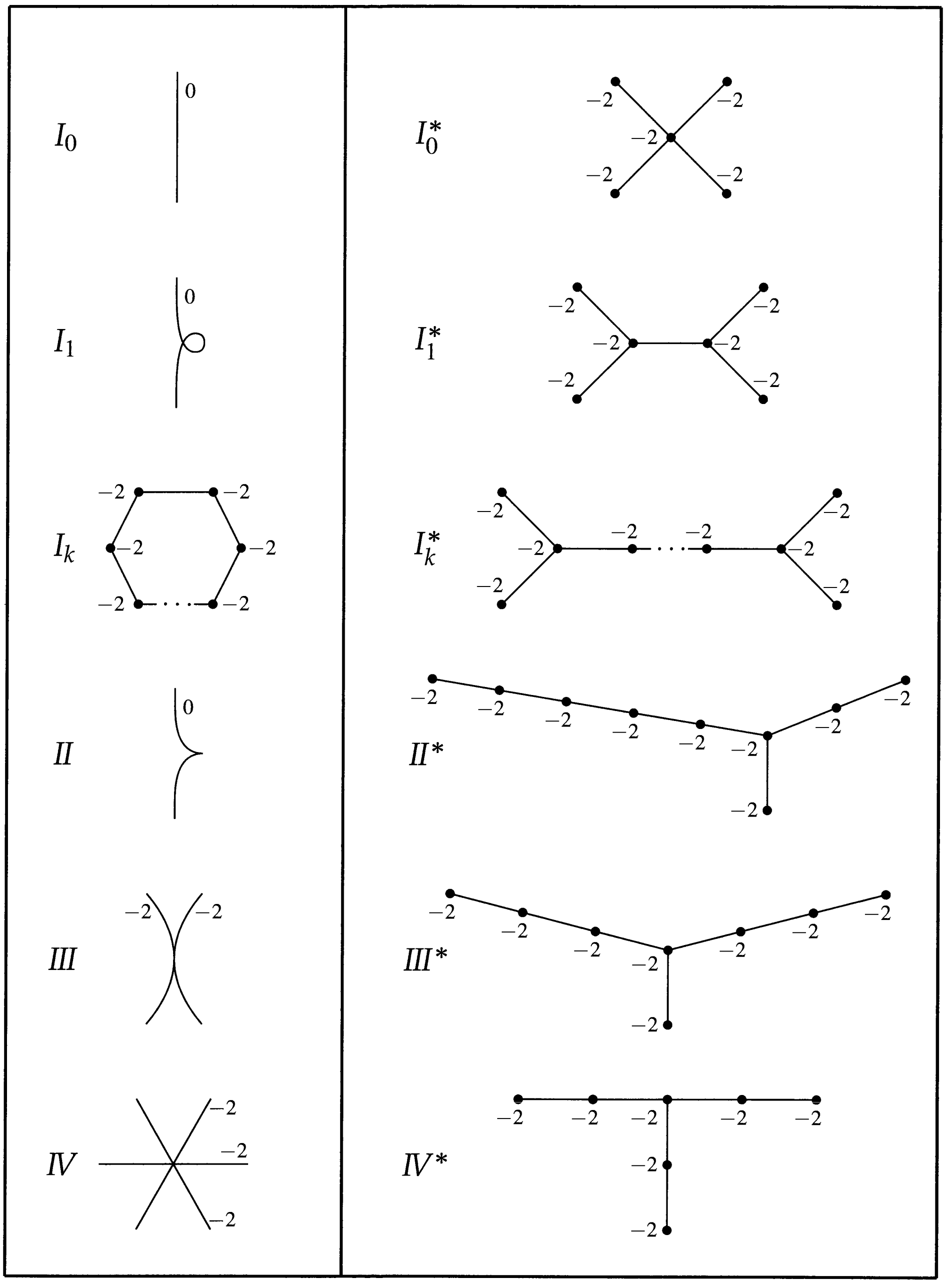


Figure 0.3: From [Sco05]: Kodaira's list of singular fibers; alternative notation for $\tilde{E}_6$ is IV^*, $\tilde{E}_7$ is III^*, and $\tilde{E}_8$ is II^*

Part I

Small exotic 4-manifolds

1 Small exotic 4-manifolds

In this part we construct potentially new manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ via *rational blowdown surgery* along certain 4-valent plumbing graphs. This way all the graph classes from [BS11] have a representative which admits a rational blowdown leading to an exotic manifold. We emphasize the simplicity of the constructions which boils down to finding a good configuration of complex lines and quadrics in $\mathbb{CP}^2$, and deciding which intersections to blow up.

1.1 Introduction

Smooth 4-manifold topology is a very intriguing field which has been shaped by many techniques and constructions in the past decades. Constructing different smooth structures on any given 4-manifold is still a challenging problem, and for many of them it is not known whether there are different smooth structures, let alone if there are an infinite number of smoothings.

The problem we will be focusing on in this part is the construction of *small* exotic 4-manifolds, meaning manifolds with small Euler characteristic and signature, homeomorphic but not diffeomorphic to some standard 4-manifolds. Donaldson first proved that a certain 4-manifold admits two different smooth structures [Don87], by using his newly constructed invariants to distinguish Dolgachev surfaces which are homeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. Since then there were several papers providing increasingly more intricate constructions of even smaller exotic 4-manifolds [AP08; AP10; Kot89; Par05; PSS05; SZ05]. In this part of the thesis we prove the following:

Theorem 1.1 (The Main Theorem I)**.** *There exists a configuration of complex lines and quadrics in $\mathbb{CP}^2$, and graphs from classes $\mathcal{B}^4$ and $\mathcal{C}^4$ shown in Figure 1.1, which can be used to produce exotic $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ via rational blowdowns.*

Examples of non-standard smooth structures on these manifolds were already known [Don87; Kot89], as well as the general technique we are using - the *rational blowdown surgery* introduced by Fintushel and Stern [FS97]. In its most general form, this surgery technique replaces an adequate embedded plumbing with some rational homology ball, changing the topology in a controlled way (see e.g. [Mic07]).

All plumbings are neighbourhoods of spheres pairwise intersecting transversely in at most one point, and the plumbing graph is a tree.

The novelty is using particular plumbings from two graph classes $\mathcal{B}^4$ and $\mathcal{C}^4$ from [BS11] shown in Figure 1.1, previously unknown to produce exotic manifolds via rational blowdown. This way we show that each class of graphs from [BS11] has a representative which admits a rational blowdown leading to an exotic manifold, which might eventually advance the understanding of smoothings of singularities discussed there.

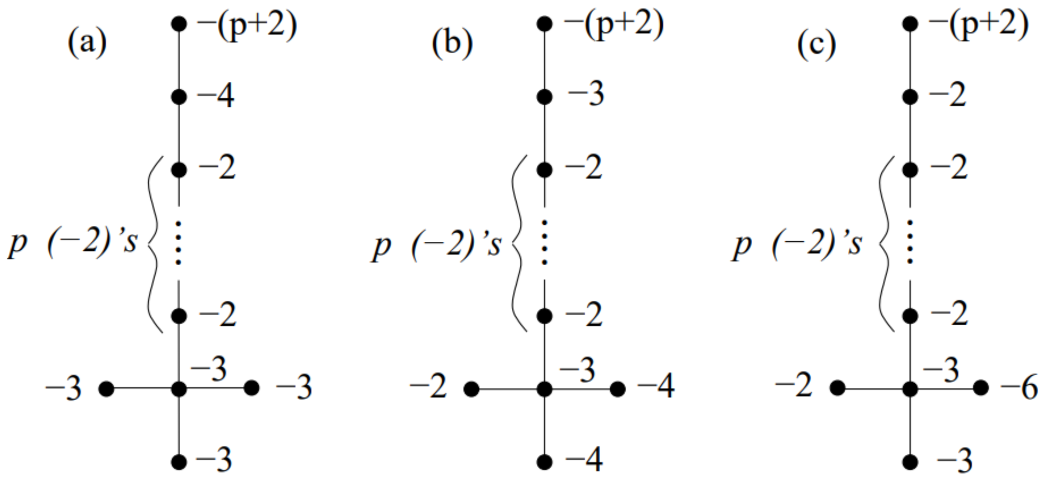


Figure 1.1: Classes $\mathcal{A}^4$, $\mathcal{B}^4$ and $\mathcal{C}^4$

Here it is worth emphasizing that we are actually not looking at a pencil of curves, blowing it up, deforming the monodromies, and rationally blowing down (see [AS19]). Rather, we start with a good configuration of degree 1 and 2 curves (complex lines and quadrics) in $\mathbb{CP}^2$ which are all already spheres by the genus-degree formula. Then we blow up some intersection points, and some additional generic points until we get a required configuration of intersecting spheres embedded in $\mathbb{CP}^2$ blown up some number of times. After rationally blowing down this configuration symplectically, we determine the homeomorphism type and show that the diffeomorphism type is not standard.

1.2 The curve configuration

The configuration of curves in $\mathbb{CP}^2$ that we start with is sketched in Figure 1.2 below. It will consist of two quadrics and four complex lines intersecting in a certain way, and it is derived by studying the configuration in the master thesis of Ta The Ahn [Anh16] where an example from class $\mathcal{A}^4$ was used in an exotic construction.

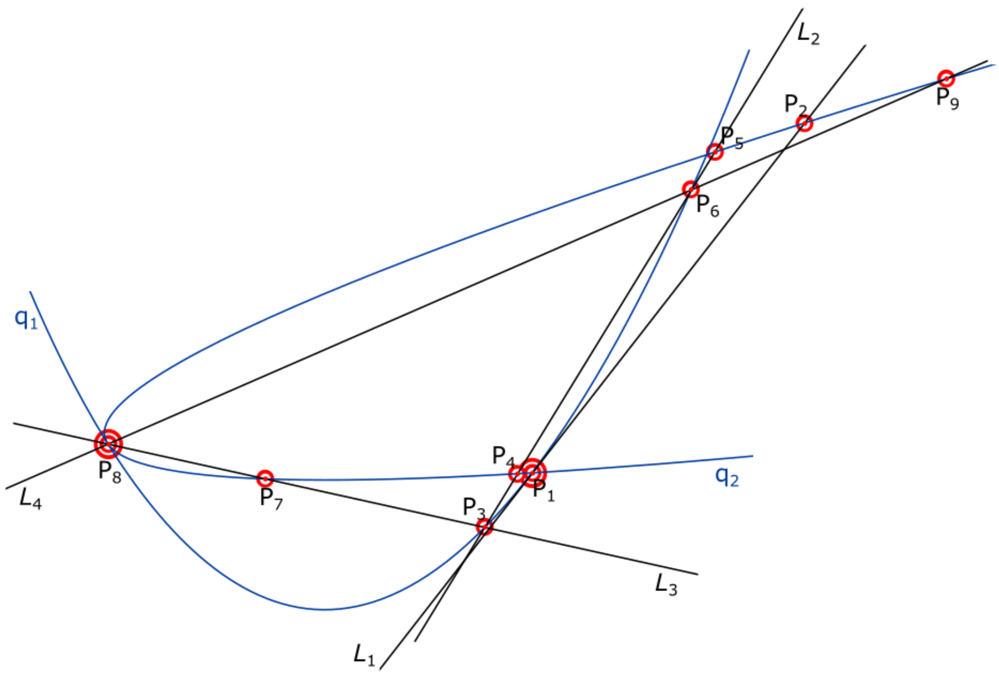


Figure 1.2: Sketch of the curve configuration

First, take two irreducible quadrics q_1 and q_2 which are tangent at one point and have two more transverse intersections. We give an example of such two quadrics, defined in standard projective coordinates in $\mathbb{CP}^2$ by homogeneous degree 2 equations:

$$z_1^2 + z_2^2 + z_3^2 = 0$$

$$z_1 z_2 + 2\sqrt{2}i \cdot z_2 z_3 + z_1 z_3 = 0$$

Their common tangency is the point $[1 : \frac{\sqrt{2}}{2}i : \frac{\sqrt{2}}{2}i]$ which we denote by P_8, and the two other intersection points of these quadrics are $[-(1+\sqrt{3})\sqrt{2}i : -(2+\sqrt{3}) : 1]$ and $[-(1-\sqrt{3})\sqrt{2}i : -(2-\sqrt{3}) : 1]$.

After constructing q_1 and q_2, we take the tangent line to q_1 at one of the transverse intersection points with q_2, denote this point by P_1 and line by L_1. This tangent line intersects q_2 in another point, denote it P_2. Now take a generic line L_2 which intersects q_1 in points we name P_3 and P_6, and intersects q_2 in P_4 and P_5. Denote by L_3 the line passing through P_8 and P_3, and by L_4 the line going through P_8 and P_6. The other intersections of L_3 and L_4 with q_2 are denoted by P_7 and P_9, respectively.

1.3 Blowing up and the incidence graph

We blow up $\mathbb{CP}^2$ as shown in Figure 1.2, starting from the point P_1 to P_9. One red circle around a point means one blow up and two circles mean we did two consecutive blow ups completely removing the intersections at the points of tangency. Exceptional curves e_1 and e_2 correspond to the point P_1, e_3 corresponds to P_2, and so on, e_9 and e_{10} correspond to P_8, and e_{11} to P_9.

In the process of blowing up a point, any curve passing through this point can be transformed in a certain way (see e.g. [GS99; Sco05]), and the result is called the *proper transform* of the curve. One effect is that proper transforms of the curves which intersect transversely in the point that is blown up, no longer intersect in that point. Another is that the homology class of the proper transform is the homology class of the initial curve minus the class of the exceptional curve. In our example, after the initial 11 blow ups, the homology classes of proper transforms of the curves and their self-intersections are as follows:

$\widetilde{q_1} = q_1 - e_1 - e_2 - e_4 - e_7 - e_9 - e_{10}$	$\widetilde{q_1} \cdot \widetilde{q_1} = -2$
$\widetilde{q_2} = q_2 - e_1 - e_3 - e_5 - e_6 - e_8 - e_9 - e_{10} - e_{11}$	$\widetilde{q_2} \cdot \widetilde{q_2} = -4$
$\widetilde{L_1} = L_1 - e_1 - e_2 - e_3$	$\widetilde{L_1} \cdot \widetilde{L_1} = -2$
$\widetilde{L_2} = L_2 - e_4 - e_5 - e_6 - e_7$	$\widetilde{L_2} \cdot \widetilde{L_2} = -3$
$\widetilde{L_3} = L_3 - e_4 - e_8 - e_9$	$\widetilde{L_3} \cdot \widetilde{L_3} = -2$
$\widetilde{L_4} = L_4 - e_7 - e_9 - e_{11}$	$\widetilde{L_4} \cdot \widetilde{L_4} = -2$

Table 1: Homology classes and self-intersections of curves after 11 blow ups

We can now form the incidence graph of the new configuration by representing curves as vertices, with an edge connecting vertices if there is an intersection between those two curves, as shown in Figure 1.3.

1.3.1 Organization

Two different ways of further blowing up intersection points in this configuration eventually give embedded plumbings from classes $\mathcal{B}^4$ and C^4 of 4-valent graphs from [BS11], and this is shown in the beginnings of Sections 1.4 and 1.5. Then we use the fact that these plumbings admit rational blowdown surgeries, and that they can be done symplectically. Finally, we find the homeomorphism types of the resulting manifolds, and prove that they are exotic. The Main Theorem stated in the introduction is comprised of Theorem 1.2 in Section 1.4 and Theorem 1.12 in Section 1.5.

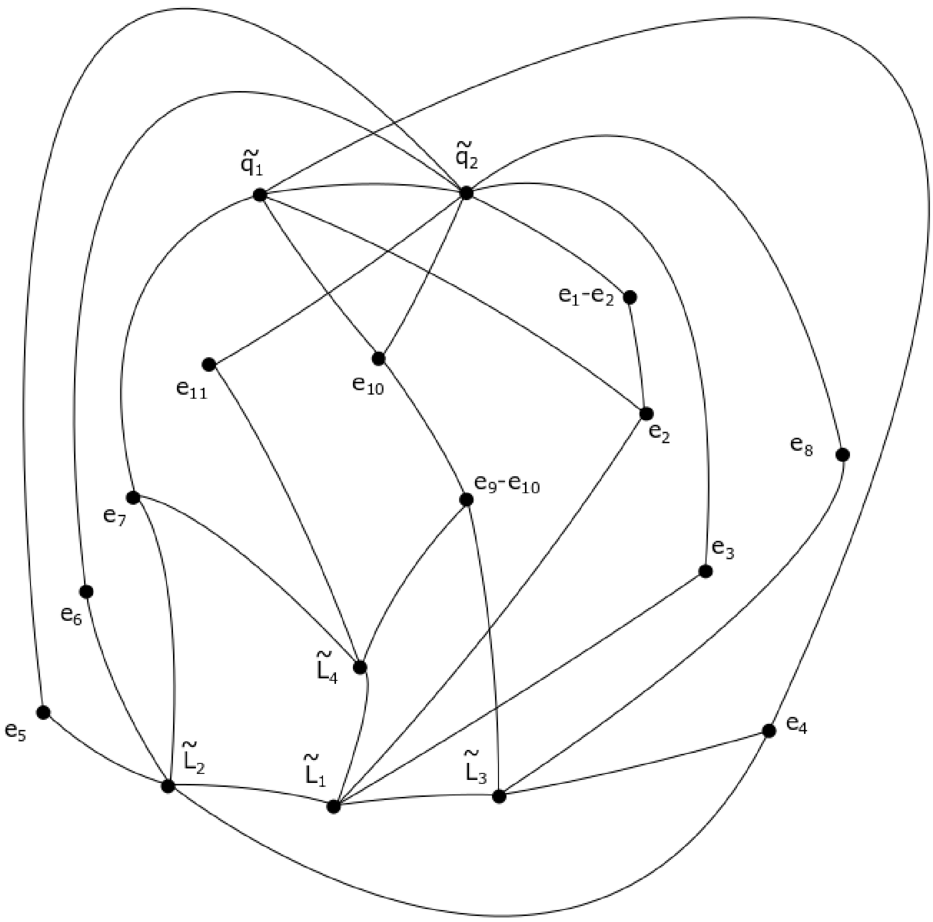


Figure 1.3: The incidence graph of the curve configuration after 11 blow ups

1.4 Exotic $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ via a graph from class $\mathcal{B}^4$

Start by Figure 1.4 where we highlighted nodes and edges which will form the required subgraph. The homology classes of curves at this point are in Table 1. Blowing up the intersection of curves $\widetilde{q_1}$ and e_2, their self-intersections drop to -3 and -2, and we get a new exceptional sphere e_{12}. Doing the same with the intersection between $\widetilde{L_2}$ and e_4, their self-intersections drop to -4 and -2 and we get e_{13}. After three additional blow ups needed to achieve the self-intersections required for the rational blowdown surgery, we arrive to the subgraph shown in Figure 1.5 which is of type $\mathcal{B}^4$ with $p = 2$ using notation of Figure 1.1: we can first blow up a generic point of $\widetilde{L_1}$, creating an exceptional curve e_{14}, and then two different generic points of $\widetilde{L_4}$, making two new exceptional curves e_{15} and e_{16}.

Denote the final classes by $u_1 = \widetilde{L_2} - e_{13}$, $u_2 = \widetilde{L_1} - e_{14}$, $u_3 = \widetilde{L_4} - e_{15} - e_{16}$, $u_4 = e_2 - e_{12}$, $u_5 = \widetilde{L_3}$, $u_6 = e_4 - e_{13}$, $u_7 = \widetilde{q_1} - e_{12}$ and $u_8 = \widetilde{q_2}$. Therefore, after 16 blow ups, we have the plumbing P from Figure 1.5 embedded in $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}$, and the homology classes of plumbing spheres are in Table 2:

$u_1 = h - e_4 - e_5 - e_6 - e_7 - e_{13}$
$u_2 = h - e_1 - e_2 - e_3 - e_{14}$
$u_3 = h - e_7 - e_9 - e_{11} - e_{15} - e_{16}$
$u_4 = e_2 - e_{12}$
$u_5 = h - e_4 - e_8 - e_9$
$u_6 = e_4 - e_{13}$
$u_7 = 2h - e_1 - e_2 - e_4 - e_7 - e_9 - e_{10} - e_{12}$
$u_8 = 2h - e_1 - e_3 - e_5 - e_6 - e_8 - e_9 - e_{10} - e_{11}$

Table 2: Homology classes of spheres of the plumbing P

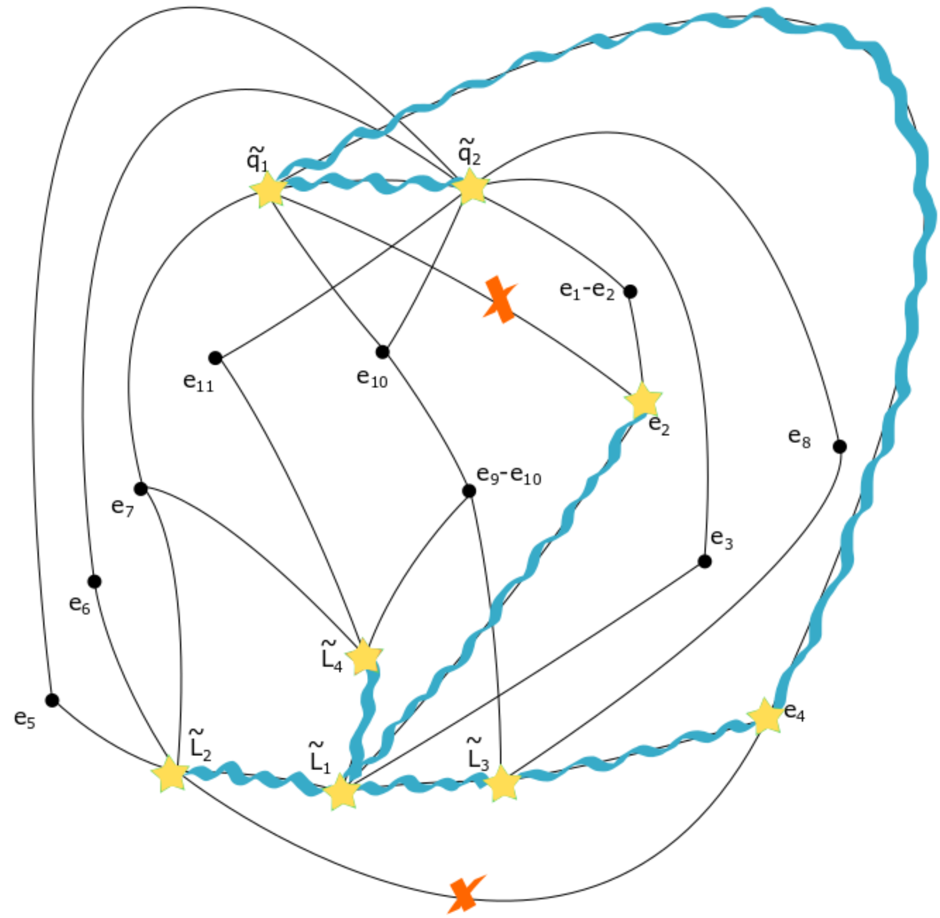


Figure 1.4: Yellow stars are vertices and blue curly lines are edges which form a subgraph from class $\mathcal{B}^4$ presented in Figure 1.5. Orange X's show which 2 intersections to blow up, whereas some additional blow ups used for adjusting the self-intersections to match the vertex markings in Figure 1.5 are not visible here but described in the main text.

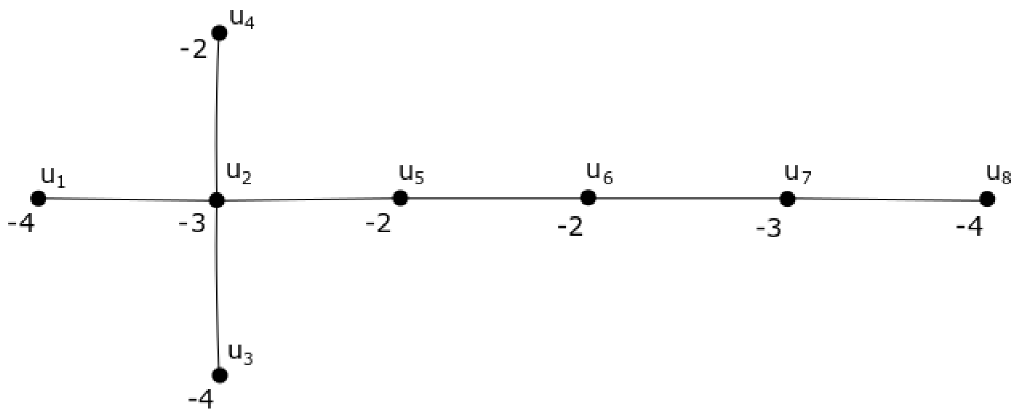


Figure 1.5: Plumbing graph P from class $\mathcal{B}^4$

As our plumbing is from the class $\mathcal{B}^4$, by [BS11, Theorem 1.6], we can perform the rational blowdown along P granting:

$$X = (\mathbb{CP}^2\#16\overline{\mathbb{CP}^2} - intP) \cup B$$

where B is the rational homology ball smoothing of the normal surface singularity defined on pp. 1296-1297 of [BS11] using results of [SSW08].

An important point is that we can assume that the rational blowdown can be performed *symplectically*, which follows from the main result of [PS14]. First, all the plumbing spheres of P can be assumed to be symplectic submanifolds as proper transforms of complex submanifolds, and second, our plumbing graph is a negative definite tree [BS11]. Then, from [PS14, Theorem 1.1], the appropriate neighbourhood of the plumbing can be replaced by B so that (X, ω_X) is symplectic, and denoting $V = \mathbb{CP}^2\#16\overline{\mathbb{CP}^2} - intP$, there is a symplectomorphism $\phi_V : (V, \omega_X|_V) \longrightarrow (V, \omega|_V)$, where ω is any symplectic structure on $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}$ that we started with.

Of course, this way we get a well-defined underlying smooth structure on the new manifold X. The main goal of this section is to prove the following:

Theorem 1.2. *X is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$.*

Proof. Propositions 1.6 and 1.11 in upcoming subsections prove the theorem. □

1.4.1 The topology of X

To find the homeomorphism type of X, we use the foundational result of Freedman [Fre82], which along with Donaldson's theorem [Don83] implies that :

Two smooth simply connected 4-manifolds are homeomorphic if and only if their Euler characteristics, signatures, and parity of the intersection forms are equal.

First we need to prove that X is simply connected, and to do so we will have three standard applications of Van Kampen's theorem. The main part is to prove that for the inclusion $i : \partial P \hookrightarrow \mathbb{CP}^2\#16\overline{\mathbb{CP}^2} - intP$, the homomorphism i_* induced on fundamental groups is a trivial map.

From [NR78, Theorem 5.1], the boundary ∂P is a Seifert fibered 3-manifold with a Seifert ivariant $\{0; (1, 3), (2, 1), (4, 1), (4, 1), (25, 18)\}$. Its fundamental group is described by [JN83, Theorem 6.1] which implies:

Lemma 1.3. *$\pi_1(\partial P)$ has a presentation given by generators $q_0, q_1, q_2, q_3, q_4, h$ and relations:*

- $q_0q_1q_2q_3q_4 = 1$
- $[h, q_i] = 1$ *for all* $i = 0, 1, 2, 3, 4$
- $q_0h^3 = 1$, $q_1^2h = 1$, $q_2^4h = 1$, $q_3^4h = 1$, $q_4^{25}h^{18} = 1$

Furthermore, the classes of q_1, q_2 and q_3 can be chosen to be normal circles to spheres u_4, u_1 and u_3, respectively.

Lemma 1.4. *$i_*(\pi_1(\partial P))$ is trivial.*

Proof. We denoted $V = \mathbb{CP}^2\#16\overline{\mathbb{CP}^2} - intP$, meaning V is the complement of the plumbing. The normal circle to the sphere u_3 can be contracted along the sphere which intersects it in a single point, and we can choose e_{15} (or e_{16}) and contract that normal circle in V. Therefore, the corresponding generator trivializes through the inclusion, $i_*(q_3) = 1$. Relation $q_3^4h = 1$ from Lemma 1.3 gives $i_*(h) = 1$ and then $q_0h^3 = 1$ implies $i_*(q_0) = 1$.

Looking at Figure 1.4, we can see that $\widetilde{L_2}$ and $\widetilde{L_4}$ do not intersect each other but intersect the sphere e_7 in one point each, and their proper transforms u_1 and u_3 do the same in the final picture. As e_7 is disjoint from the rest of the plumbing, normal circles to u_1 and u_3, namely q_2 and q_3, can be isotoped in e_7 to bound an annulus in V. Therefore, $i_*(q_2) = i_*(q_3)$, so $i_*(q_2) = 1$ as well.

From $q_0q_1q_2q_3q_4 = 1$ we are left with $i_*(q_1q_4) = 1$, which we multiply by $i_*(q_1)$ on the left. Using $i_*(q_1)^2 = 1$ which holds since $q_1^2h = 1$ and $i_*(h) = 1$, we get

$i_*(q_4) = i_*(q_1)$. Thus we have $i_*(q_4)^2 = 1$ as well, and by deducing $i_*(q_4)^{25} = 1$ from the last relation in Lemma 1.3, it follows that $i_*(q_4) = 1$. Finally, $i_*(q_1) = i_*(q_4) = 1$ concludes the result. □

Lemma 1.5. *X is simply connected.*

Proof. X is constructed as the union of $V = \mathbb{CP}^2\#16\overline{\mathbb{CP}^2} - intP$ and some rational homology ball B glued along ∂P. Therefore Van Kampen's theorem gives us a presentation of its fundamental group through fundamental groups of the two pieces.

To determine $\pi_1(V)$ we also apply Van Kampen's theorem, this time to the decomposition $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2} = V \cup P$. The fundamental group of the plumbing P is trivial because it is homotopic to a wedge sum of several spheres. Also, $\pi_1(\mathbb{CP}^2\#k\overline{\mathbb{CP}^2})$ is trivial for any k because it can be built without 1-handles, so from $\pi_1(\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}) = \pi_1(V) *_{\pi_1(\partial P)} \pi_1(P)$ we get $1 = \pi_1(V)/i_*(\pi_1(\partial P))$. Now Lemma 1.4 concludes that $\pi_1(V)$ is a trivial group.

We denote the inclusion of the boundary ∂B into the rational homology ball B by $j : \partial B \hookrightarrow B$, and $N := \langle i_*(x) \cdot j_*(x)^{-1} | x \in \pi_1(\partial B)\rangle$. From Van Kampen's theorem and the triviality of $\pi_1(V)$, we have that $\pi_1(X) = \pi_1(V) *_N \pi_1(B) = \pi_1(B)/\langle j_*(x) | x \in \pi_1(\partial B)\rangle$. However, surjectivity of j_* comes from the fact that our rational homology ball was constructed as a complement of a certain (dual) plumbing P' from $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ for some $k > 0$ ([SSW08, section 8.1] and [BS11, pp. 1296-1297]). More precisely, from another application of Van Kampen's theorem on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2} = B \cup P'$, we get $1 = \pi_1(B)/\langle j_*(x) | x \in \pi_1(\partial B)\rangle$. Therefore, X is simply connected. □

Proposition 1.6. *X is homeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$.*

Proof. To calculate $\chi(X)$ and $\sigma(X)$ we use the formulas:

$$\chi(X) = \chi(\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}) - \chi(P) + \chi(B) = 19 - 9 + 1 = 11$$

$$\sigma(X) = \sigma(\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}) - \sigma(P) + \sigma(B) = -15 - (-8) = -7$$

Rokhlin's theorem [Rok52] implies that if the signature of a smooth simply connected 4-manifold is not divisible by 16, its intersection form must be odd, so this is the case for X. Therefore, the three invariants of X match the corresponding invariants of $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$. As X is simply connected by Lemma 1.5, it is homeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ as a consequence of Freedman's theorem. □

1.4.2 Exoticness of X

To prove that X is not diffeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, we will use its symplectic structure ω_X explained earlier (coming from [PS14]), and the following result:

Lemma 1.7 (**[LL95, Theorem D]**). *There is a unique symplectic structure on $\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$ for all $2 \le m \le 9$ up to diffeomorphism and deformation.*

Remark 1.8. *We will slightly abuse notation denoting symplectic forms as their cohomology classes. Poincaré dual of α will be denoted by $PD(\alpha)$.*

A symplectic structure Ω on a 4-manifold M determines a contractible family $\mathcal{J}$ of Ω-compatible almost complex structures J on the cotangent bundle T^*M. The first Chern class is the same for all $J \in \mathcal{J}$ and it is called the symplectic canonical class $K_\Omega = c_1(T^*M, J)$.

The strategy of proving that X is exotic is as in [Par05], to calculate the cup product of the symplectic class and a compatible canonical class on both $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ and X, see that the signs of these products differ, and prove that this is impossible because of the uniqueness result stated in Lemma 1.7.

Lemma 1.9 essentially stated as [KS16, Lemma 5.4] presents a standard symplectic structure on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ and calculates the sign of the required cup product to be negative. Lemma 1.10 follows from Lemmas 1.7 and 1.9, and shows that this cup product has to be negative for any symplectic structure on $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ or $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. This is a rather special result for $\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$ given $2 \le m \le 9$; in general, the sign $K_\omega \cdot \omega$ can be used as a smooth invariant on a symplectic manifold only when we know the manifold in question is minimal, and this is called *the symplectic Kodaira dimension.*

Lemma 1.9 (**[KS16, Lemma** 5.4**]**). *For every $k > 0$, $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ admits a symplectic structure ω that satisfies $PD(\omega) = ah - b_1e_1 - \dots - b_ke_k$ for some positive rational numbers $a, b_1, \dots, b_k$. For fixed $a > 0$, b_i's can be chosen to be arbitrarily small. The induced canonical class $K := K_\omega$ satisfies $PD(K) = -3h + e_1 + \dots + e_k$ and for small enough b_i's, we have $K \cdot \omega < 0$.*

Proof. In $\mathbb{CP}^2$, the dual of the cohomology class of ω is ah for some $a > 0$ and we can choose it to be rational - this is because the symplectic area of $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is a positive number a and it can be normalized to be rational (we could normalize it so that $a = 1$, but keep "a" to see its importance). The proof of this lemma follows from [MS98, section 7.1], and more precisely from Theorem 7.1.21 on the existence and properties of the symplectic blow up. Namely, part (v) of that theorem implies that after the blow up, the cohomology class of the symplectic form changes as $\omega_{\tilde{M}} = \omega_M - \pi\lambda^2 PD(e)$. Here e denotes the homology class of the exceptional curve and λ is the radius of the ball removed in the process of the symplectic blow up as explained in [MS98]. Choosing the ball in Darboux's chart to be as small as needed and $\pi\lambda^2$ rational, and repeating the procedure k times, gives us $PD(\omega) = ah - b_1e_1 - \ldots - b_ke_k$ as required.

Formula (7.1.31) in [MS98] shows the canonical class of the blow up $\tilde{M}$ to be $c_1(T^*\tilde{M}) = c_1(T^*M) + PD(e)$. From the previous and $PD(K_{\mathbb{CP}^2}) = -3h$, we get $PD(K) = -3h + e_1 + \ldots + e_k$. Finally, $K \cdot \omega = -3a + b_1 + \ldots + b_k$ is negative for b_i's small enough. □

Lemma 1.10. *For any symplectic structure $\overline{\omega}$ on $M = \mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$ for $2 \le m \le 9$:*

$$K_{\overline{\omega}} \cdot \overline{\omega} < 0$$

Proof. This result essentially follows from Lemma 1.7 ([LL95, Theorem D]), as $\overline{\omega}$ has to be deformation equivalent to the standard symplectic structure ω, meaning that up to diffeomorphism, there is a path of symplectic forms on M connecting them.

This means there is a symplectomorphism $\psi : (M, \omega_M) \longrightarrow (\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}, \omega)$ such that there is a path of symplectic forms ω_t connecting $\omega_0 = \overline{\omega}$ and $\omega_1 = \omega_M$. Naturality of Chern classes gives $K_{\omega_M} = \psi^*(K)$ so $K_{\omega_M} \cdot \omega_M = \psi^*(K) \cdot \psi^*(\omega) = \psi^*(K \cdot \omega) = K \cdot \omega$, saying that symplectomorphism does not change this product.

Assume that $K_{\overline{\omega}} \cdot \overline{\omega} \ge 0$. Firstly, the canonical class $K_{\overline{\omega}}$ does not change by deformation so $PD(K_{\overline{\omega}}) = -3h + e_1 + \ldots + e_m$. Now $PD(\overline{\omega}) = a_0h + a_1e_1 + \ldots + a_me_m$ for some numbers $a_i \in \mathbb{R}$. However, as $\overline{\omega}$ is symplectic, we must have $\overline{\omega} \cdot \overline{\omega} > 0$ so $a_0^2 > \sum_{i=1}^m a_i^2$. Having $K_{\overline{\omega}} \cdot \overline{\omega} = -3a_0 - a_1 - \ldots - a_m \ge 0$, we get $3a_0 \le -(\sum_{i=1}^m a_i)$. If $a_0 \le 0$, from the path of symplectic forms with $PD(\omega_t) = a_0^th + a_1^te_1 + \ldots + a_m^te_m$, we would have a continuous funcition a_0^t connecting $a_0^0 = a_0 \le 0$ and $a_0^1 > 0$ (as $a > 0$ for symplectomorphic ω). Then there would be τ for which $a_0^\tau = 0$ and thus $\omega_\tau \cdot \omega_\tau \le 0$, which is not possible. Therefore, $a_0 > 0$ and from earlier we have $0 < 3a_0 \le -(\sum_{i=1}^m a_i)$ so:

$$9a_0^2 \le (\textstyle\sum_{i=1}^m a_i)^2 \le m(\sum_{i=1}^m a_i^2) \le 9(\sum_{i=1}^m a_i^2) < 9a_0^2$$

provides the required contradiction using the Cauchy–Schwarz inequality. □

Proposition 1.11. *X is not diffeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$.*

Proof. As mentioned, the strategy is to calculate the cup product of the symplectic class and a compatible canonical class for X, and see that the sign of this product is positive, which proves exoticness of X using Lemma 1.10.

Let ω denote the symplectic form on $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}$ provided by Lemma 1.9, whose Poincaré dual is equal to:

$$PD(\omega) = ah - b_1e_1 - \ldots - b_{16}e_{16}$$

and let K denote the corresponding canonical class:

$$PD(K) = -3h + e_1 + \ldots + e_{16}$$

From the previous two we have:

$$K \cdot \omega = -3a + b_1 + \ldots + b_{16}$$

The symplectic structure ω_X on X obtained after the rational blow down, was defined earlier in Section 1.4, and it has a compatible symplectic canonical class K_X coming from a generic almost complex structure compatible with ω_X.

To be able to calculate $K_X \cdot \omega_X$, we will decompose the cohomology classes K and ω. Denoting again $V = \mathbb{CP}^2\#16\overline{\mathbb{CP}^2} - intP$, we have decompositions $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2} = V \cup P$ and $X = V \cup B$.

As a first step, note that the boundary Seifered fibered 3-manifold $\partial P = -\partial B$ is a *rational homology sphere* because $\frac{3}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{18}{25} \neq 0$ (see Section 1.2.3 in [Sav02]). To prove it directly, we can calculate $H_1(\partial P; \mathbb{Z})$ from Lemma 1.3 and see that it is a finite group, which then implies $H^*(\partial P; \mathbb{Q}) = H^*(S^3; \mathbb{Q})$.

From the Mayer-Vietoris sequences for decompositions $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2} = V \cup P$ and $X = V \cup B$, we get exact sequences:

$$H^1(\partial P; \mathbb{Q}) \longrightarrow H^2(\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}; \mathbb{Q}) \longrightarrow H^2(V; \mathbb{Q}) \oplus H^2(P; \mathbb{Q}) \longrightarrow H^2(\partial P; \mathbb{Q})$$

$$H^1(\partial B; \mathbb{Q}) \longrightarrow H^2(X; \mathbb{Q}) \longrightarrow H^2(V; \mathbb{Q}) \oplus H^2(B; \mathbb{Q}) \longrightarrow H^2(\partial B; \mathbb{Q})$$

The triviality in $\mathbb{Q}$-cohomology gives $H^1(\partial P; \mathbb{Q}) = 0 = H^2(\partial P; \mathbb{Q})$ and $H^1(\partial B; \mathbb{Q}) = 0 = H^2(\partial B; \mathbb{Q})$, so both middle arrows are isomorphisms. From the first sequence, we can decompose the cohomology classes:

$$K = K|_V + K|_P \text{ and } \omega = \omega|_V + \omega|_P$$

As B is a rational homology 4-ball, $H^2(B; \mathbb{Q}) = 0$ so the second sequence gives that classes K_X and ω_X satisfy:

$$K_X = K_X|_V = \phi_V^*(K|_V) \text{ and } \omega_X = \omega_X|_V = \phi_V^*(\omega|_V)$$

where ϕ_V is the symplectomorphism from the beginning of Section 1.4. So:

$$K_X \cdot \omega_X = \phi_V^*(K|_V) \cdot \phi_V^*(\omega|_V) = \phi_V^*(K|_V \cdot \omega|_V) = K|_V \cdot \omega|_V = K \cdot \omega - K|_P \cdot \omega|_P$$

$$\boxed{K_X \cdot \omega_X = K \cdot \omega - K|_P \cdot \omega|_P}$$

The intersection matrix M of the plumbing P is defined by the intersections $[u_i \cdot u_j]$ as in Figure 1.5:

$$M = \begin{bmatrix} -4 & 1 & & & & & & \\ 1 & -3 & 1 & 1 & 1 & & & \\ & 1 & -4 & & & & & \\ & 1 & & -2 & & & & \\ & 1 & & & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -3 & 1 \\ & & & & & & 1 & -4 \end{bmatrix}$$

Let $\{\gamma_i\}_{i=1}^8$ be the basis of $H^2(P;\mathbb{Q})$ which is dual to the basis $\{u_i\}_{i=1}^8$, meaning $\gamma_i(u_j) = \delta_{ij}$. Then the intersections $[\gamma_i \cdot \gamma_j]$ are given by $[M^{-1}]_{ij}$:

$$M^{-1} = -\frac{1}{512} \cdot \begin{bmatrix} 153 & 100 & 25 & 50 & 72 & 44 & 16 & 4 \\ 100 & 400 & 100 & 200 & 288 & 176 & 64 & 16 \\ 25 & 100 & 153 & 50 & 72 & 44 & 16 & 4 \\ 50 & 200 & 50 & 356 & 144 & 88 & 32 & 8 \\ 72 & 288 & 72 & 144 & 576 & 352 & 128 & 32 \\ 44 & 176 & 44 & 88 & 352 & 528 & 192 & 48 \\ 16 & 64 & 16 & 32 & 128 & 192 & 256 & 64 \\ 4 & 16 & 4 & 8 & 32 & 48 & 64 & 144 \end{bmatrix}$$

From $K|_P = \sum_{i=1}^8 (K|_P(u_i))\gamma_i$, and $K|_P(u_i) = K(u_i) = PD(K) \cdot u_i$, we have $K|_P = \sum_{i=1}^8 (PD(K) \cdot u_i)\gamma_i$. Taking the values of u_i's from Table 2:

$$K|_P = 2\gamma_1 + \gamma_2 + 2\gamma_3 + \gamma_7 + 2\gamma_8$$

Analogously, we get $\omega|_P = \sum_{i=1}^8 (PD(\omega) \cdot u_i)\gamma_i$:

$\omega|_P = (a - b_4 - b_5 - b_6 - b_7 - b_{13})\gamma_1 + (a - b_1 - b_2 - b_3 - b_{14})\gamma_2 + (a - b_7 - b_9 - b_{11} - b_{15} - b_{16})\gamma_3 + (b_2 - b_{12})\gamma_4 + (a - b_4 - b_8 - b_9)\gamma_5 + (b_4 - b_{13})\gamma_6 + (2a - b_1 - b_2 - b_4 - b_7 - b_9 - b_{10} - b_{12})\gamma_7 + (2a - b_1 - b_3 - b_5 - b_6 - b_8 - b_9 - b_{10} - b_{11})\gamma_8$

After calculating $K|_P \cdot \omega|_P$, we use $K_X \cdot \omega_X = K \cdot \omega - K|_P \cdot \omega|_P$ to get:

$K_X \cdot \omega_X = 5.625a - 2.5b_1 - 0.875b_2 - 1.5b_3 - 1.1875b_4 - 0.6875b_5 - 0.6875b_6 - 1.875b_7 - 1.25b_8 - 3.1875b_9 - 0.75b_{10} - 0.6875b_{11} - 0.875b_{12} - 1.1875b_{13} - 0.75b_{14} + 0.0625b_{15} + 0.0625b_{16}$

We have $K_X \cdot \omega_X > 0$ because a is positive and we can choose b_i's to be arbitrarily small. If X was diffeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, Lemma 1.10 would imply $K_X \cdot \omega_X < 0$ so this concludes that X is exotic. □

1.5 Exotic $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ via a graph from class C^4

In this section we construct a different plumbing from the one in Section 1.4, again starting with the construction in Section 1.3. We keep the notation of some auxiliary objects as in the previous sections to simplify the exposition. Apart from the construction of the plumbing, all calculations are similar so we only emphasize the differences.

Starting from the incidence graph in Figure 1.3, in Figure 1.6 we highlight nodes and edges which will form the required subgraph from C^4.

We first blow up the intersection between e_7 and $\widetilde{L_2}$ and denote the exceptional curve by e_{12}. This way the proper transform of $\widetilde{L_2}$ gets self-intersection -4. With two further blow ups of different generic points of $\widetilde{L_2}$, we transform it into a curve of self-intersection -6, getting curves e_{13} and e_{14} in the process. Then blow up a generic point of the curve $\widetilde{L_1}$ getting e_{15}, and setting the self-intersection of the proper transform of $\widetilde{L_1}$ to -3. Now blow up a generic point of e_{15}, allowing its self-intersection to drop to -2, and name the exceptional curve e_{16}. Lastly, blow up a generic point of $\widetilde{L_3}$ dropping its self-intersection to -3 via the curve e_{17}.

Denote the classes by $v_1 = e_{15}-e_{16}, v_2 = \widetilde{L_1}-e_{15}, v_3 = \widetilde{L_3}-e_{17}, v_4 = \widetilde{L_2}-e_{12}-e_{13}-e_{14}$, $v_5 = \widetilde{L_4}$, $v_6 = e_7 - e_{12}$, $v_7 = \widetilde{q_1}$ and $v_8 = \widetilde{q_2}$. These curves form the plumbing Q embedded in $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$, and its graph is presented in Figure 1.7. Therefore, the homology classes of spheres in the plumbing Q are:

$$\boxed{\begin{gathered}
v_1 = e_{15} - e_{16}\\
v_2 = h - e_1 - e_2 - e_3 - e_{15}\\
v_3 = h - e_4 - e_8 - e_9 - e_{17}\\
v_4 = h - e_4 - e_5 - e_6 - e_7 - e_{12} - e_{13} - e_{14}\\
v_5 = h - e_7 - e_9 - e_{11}\\
v_6 = e_7 - e_{12}\\
v_7 = 2h - e_1 - e_2 - e_4 - e_7 - e_9 - e_{10}\\
v_8 = 2h - e_1 - e_3 - e_5 - e_6 - e_8 - e_9 - e_{10} - e_{11}
\end{gathered}}$$

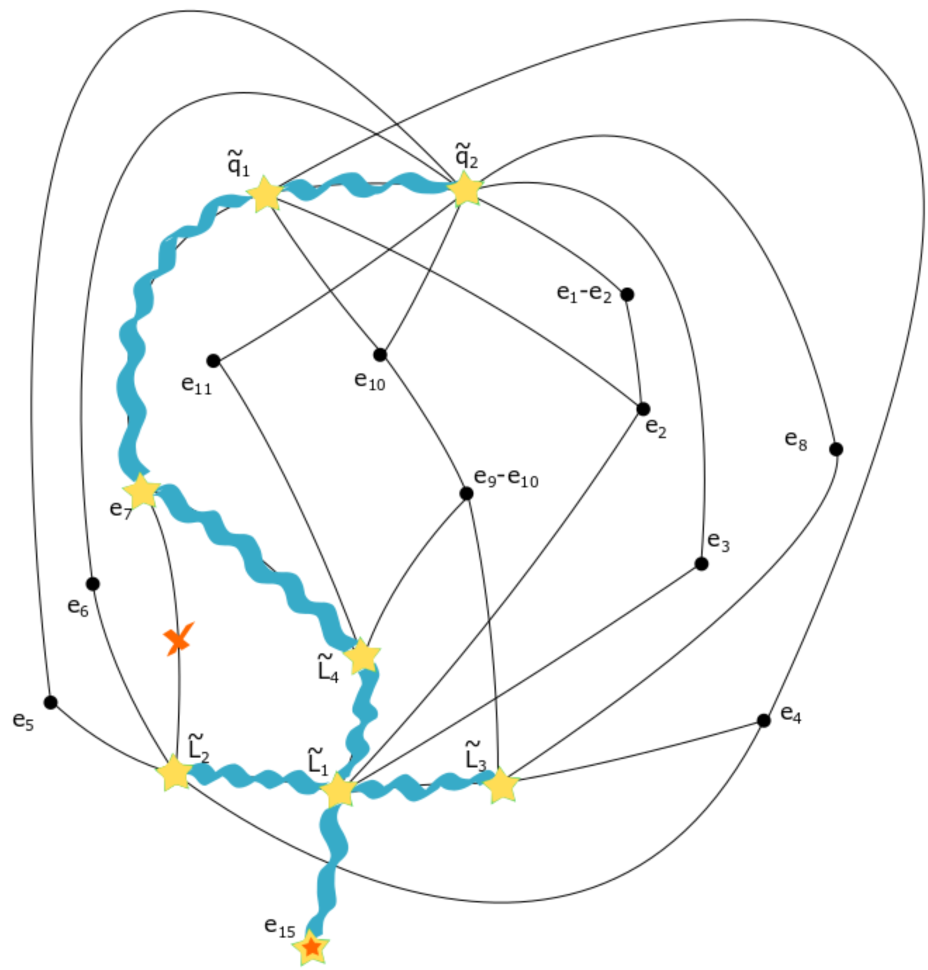


Figure 1.6: Yellow stars are vertices and blue curly lines are edges which form a subgraph from class C^4 presented in Figure 1.7. Note that e_{15} is a new vertex compared to the starting Figure 1.3, marked with a smaller orange star because it comes from a new blow up. To arrive to an embedding, orange X shows which intersection to blow up. Some additional blow ups used for adjusting the self-intersections to match the vertex markings in Figure 1.7 are not visible here but are described in the main text.

Table 3: Homology classes of spheres of the plumbing Q

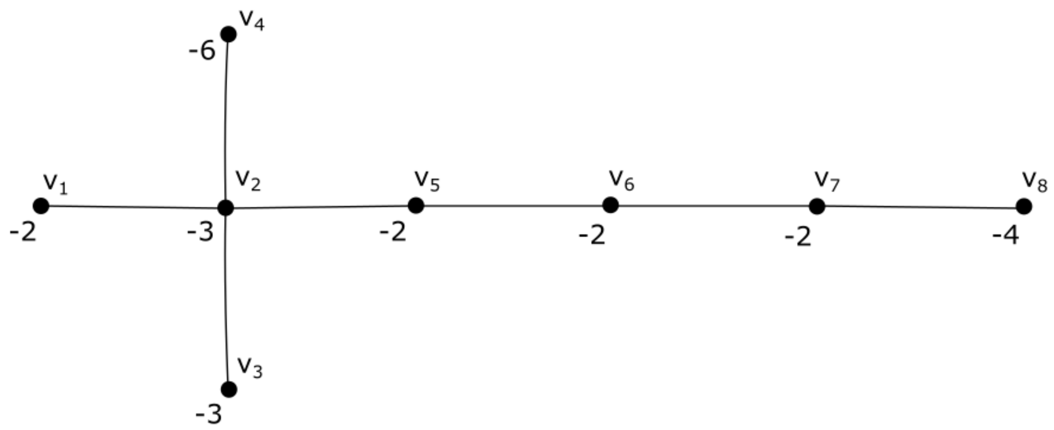


Figure 1.7: Plumbing graph Q from class C^4

We can rationally blow down Q by [BS11] and get the manifold:

$$Y = (\mathbb{CP}^2\#17\overline{\mathbb{CP}^2} - intQ) \cup D$$

where D is a different rational homology ball than the one from Section 1.4. Details are very similar to the ones in the previous section and we only emphasize the differences, showing this time:

Theorem 1.12. *Y is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$.*

Proof. Propositions 1.16 and 1.17 together will complete the proof. □

1.5.1 The topology of Y

In this example, the boundary ∂Q is a Seifert fibered 3-manifold [NR78] with Seifert ivariant $\{0; (1,3), (6,1), (3,1), (2,1), (13,10)\}$. Analagously to Lemma 1.3, by [JN83] we have:

Lemma 1.13. *$\pi_1(\partial Q)$ has a presentation given by generators $q_0, q_1, q_2, q_3, q_4, h$ and relations:*

- $q_0q_1q_2q_3q_4 = 1$
- $[h, q_i] = 1$ *for all* $i = 0, 1, 2, 3, 4$

- $q_0h^3 = 1$, $q_1^6h = 1$, $q_2^3h = 1$, $q_3^2h = 1$, $q_4^{13}h^{10} = 1$

Furthermore, the classes of q_1, q_2 and q_3 can be chosen to be normal circles to spheres v_4, v_3 and v_1, respectively.

Lemma 1.14. *$i_*(\pi_1(\partial Q))$ is trivial.*

Proof. In this case, compared to the previous section, it is easier to deduce the triviality of $i_*(\pi_1(\partial Q))$, as we made a lot of generic blow ups. More precisely, each of the three leaves of the plumbing graph Q in Figure 1.7, that is v_4, v_3 and v_1, is intersecting a different exceptional sphere otherwise disjoint from the plumbing. As in the proof of Lemma 1.4, the normal circles can be contracted in the complement of Q, so we can deduce $i_*(q_1) = 1$, $i_*(q_2) = 1$ and $i_*(q_3) = 1$. From $q_1^6h = 1$, we get $i_*(h) = 1$ and then $q_0h^3 = 1$ implies $i_*(q_0) = 1$. The first relation of Lemma 1.13 now gives $i_*(q_4) = 1$ and concludes that $i_*(\pi_1(\partial Q))$ is a trivial group. □

Lemma 1.15. *Y is simply connected.*

Proof. Using Lemma 1.14 instead of Lemma 1.4, the proof is analogous to the proof of Lemma 1.5. □

Proposition 1.16. *Y is homeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$.*

Proof. As before we have:

$$\chi(Y) = \chi(\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}) - \chi(Q) + \chi(D) = 20 - 9 + 1 = 12$$

$$\sigma(Y) = \sigma(\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}) - \sigma(Q) + \sigma(D) = -16 - (-8) = -8$$

Y has an odd intersection form by Rohlkin's theorem [Rok52] and thus, all the invariants match the ones of $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. From Lemma 1.15, these 4-manifolds are both simply connected, and so by Freedman's theorem we get that they must be homeomorphic. □

1.5.2 Exoticness of Y

Proposition 1.17. *Y is not diffeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$.*

Proof. The proof is essentially the same as the proof of Proposition 1.11. Start by introducing a symplectic form on $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$ using Lemma 1.9:

$$PD(\omega) = ah - b_1e_1 - \ldots - b_{17}e_{17}$$

This time, let K be the standard canonical class of $\mathbb{CP}^2\#17\overline{\mathbb{CP}^2}$:

$$PD(K) = -3h + e_1 + \ldots + e_{17}$$

From these two we have:

$$K \cdot \omega = -3a + b_1 + \ldots + b_{17}$$

The intersection matrix of the plumbing Q is $[v_i \cdot v_j]$:

$$N = \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -3 & 1 & 1 & 1 & & & \\ & 1 & -3 & & & & & \\ & 1 & & -6 & & & & \\ & 1 & & & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -4 \end{bmatrix}$$

The intersection matrix of the basis $\{\gamma_i\}_{i=1}^8$ dual to $\{v_i\}_{i=1}^8$ is:

$$N^{-1} = -\frac{1}{576} \cdot \begin{bmatrix} 405 & 234 & 78 & 39 & 180 & 126 & 72 & 18 \\ 234 & 468 & 156 & 78 & 360 & 252 & 144 & 36 \\ 78 & 156 & 244 & 26 & 120 & 84 & 48 & 12 \\ 39 & 78 & 26 & 109 & 60 & 42 & 24 & 6 \\ 180 & 360 & 120 & 60 & 720 & 504 & 288 & 72 \\ 126 & 252 & 84 & 42 & 504 & 756 & 432 & 108 \\ 72 & 144 & 48 & 24 & 288 & 432 & 576 & 144 \\ 18 & 36 & 12 & 6 & 72 & 108 & 144 & 180 \end{bmatrix}$$

To calculate $K_Y \cdot \omega_Y$, we can aquire $K|_Q$ and $\omega|_Q$ decomposing the second cohomology classes as before. Again, this is possible because the boundary manifold ∂Q is Seifert fibered and $\frac{3}{1} + \frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{10}{13} \neq 0$, so it is a rational homology sphere (see [Sav02]). We have $K|_Q = \sum_{i=1}^{8} (PD(K) \cdot v_i)\gamma_i$, and by using the values of $PD(K)$ and v_i's from Table 3:

$$K|_Q = \gamma_2 + \gamma_3 + 4\gamma_4 + 2\gamma_8$$

A similar formula $\omega|_Q = \sum_{i=1}^{8} (PD(\omega) \cdot v_i)\gamma_i$ gives:

$\omega|_Q = (b_{15} - b_{16})\gamma_1 + (a - b_1 - b_2 - b_3 - b_{15})\gamma_2 + (a - b_4 - b_8 - b_9 - b_{17})\gamma_3 + (a - b_4 - b_5 - b_6 - b_7 - b_{12} - b_{13} - b_{14})\gamma_4 + (a - b_7 - b_9 - b_{11})\gamma_5 + (b_7 - b_{12})\gamma_6 + (2a - b_1 - b_2 - b_4 - b_7 - b_9 - b_{10})\gamma_7 + (2a - b_1 - b_3 - b_5 - b_6 - b_8 - b_9 - b_{10} - b_{11})\gamma_8$

And once again, from $K_Y \cdot \omega_Y = K \cdot \omega - K|_Q \cdot \omega|_Q$:

$K_Y \cdot \omega_Y = 5.625a - 2.5b_1 - 1.75b_2 - 1.5b_3 - 1.875b_4 - 0.708\overline{3}b_5 - 0.708\overline{3}b_6 - 1.208\overline{3}b_7 - 0.\overline{6}b_8 - 3.1\overline{6}b_9 - 0.69791\overline{6}b_{10} - 1.25b_{11} - 1.208\overline{3}b_{12} + 0.041\overline{6}b_{13} + 0.041\overline{6}b_{14} + 0.125b_{15} + 0.125b_{16} + 0.08\overline{3}b_{17}$

$K_Y \cdot \omega_Y > 0$ because a is positive and b_i's can be arbitrarily small. By Lemma 1.10, this is impossible unless Y is exotic. □

Remark 1.18. *Finding interesting configurations of lines and quadrics could produce even smaller exotic* 4*-manifolds via suitable rational blowdowns, so this is one upcoming challenge. It seems that the exoticness proof will remain true if enough curves from the initial configuration are used in the plumbing, so it would only remain to take care of simple connectedness.*

Part II

Moduli spaces of Higgs bundles

2 Moduli spaces of Higgs bundles

In this part we analyze the moduli space of meromorphic Higgs bundles over $\mathbb{CP}^1$ with 3 simple poles and structure group $\mathbb{GL}(3,\mathbb{C})$. Even though in general these moduli spaces can be of a larger dimension, we focus on spaces of real dimension 4, and initial parameters that lead to this case (Subsection 2.2.1). Moduli spaces of Higgs bundles come with very rich geometry - they admit hyperkähler structures [Hit87b] - and here we look at one such moduli space through the lens of the *Hitchin fibration* [Hit87a]. After adequate compactifications and a birational transformation, this map turns out to be an elliptic fibration and thus stratifies the moduli space into a 2-dimensional family of tori along with a few isolated singular fibers. Our particular case of interest is when this fibration has exactly 2 singular fibers.

The first of these fibers is added when we compactify the moduli space, and we call it the *fiber at infinity*. In our case it consist of three -2-framed spheres intersecting in one point, known as the *type IV fiber*, shown in the left part of Figure 2.1. Assuming there is only one other fiber, it has to be a $\tilde{E}_6$ *fiber* - a configuration of seven -2-framed spheres intersecting in a tree-like pattern shown on the right part of Figure 2.1, and discussed in Example in Figure 0.2.

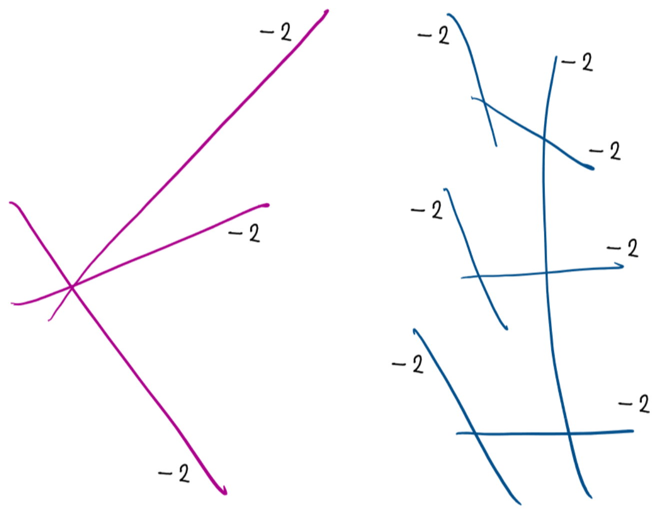


Figure 2.1: Type IV fiber (left) and $\tilde{E}_6$ fiber (right)

2.1 Introduction

Moduli spaces of geometric structures have been an interesting topic since it was shown by Riemann that 2-dimensional surfaces of genus g admit a $6g-6$-dimensional family of complex structures. The topic that eventually grew starting with this, is the theory of *moduli spaces of stable vector bundles of fixed rank over a Riemann surface* Σ, put into its modern form by Mumford via his Geometric Invariant Theory. In the book with the same title, he introduced a notion of *stability* which in our context means that for every proper sub-bundle $\mathcal{E}'$ of a vector bundle $\mathcal{E}$, we have:

$$\frac{deg(\mathcal{E}')}{rank(\mathcal{E}')} < \frac{deg(\mathcal{E})}{rank(\mathcal{E})}$$

The point of view that Hitchin introduced in his 1987 paper [Hit87a] is to examine the symplectic geometry of the cotangent bundle of these moduli spaces. From this perspective, through *Serre duality*, the cotangent vectors turn out to be elements of the form $\phi : \mathcal{E} \to \mathcal{E} \otimes T^*\Sigma$. Assembling these *Higgs pairs* $(\mathcal{E}, \theta)$ and quotienting by the natural isomorphism relation gives us the *moduli space of Higgs bundles*. The map introduced in [Hit87a] essentially maps a Higgs pair into the eigenvalues of θ, and is nowdays called the *Hitchin fibration*. This turns out to be an *algebraically completely integrable Hamiltonian system*, which means that the generic fiber is an open set in an abelian variety, and the Hamiltonian vector fields are linear.

This theory arose as a consequence of Hitchin's two-dimensional reduction of Yang-Mills equations [Hit87b], and this gauge theoretic approach served to prove that the moduli spaces of Higgs bundles have a *hyperkähler structure*. Yang-Mills equations are a certain system of partial differential equations for a connection on a vector bundle or a principle bundle over a 4-manifold. As an interesting point related to Part I of this thesis - in mathematics, the moduli space of solutions to these equations, was used by Donaldson to prove his famous result stating that the intersection form of a smooth compact oriented *definite* 4-manifold is unique [Don83]. In physics, these equations constitute Yang-Mills theory - a gauge theoretic approach to understanding elementary particles, and a part of the mathematical basis for the *Standard model*. A famous open problem in this area is the mass-gap problem listed as one of the 7 Millenium Prize problems.

The big bicture of moduli spaces of Higgs bundles shows us how incredibly rich this field became. An excellent survey article of Boalch [Boa12] starts by explaining the three different perspectives of this hyperkähler manifold: moduli spaces of Higgs bundles $\mathcal{M}_{Dol}$, moduli spaces of holomorphic connections $\mathcal{M}_{DR}$, and as a character variety $\mathcal{M}_B$ - conjugacy classes of irreducible representations of the fundemantal group of the base surface. The unification usually follows from the initial gauge theoretic perspective and Yang-Mills equations which also provides the proof of the hyperkähler structure. For $G = \mathbb{GL}(n, \mathbb{C})$-Higgs bundles over a surface Σ, the character variety viewpoint gives us the simplest description of the smooth manifold [Boa18], $\mathcal{M}_B \cong Hom^{irr}(\pi_1(\Sigma), G)/G$ taking the irreducible representations of $\pi_1(\Sigma)$ into G among all representations:

$$Hom(\pi_1(\Sigma), G)/G \cong \{A_1, B_1, ..., A_g, B_g \in G | [A_1, B_1] \cdots [A_g, B_g] = I\}/G$$

The theory got extended in the meantime to punctured surfaces and meromorphic instead of holomorphic bundles. In 3 previous papers by Ivanics, Stipsicz and Szabó [ISS18], [ISS19a], [ISS19b], when moduli spaces turn out to be of real dimension 4, several interesting instances of the problem of determining singular fibers of Hitchin fibration were treated. The focus was on rank 2 holomorphic bundles over $\mathbb{CP}^1$ with structure group $\mathbb{GL}(2, \mathbb{C})$ and several poles. In [ISS18] they first treat all possible cases when there is 1 pole, in [ISS19a] when there are exactly 2 poles, and in [ISS19b] the authors focus on the question when, regardless of the number of poles, the fibration has exactly 1 singular fiber excluding the compactifying fiber at infinity. This means determining what some additional parameters, non-existent in the classical case, must satisfy in order for the Hitchin fibration to have some combination of singular fibers.

In present work, we will fix the number of poles and request exactly two singular fibers. The case we treat is of moduli spaces of meromorphic Higgs bundles $(\mathcal{E}, \theta)$ with 3 simple poles and structure group $\mathbb{GL}(3, \mathbb{C})$. Thus we examine under which conditions the 'most singular' case appears, meaning that other than the compactifying singular fiber, the Hitchin fibration exhibits a unique singular fiber. To enable the moduli space to have nice properties such as being hyperkähler [KON93; Nak96; Sim90], it is necessary to fix some additional parameters. These parameters are leading terms in the local representation of the Higgs field θ near each of the poles, and in the following sections we give necessary and sufficient conditions on these parameters under which only one singular fiber - $\tilde{E}_6$ - appears, and prove:

Theorem 2.1 (The Main Theorem II). *Let $D = t_1 + t_2 + t_3$ be a simple effective divisor on $\mathbb{CP}^1$. Hitchin fibration on $\mathcal{M}(\mathbb{CP}^1, D, 3, 0)$ with a generic parabolic structure has a unique singular fiber (aside from the type IV fiber at infinity) if and only if the specified residue matrices at the 3 poles, each have a triple eigenvalue.*

In this case the singular fiber is necessarily $\tilde{E}_6$ and the 3 triple eigenvalues a, b and c satisfy $a + b + c = 0$.

One specialty of moduli spaces having this special form with only two singular

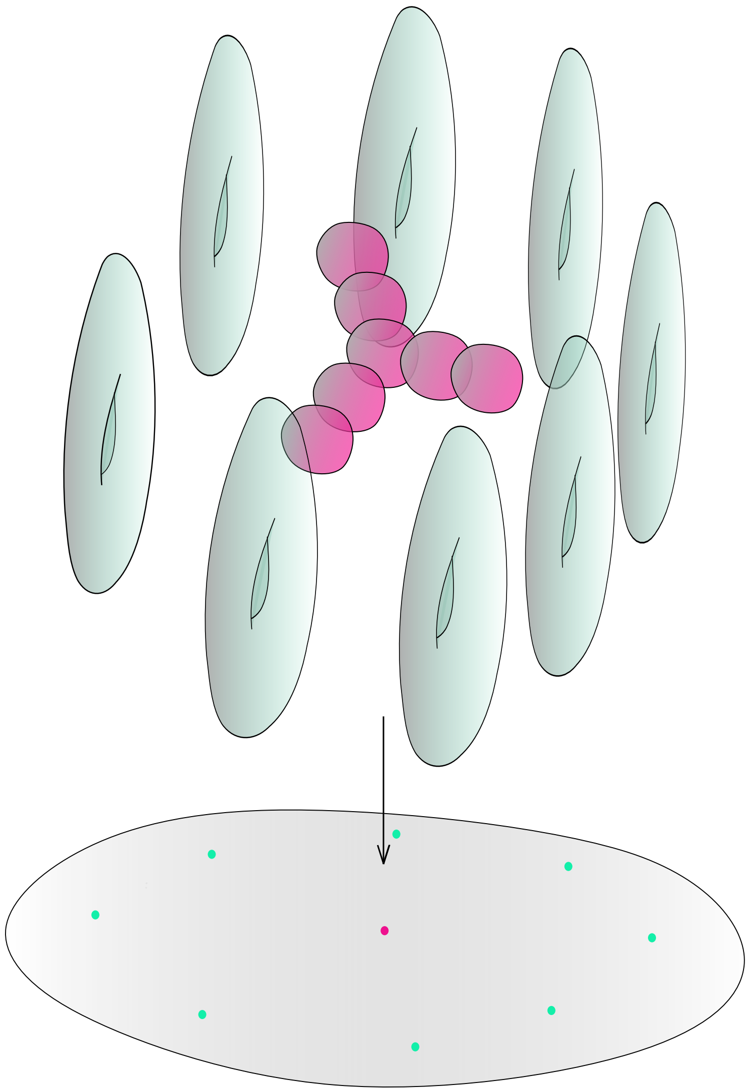


Figure 2.2: Sketch of the Hitchin fibration with a singular $\tilde{E}_6$ fiber

fibers is that there is a natural $\mathbb{C}^*$-action on the moduli space. This action can be visualized as the one which rotates and contracts the fibers around the central singular fiber which is fixed, see Figure 2.2. In the standard regular case, this action is always present and acts by multiplying the Higgs field by a complex number: $w \cdot (\mathcal{E}, \theta) \mapsto (\mathcal{E}, w\theta)$. Even though it might seem at the first glance that the action will always exist in the irregular case as well, in fact it does not. The reason is that now we fix certain subsets (coadjoint orbits of the moment map) and the multiplication by w does not have to keep the point in the same orbit, so this action does not necessarily descend to an action on the irregular moduli space. As for one direct use of this treatment, authors in [ISS19b] point out that it is simpler to treat the so called *P=W conjecture* in the presence of an $\mathbb{C}^*$-action.

2.1.1 Organization

In Section 2.2 we introduce the technical notions for the particular case of the irregular moduli space we study here, with many references to additional materials and clarifications. We also restate the main theorem as Theorem 2.7, and give the proof conditional on Theorem 2.8 of Section 2.3. This section analyzes pencils on the first Hirzebruch surface $\mathbb{F}_1$ under some assumtions coming from the geometrical perspective, but is otherwise independent from the moduli of Higgs bundles story.

2.1.2 Conventions

We will denote the space of holomorphic sections of a V-vector bundle over Σ by $H^0(\Sigma, V)$ emphasizing their sheaf theoretical nature, as is standard when working with Higgs bundles. In this part, the symbol $\otimes$ for tensor product will denote $\otimes_{\mathcal{O}^1_{\mathbb{CP}}}$.

2.2 The Geometrical part

The focus of this work are specific meromorphic Higgs bundles over $\mathbb{CP}^1$ with structure group $G = \mathbb{GL}(3, \mathbb{C})$. Let $D = t_1 + t_2 + t_3$ be a simple effective divisor over $\mathbb{CP}^1$, meaning a linear combination of 3 distinct points on the surface $\mathbb{CP}^1$, and let $K_{\mathbb{CP}^1} = \Omega^1_{\mathbb{CP}^1}$ be the canonical (cotangent) bundle of $\mathbb{CP}^1$.

Definition 2.2. *A meromorphic Higgs bundle with a polar divisor D over $\mathbb{CP}^1$ is a pair $(\mathcal{E}, \theta)$ where $\mathcal{E}$ is a rank 3 holomorphic vector bundle over $\mathbb{CP}^1$, and θ is a $\mathcal{O}_{\mathbb{CP}^1}$-linear vector bundle morphism called the* Higgs field*:*

$$\theta : \mathcal{E} \to \mathcal{E} \otimes K_{\mathbb{CP}^1}(D)$$

where $K_{\mathbb{CP}^1}(D)$ denotes the meromorphic differentials over $\mathbb{CP}^1$ with poles at D.

Note we can also represent θ as a section of a bundle using the isomorphism $\mathcal{E} \otimes \mathcal{E}^* \cong End(\mathcal{E})$, which gives $\theta \in H^0(\mathbb{CP}^1, End(\mathcal{E}) \otimes K_{\mathbb{CP}^1}(D))$.

2.2.1 The definition and the dimension of the moduli space

To define the moduli space of Higgs pairs, we have to be more careful than in the standard, holomorphic case. It is not enough to take all the stable Higgs bundles and quotient, but we also have to fix the behaviour of the Higgs field near the poles in order for our moduli space to have nice properties and be hyperkähler as in the standard case. More precisely, we request that in some local coordinates, near say pole t_1, the leading term of the Higgs field $\theta = \Sigma_{k=-1}^{\infty} A_k z^k$ has a fixed *coadjoint orbit* for the leading term A_{-1}. In practice, we fix the parameters $a_i \in \mathbb{C}$, so that:

$$\theta = \left[\begin{pmatrix} a_1 & * & * \\ 0 & a_2 & * \\ 0 & 0 & a_3 \end{pmatrix} \cdot z^{-1} + O(1)\right] \otimes dz$$

near $z = 0$. This is a form that can always be chosen by starting with an arbitrary regular matrix in $\mathfrak{gl}^*(3, \mathbb{C})$ and acting by the *coadjoint action*. Analogously we choose matrices with parameters b_i and c_i for poles t_2 and t_3.

There is a notion of *parabolic stability* that needs to be treated carefully in order to compactify these spaces properly. For this we refer to Section 3 of [ISS19a], and [KON93; Nak96]. In our cases, it is enough to assume a generic parabolic structure and parabolic degree being 0. When the local conditions at poles are satisfied, as well as the parabolic stability conditions, the space of Higgs pairs quotiented by bundle isomorphisms becomes *the moduli space of meromorphic Higgs bundles*. For the general and precise construction of the moduli space, we refer to papers of Simpson[Sim90], Konno [KON93], and Boalch's survey [Boa12].

Definition 2.3. *Denote by $\mathcal{M}(\mathbb{CP}^1, D, 3)$ the moduli space of Higgs bundles $(\mathcal{E}, \theta)$ of rank 3 over $\mathbb{CP}^1$ with 3 simple poles at D.*

This moduli space decomposes into $\mathbb{Z}$ connected components according to the degree of $\mathcal{E}$: $\mathcal{M}(\mathbb{CP}^1, D, 3) = \sqcup_{d \in \mathbb{Z}} \mathcal{M}(\mathbb{CP}^1, D, 3, d)$, and from now on we chose a component - assume the degree is $d = 0$, and simply denote it $\mathcal{M}$.

To demonstrate at least one important property of our moduli space, we now calculate its dimension, by using the complex symplectic quotient perspective just outlined (see also Section 3.1 of [Boa12]). Take 3 different coadjoint orbits $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ of some regular elements in $\mathfrak{gl}^*(3, \mathbb{C})$ subject to the Fuch's relation

$\Sigma_1^3 tr(O_i) = 0$. This relation goes in parallel with the fact that the group we quotient out by is in fact $\mathbb{PGL}(3,\mathbb{C})$. Ignoring stability conditions, the moduli space $\mathcal{M}$ can now be writen as the (complex symplectic) quotient:

$$\mathcal{M} = O_1 \times O_2 \times O_3 // \mathbb{PGL}(3,\mathbb{C})$$

This is called the *GIT quotient* and is a construction typical for Geometric Invariant Theory mentioned in the introduction. The double 'slash' means that we first take the inverse image of zero for the *moment map*, and then further quotient by the diagonal action of $\mathbb{PGL}(3,\mathbb{C})$, so

$$\mathcal{M} = \{A_1, A_2, A_3 | A_1 + A_2 + A_3 \equiv 0\} / \mathbb{PGL}(3,\mathbb{C})$$

Note we will refer to complex dimension of objects for the rest of this subsection. To calculate the dimension of $\mathcal{M}$ we first calculate that each of the coadjoint orbits O_i has dimension 6: Start from the full dimension $dim(\mathfrak{gl}^*(3,\mathbb{C})) = 9$. As the action is coadjoint, the dimension of the stabilizer of a regular element A_i is the dimension of the centralizer $dim(Stab(A_i)) = dim(Z(A_i))$, and we have $dim(Z(A_i)) = 3$ which in this case is an exercise in putting our matrix into the Jordan normal form and checking 3 different cases that arise depending on the equalities between eigenvalues. Therefore:

$$dim(O_i) = dim(\mathfrak{gl}^*(3,\mathbb{C})) - dim(Stab(A_i)) = 6$$

The quotient at the 0 value of the moment map means we first took a co-dimension 8 subspace of $O_1 \times O_2 \times O_3$, and then quotiented by the action of $\mathbb{PGL}(3,\mathbb{C})$ finally getting:

$$dim(\mathcal{M}^*) = dim(O_1 \times O_2 \times O_3) - 2 \cdot dim(\mathbb{PGL}(3,\mathbb{C})) = 3 \cdot 6 - 8 - 8 = 2$$

Therefore, our moduli space has complex dimension 2 as claimed.

2.2.2 Hitchin fibration

Before introducing the main object of this subsection, we first introduce $Z_{\mathbb{CP}^1}(D)$ as an adequate compactification of the total space of the line bundle $K_{\mathbb{CP}^1}(D)$, which we get by a projectivization construction:

$$Z_{\mathbb{CP}^1}(D) = \mathbb{P}_{\mathbb{CP}^1}(K_{\mathbb{CP}^1}(D) \oplus \mathcal{O}_{\mathbb{CP}^1})$$

In our case $K_{\mathbb{CP}^1}(D) \cong \mathcal{O}(-2) \otimes \mathcal{O}(3) = \mathcal{O}(1)$, so $Z_{\mathbb{CP}^1}(D) = \mathbb{F}_1$ where $\mathbb{F}_1$ denotes the first Hirzebruch surface, diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Denote the ruling coming from the projectivization by $p : \mathbb{F}_1 \to \mathbb{CP}^1$, and take the pull-back of p to itself:

$$\begin{tikzcd}
p^* K_{\mathbb{CP}^1}(D) \arrow[d, "p^*p"'] & K_{\mathbb{CP}^1}(D) \arrow[d, "p"] \\
Z_{\mathbb{CP}^1}(D) \arrow[r, "p"] & \mathbb{CP}^1
\end{tikzcd}$$

Define ζ to be the canonical section of $p^* K_{\mathbb{CP}^1}(D)$, meaning away from the -1 *infinity section*, ζ is the Liouville form that defines the symplectic structure of the cotangent bundle $K_{\mathbb{CP}^1}(D)$.

Definition 2.4. *Following the previous discussion, define the characteristic polynomial for a Higgs field $\theta \in H^0(\mathbb{CP}^1, End(\mathcal{E}) \otimes K_{\mathbb{CP}^1}(D))$ to be:*

$$\chi_\theta(\zeta) = det(\zeta I_{\mathcal{E}} - \theta) = \zeta^3 + F_\theta \zeta^2 + G_\theta \zeta + H_\theta$$

where F_θ, G_θ and H_θ are some meromorphic differentials:

$$F_\theta \in H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}(D)), G_\theta \in H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}(D)^{\otimes 2}), H_\theta \in H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}(D)^{\otimes 3}).$$

Remark 2.5. *We simplified the notation but formally, the previous definition is not fully correct - this can be seen comparing ζ^3 and H_θ which we add together even though one is a section with base $Z_{\mathbb{CP}^1}(D) = \mathbb{F}_1$ and the other one has base $\mathbb{CP}^1$. To formalize, we can take $\zeta \in H^0(\mathbb{F}_1, p^* K_{\mathbb{CP}^1}(D) \otimes \mathcal{O}_{\mathbb{F}_1|\mathbb{CP}^1}(1))$, and introduce a further canonical section $\xi \in H^0(\mathbb{F}_1, \mathcal{O}_{\mathbb{F}_1|\mathbb{CP}^1}(1))$ so that χ_θ becomes $\chi_\theta(\zeta) = det(\zeta I_{\mathcal{E}} - \xi p^* \theta) \in H^0(\mathbb{F}_1, p^* K_{\mathbb{CP}^1}(D)^{\otimes 3} \otimes \mathcal{O}_{\mathbb{F}_1|\mathbb{CP}^1}(3))$ (see Section 2 of [ISS19b]).*

Let's denote

$$\mathcal{B} := H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}(D)) \oplus H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}(D)^{\otimes 2}) \oplus H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}(D)^{\otimes 3})$$

and call it the *Hitchin base*. Using the previous paragraphs:

Definition 2.6. Hitchin map *on $\mathcal{M}$ is:*

$$h : \mathcal{M} \to \mathcal{B}$$
$$(\mathcal{E}, \theta) \mapsto (F_\theta, G_\theta, H_\theta).$$

2.2.3 Spectral correspondence

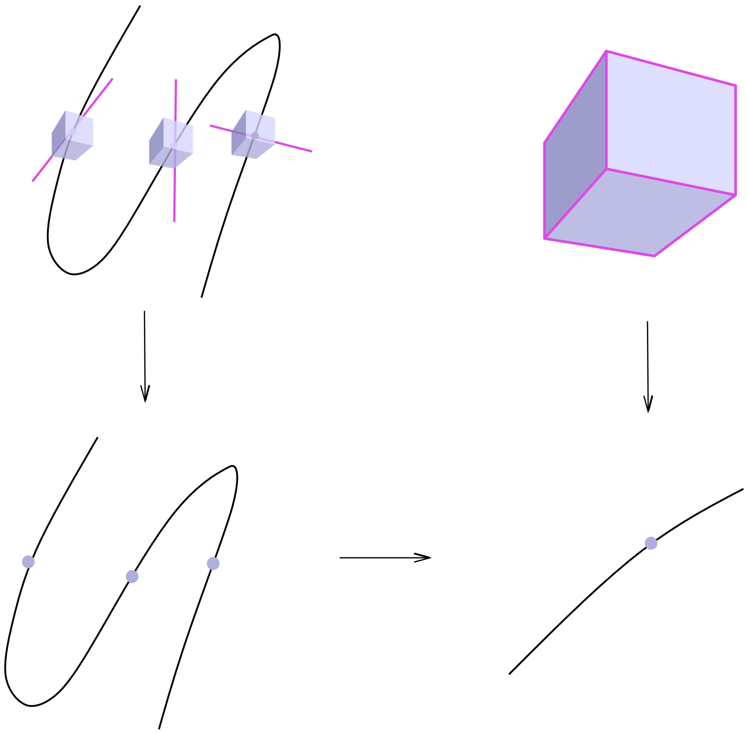


Figure 2.3: A sketch of the spectral curve (lower left) and the line bundle associated to a Higgs pair via the spectral correspondence

The precise version of this correspondence is spelled out in Proposition 3.1 [Sza17], but in this subsection we give an idea of why Hitchin fibration can be derived from a pencil of cubics on $\mathbb{F}_1$. For a generic point (F, G, H) in the Hitchin base $\mathcal{B}$, define the curve $\Sigma_{(F,G,H)}$ by taking all the points in the base $Z_{\mathbb{CP}^1}(D) = \mathbb{F}_1$ that satisfy:

$$\zeta^3 + F\zeta^2 + G\zeta + H = 0$$

for a subset of the bundle $p^* K_{\mathbb{CP}^1}(D)^{\otimes 3} \otimes \mathcal{O}_{\mathbb{F}_1|\mathbb{CP}^1}(3)$ given by the intersection between the 0-section and the section $\zeta^3 + F\zeta^2 + G\zeta + H$. This is a smooth curve we call the *spectral curve*. It is a zero set of a degree 3 polynomial and it is exactly a kind of a curve in the pencil on $\mathbb{F}_1$ which after a birational transformation becomes a fiber of the Hitchin fibration.

The fiber in $h : \mathcal{M} \to \mathcal{B}$ over (F, G, H) is made of pairs $(\mathcal{E}, \theta)$ for which the characteristic polynomial is:

$$\chi_\theta(\zeta) = \zeta^3 + F\zeta^2 + G\zeta + H$$

Taking an arbitrary rank 3 holomorphic vector bundle $\mathcal{E} \to \mathbb{CP}^1$, we can pull it back to $\Sigma_{(F,G,H)} \subset Z_{\mathbb{CP}^1}(D)$ via map p, see Figure 2.3. Generically, the pre-image of a point in $\mathbb{CP}^1$ are 3 points, its eigenvalues in $K_{\mathbb{CP}^1}(D) \subset Z_{\mathbb{CP}^1}(D)$. The associated eigenspaces make up a subbundle of a pullback vector bundle we just defined. This way we get a spectral curve $\Sigma_{(F,G,H)}$ which is a 3-fold (ramified) cover over $\mathbb{CP}^1$, along with a line bundle on it. This is exactly the *spectral correspondence*: an element of the Higgs moduli space $(\mathcal{E}, \theta)$ generically defines a spectral curve and a line bundle on it, and vice-versa.

This way we see a fiber of the Hitchin fibration is defined by a spectral curve together with its Jacobian, the compact abelian variety comprised of all line bundles of degree 0 on the curve [AHH90; Bea90; Nit91]. In this case this is of complex dimension 1 - therefore a torus $\mathbb{T}^2$ - and as h is a holomorphic map, our Hitchin fibration is really an elliptic fibration.

2.2.4 The main theorem

Theorem 2.7. *Let $D = t_1 + t_2 + t_3$ be a simple effective divisor on $\mathbb{CP}^1$. Hitchin fibration on $\mathcal{M}(\mathbb{CP}^1, D, 3, 0)$ with a generic parabolic structure has a unique singular fiber (aside from the type IV fiber at infinity) if and only if the specified residue matrices at the 3 poles, each have a triple eigenvalue.*

In this case the singular fiber is necessarily $\tilde{E}_6$ and the 3 triple eigenvalues a, b and c satisfy $a + b + c = 0$.

Proof. Firstly, our compactified moduli space is isomorphic to $\mathbb{F}_1$ and it contains a pencil of spectral curves, where one of the curves C_1 consists of the union of the -1-section 'at infinity' and three special fibers of the ruling. After the resolution of this pencil, by blowing down the -1-section and blowing up base points, the manifold becomes $X = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. The pencil transforms into an elliptic fibration which is our Hitchin fibration assigned to the compactified moduli space. One of the singular fibers is a transformation of C_1 - the curve made of the -1-section and 3 fibers in $\mathbb{F}_1$. In X this curve becomes the type IV fiber seen in the Example in Figure 0.2. In order to have only one more singular fiber, this fiber has to be an $\tilde{E}_6$ fiber because monodromy around singular fibers uniquely determines them, as explained in [SSS07].

The pencil resolution process in our case starts by blowing down once away from all the base points of the pencil, and then blowing up 9 times - therefore we can see our pencil in $\mathbb{F}_1$ as a pencil in $\mathbb{CP}^2$. From understanding the spectral correspondence [Sza17], it follows that the fibers over the 3 poles comprising divisor D all contain a base point. Thus there are at least 3 basepoints coming from 3 different fibers. Since we are only blowing up from here, and since we do not have base points at the end, it directly follows that base points got transformed into at least 3 sections for the fibration [SSS07].

Proposition 2.17 establishes that if we get a combination of a type IV and $\tilde{E}_6$ fibers, and start from at least 3 sections, then there are *exactly 3 sections* and therefore *exactly 3 basepoints* for the pencil. This means that at each of the poles, there is only one triple eigenvalue, and the residue forms of Higgs fields θ are:

$$\begin{pmatrix} a & * & * \\ 0 & a & * \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} b & * & * \\ 0 & b & * \\ 0 & 0 & b \end{pmatrix}, \begin{pmatrix} c & * & * \\ 0 & c & * \\ 0 & 0 & c \end{pmatrix}$$

for some eigenvalues $a, b, c \in \mathbb{C}$.

Having established there are 3 basepoints on 3 fibers in $\mathbb{F}_1$, we upgrade the Proposition 2.17 into Theorem 2.8 from Subsection 2.3.2, to show that there is only one possible pencil that leads to the desired combination of singular fibers. This is exactly the pencil made up of two curves: C_1 - the -1-section union 3 fibers of p, and C_2 - a multiplicity 3 generic +1-section of p.

This implies that our three base points have to belong to the same +1-section in $\mathbb{F}_1$, or equivalently, the same complex line in $\mathbb{CP}^2$. So what is left is to understand the relation between parameters coming from different poles - a, b and c, that imply that the base points belong to the same section.

As any θ is a meromorphic form, the same is true for its symmetric polynomials, so especially $tr(\theta)$ is meromorphic, and then the residue theorem tells us that:

$$tr(Res_{t_1}(\theta)) + tr(Res_{t_2}(\theta)) + tr(Res_{t_3}(\theta)) = 0$$

This translates to $3a + 3b + 3c = 0$, and as these are complex numbers finally to $a + b + c = 0$. This is enough to finish the proof of the theorem as the condition of belonging to the same complex line is then automatically fullfilled.

To see it using a different language, we take a brief tour into charts. First cover $\mathbb{CP}^1$ with two standard charts: U_1 with coordinate $z_1 \in \mathbb{C}$ where $z_1 = 0$ at the point $t_1 = [0 : 1] \in \mathbb{CP}^1$; and U_2 with coordinate $z_2 \in \mathbb{C}$ where $z_2 = 0$ at the point $t_2 = [1 : 0] \in \mathbb{CP}^1$. By acting with a Möbius transformation, we can assume the remaining pole is $t_3 = [1 : 1] \in \mathbb{CP}^1$, and in the U_1 chart, this pole corresponds to $z_1 = 1$.

The bundle $K_{\mathbb{CP}^1}(D)$ has the following trivializing sections on U_1 and U_2:

$$\kappa_1 = \frac{dz_1}{z_1(z_1-1)}, \quad \kappa_2 = \frac{dz_2}{z_2}$$

The conversion between charts is: $z_1 = \frac{1}{z_2}$ and then the conversion between trivializations becomes: $\kappa_2 = (1-z_1)\kappa_1$.

Take an arbitrary section of $K_{\mathbb{CP}^1}(D)$, denote it σ. In chart U_1, $\sigma = (a_0 + a_1 z_1)\kappa_1$ for some $a_0, a_1 \in \mathbb{C}$, or otherwise we would have a pole of order > 1 at $[1:0]$.

To find the relation between eigenvalues a, b and c, we need to compare the $Res_{t_i}(\sigma)$ at all 3 poles t_1, t_2 and t_3.

$$Res_{t_1}(\sigma) = Res_{z_1=0}((a_0+a_1z_1)\frac{dz_1}{z_1(z_1-1)}) = Res_{z_1=0}(-(a_0+a_1z_1)(1-z_1)^{-1}\frac{dz_1}{z_1})$$
$$= Res_{z_1=0}(-(a_0+a_1z_1)(1+z_1+z_1^2+\ldots)\frac{dz_1}{z_1}) = Res_{z_1=0}((-a_0-(a_0+a_1)z_1+\ldots)\frac{dz_1}{z_1}) = -a_0$$

$$Res_{t_2}(\sigma) = Res_{z_1=\infty}((a_0+a_1z_1)\frac{dz_1}{z_1(z_1-1)}) = \{change : z_1 = \frac{1}{z_2}\}$$
$$= Res_{z_2=0}((a_0+a_1\frac{1}{z_2})\frac{d(\frac{1}{z_2})}{\frac{1}{z_2}(\frac{1}{z_2}-1)}) = Res_{z_2=0}(\frac{a_0z_2+a_1}{(z_2-1)z_2}dz_2)$$
$$= Res_{z_2=0}((-a_1-a_0z_2)(1+z_2+z_2^2+\ldots)\frac{dz_2}{z_2}) = -a_1$$

$$Res_{t_3}(\sigma) = Res_{z_1=1}((a_0+a_1z_1)\frac{dz_1}{z_1(z_1-1)}) = \{change : z_3 = z_1 - 1\}$$
$$= Res_{z_3=0}(\frac{a_0+a_1+a_1z_3}{1+z_3}\frac{dz_3}{z_3}) = Res_{z_3=0}((a_0+a_1+a_1z_3)(1-z_3+z_3^2-..)\frac{dz_3}{z_3})$$
$$= a_0 + a_1$$

Therefore, adding 3 expressions, we conclude the residue theorem, and again that the 3 triple eigenvalues at the 3 poles satisfy $a+b+c=0$.

□

2.3 The Topological part

In this section we examine pencils of curves in the first Hirzebruch surface $\mathbb{F}_1$ which lead to a fibration in the elliptic surface $E(1) \cong \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ having exactly two singular fibers, one of *type IV* and one of type $\tilde{E}_6$, as shown in Figure 2.1.

More precisely, this means that we are examining which pencils in $\mathbb{F}_1 \cong \mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ after a minimal elimination of their base points (which comprises of a certain combination of blow-ups and blow-downs), contain only these 2 singular fibers in an elliptic fibration on $E(1) \cong \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$.

Additionally, in the previous section, we explained that our pencil has to have at least 3 base points, one on each fiber of the usual $\mathbb{CP}^1$- ruling on $\mathbb{F}_1$, that we denoted p. Therefore, if we blow-down the -1 section of $\mathbb{F}_1$, after 9 blow-ups, we will have at least 3 sections of the elliptic fibration, intersecting each of the 3 leaves of the type IV fiber as in Figure 2.4.

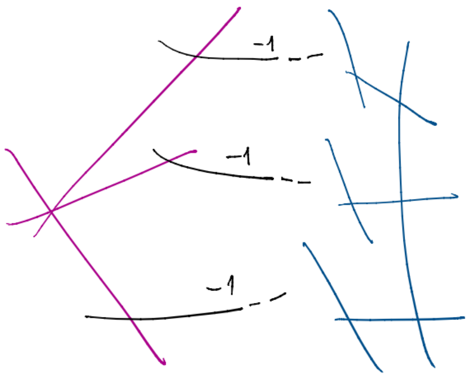


Figure 2.4: 3 sections on 3 different fibers of the type IV fiber

For further reference let us denote the fiber components as in Figure 2.5.

The goal of this section is to show Theorem 2.8:

Theorem 2.8. *Take a pencil in $\mathbb{F}_1$ whose one defining curve is a curve consisting of 3 distinct fibers of the ruling and the -1 section. Assume that each of these 3 fibers contains a base point of the pencil, and that by a minimal resolution of this pencil we get an elliptic fibration with exactly 2 singular fibers: a type IV fiber and an $\tilde{E}_6$ fiber.*

Then this pencil has exactly 3 base points, and is defined by 2 curves:

1. C_1 = 3 distinct fibers of the ruling on $\mathbb{F}_1$ and the -1 section of p

2. C_2 = +1 section of p counted with multiplicity 3

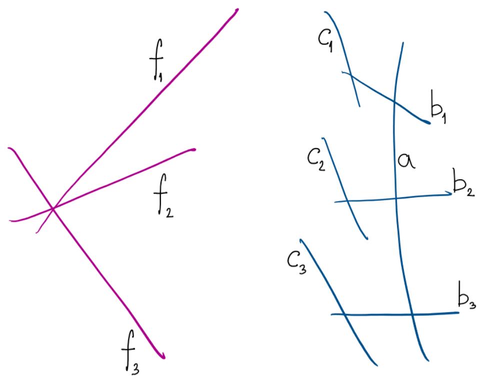


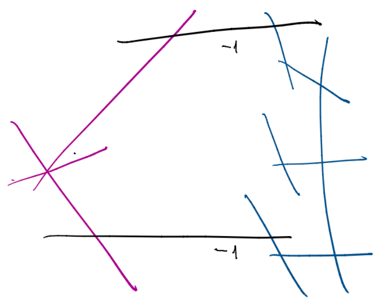
Figure 2.5: Notation of fiber components

In order to prove it, we first show in Subsection 2.3.1 that starting from at least 3 sections (each intersecting one leaf of the type IV fiber), we cannot have more than 3 sections in the elliptic fibration and therefore not more than 3 base points in the starting pencil. Then in Subsection 2.3.2 we examine all the pencils of degree 3 curves that have only 3 base points and lead to an elliptic fibration with exactly 2 previously specified singular fibers.

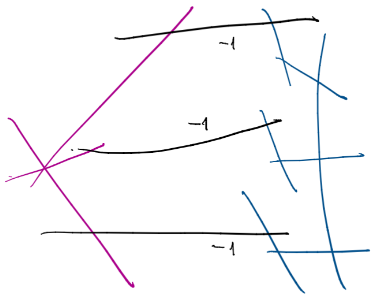
2.3.1 At least 3 base points implies at most 3 base points

The goal of this subsection is to prove Proposition 2.17, so we first exclude various impossible configurations of curves in $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. Assume this is the ambient manifold in the formulation of lemmas for the entire subsection.

Lemma 2.9. *If there are two sections in the following configuration (intersecting different pairs of leaves in both fibers):*



then if there is an additional section, it has to intersect exactly the remaining pair of leaves, one in each fiber:



Proof. Apart from the claimed configuration, by symmetry, these are all the other possibilities for an additional section:

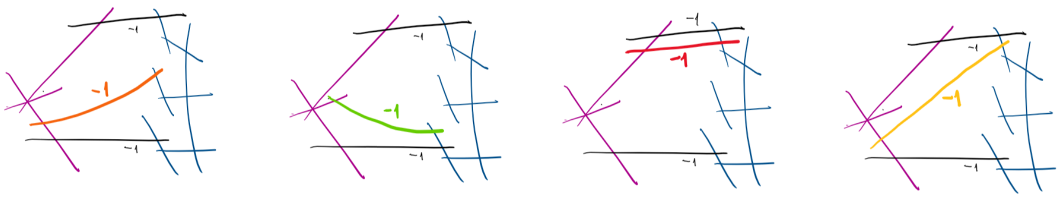


Figure 2.6: 4 possibilities

Lemmas 2.11-2.14 prove these configurations are impossible. □

Remark 2.10. *In all the flowcharts, the number next to a curve denotes its self-intersection, and whenever there is no number, the self-intersection is assumed to be -2. In the beginning figure of every flowchart we will name all the curves involved but will use only the relevant ones in the figures that follow. The exception is Lemma 2.11 where we denote all the curves affected by the transformations in that step.*

Lemma 2.11. *The configuration in Figure 2.7 is impossible.*

Proof. As per Figure 2.9, we first blow down all 3 sections e_1, e_2, e_3, and subsequently the proper transforms of c_1, c_2 and c_3. This is followed by blowing down the proper transforms of b_1 and then a, after which we blow up the common intersection of spheres f_1, f_2 and f_3. Regardless of the blow-ups and blow-downs, we are in a 4-manifold with $b_1^+ = 1$ but at this point found two disjoint curves: $\tilde{\tilde{f}}_1$ and $\tilde{\tilde{f}}_2$ both with self-intersections +1, and this is impossible. □

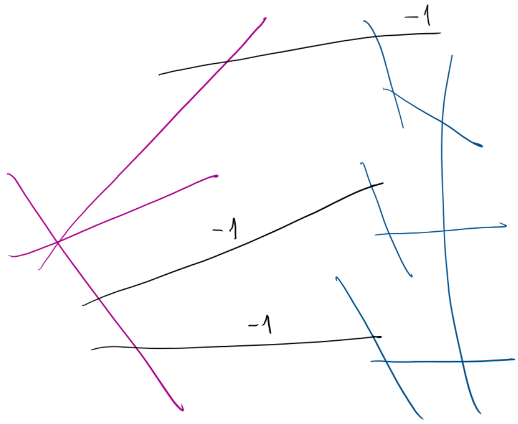


Figure 2.7: Impossible configuration in Lemma 2.11, proof in Figure 2.9

Lemma 2.12. *The configuration in Figure 2.8 is impossible.*

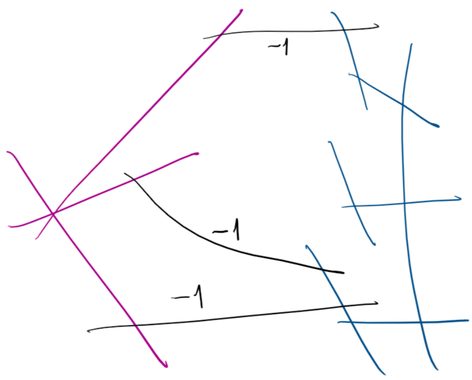


Figure 2.8: Impossible configuration in Lemma 2.12, proof in Figure 2.10

Proof. After a series of blow-downs represented in the flowchart of Figure 2.10, the last figure shows two disjoint curves, $\tilde{f_1}$ and $\tilde{e_3}$, both of self-intersection +1. They therefore cannot represent the same homology class and the ambient manifold has $b_2^+ = 1$ as before, so this is contradictory.

Notice that we could have also blown down two more curves and transformed $\tilde{f_1}$ further into a +3 curve. In any case, we get two different positive self-intersection homology classes meaning positive definite subspace of dimension 2 for a manifold which has $b_2^+ = 1$. □

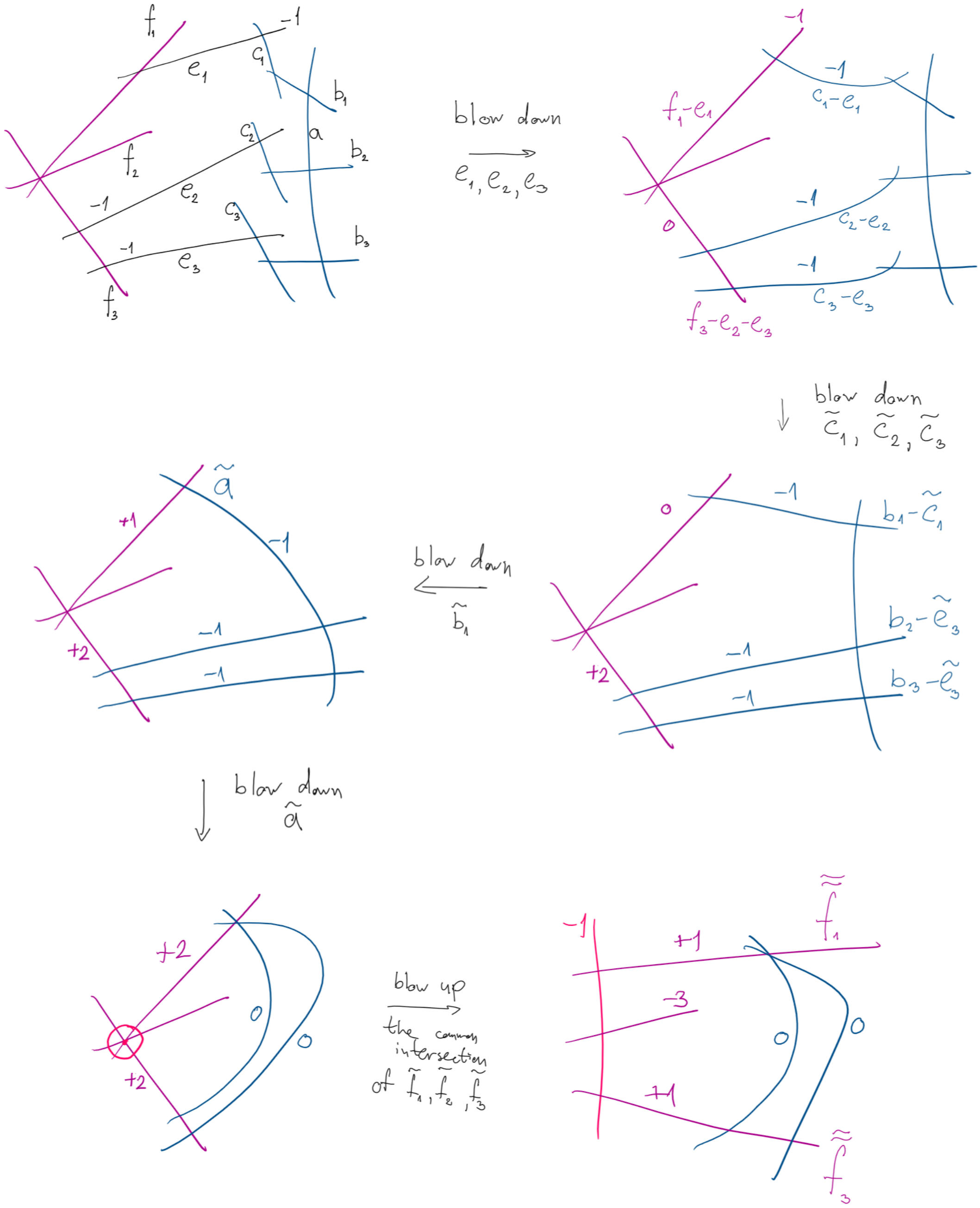


Figure 2.9: Flowchart of transformations in Lemma 2.11

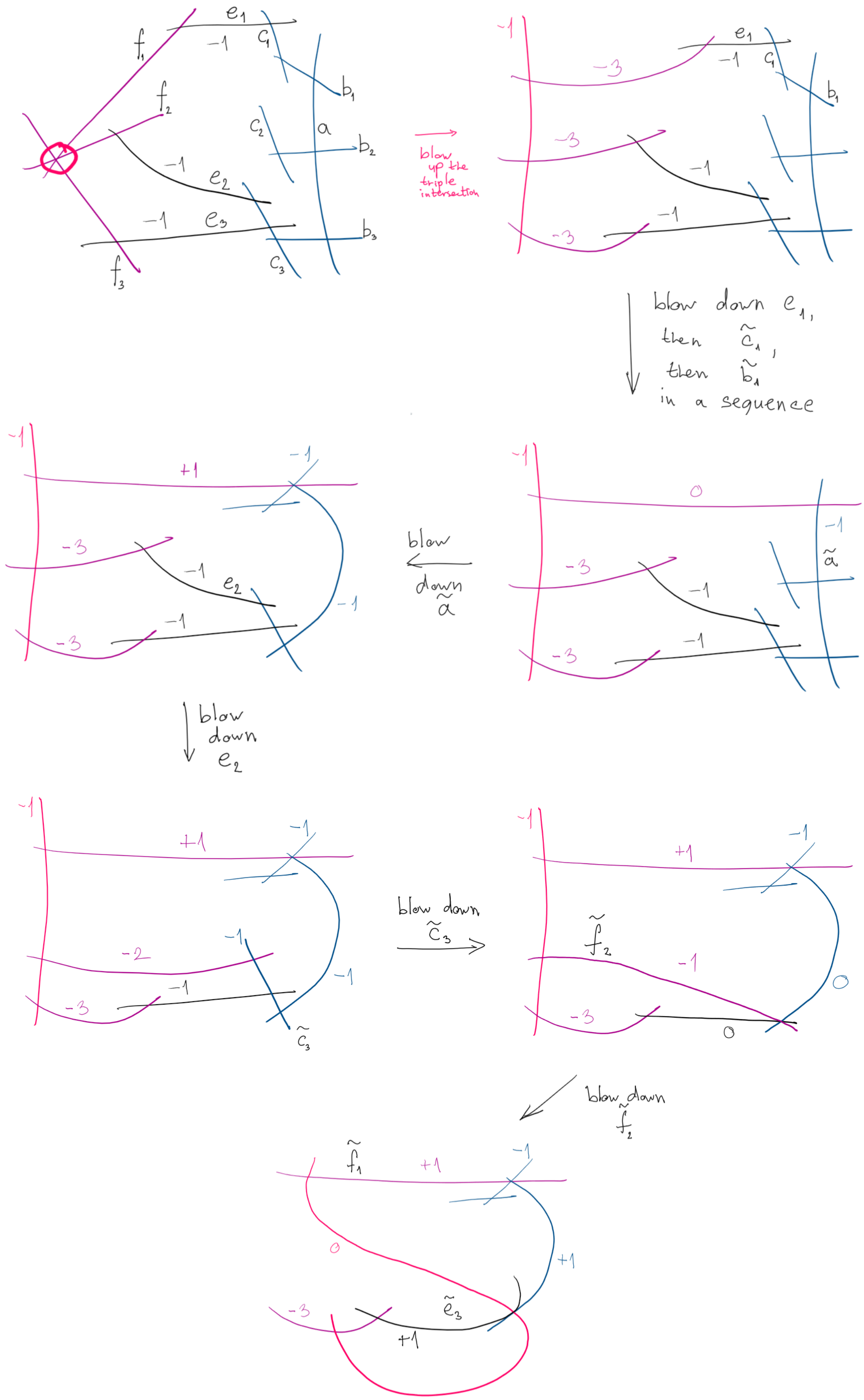


Figure 2.10: Flowchart of transformations in Lemma 2.12

Lemma 2.13. *The configuration in Figure 2.11 is impossible.*

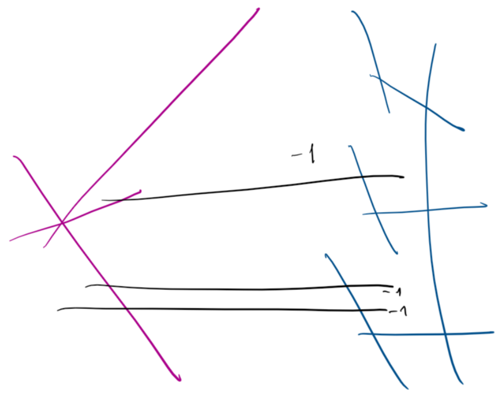


Figure 2.11: Impossible configuration in Lemma 2.13, proof in Figure 2.13

Proof. Figure 2.13 presents the blow down procedure leading to a pair of curves which again cannot be in a manifold with $b_2^+ = 1$.

In the last part of the flowchart in Figure 2.13 we have two disjoint curves, $\tilde{c_3}$ of self-intersection +4 and $\tilde{f_2}$ of self-intersection +1, and as before, this is impossible. □

Lemma 2.14. *The configuration in Figure 2.12 is impossible.*

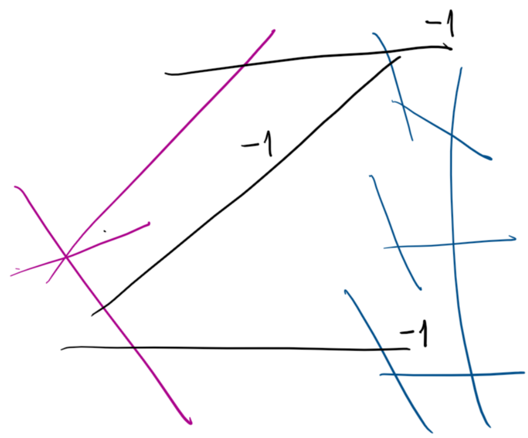


Figure 2.12: Impossible configuration in Lemma 2.14, proof in Figure 2.14

Proof. This time we use the same techniques but do not prove contradiction by observing two disjoint positive self-intersection curves, but rather a dimension 10 negative definite subspace of a space having $b_2^- = 9$. This is done by showing there are 10 consecutive blow downs, as the flowchart in Figure 2.14 shows.

In the last figure of the flowchart we still have a -1 curve $\tilde{f}_2$ which is impossible as the ambient manifold has $b_2^- = 0$. □

Remark 2.15. *In the proof of Lemma 2.14, the numbers in the upper right corners of the flowchart 2.14 denote the current b_2^- which changes (drops) as we continue the blow-down procedure.*

Lemma 2.16. *The configuration in Figure 2.15 is impossible.*

Proof. First blow down 3 sections e_1, e_2, e_3, and subsequently $\tilde{f}_1$ as in Figure 2.16.

Now notice we have the configuration of curves as in Figure 2.17 inside our 4-manifold with $b_2^- = 5$.

The intersection form of curves in Figure 2.17 is:

$$\begin{pmatrix} -2 & 1 & 1 & 1 & & \\ 1 & -2 & & & 1 & \\ 1 & & -2 & & & 1 \\ 1 & & & -2 & & \\ & 1 & & & -2 & \\ & & 1 & & & -2 \end{pmatrix}$$

Doing simple Gaussian elimination we get:

$$\begin{pmatrix} -2 & 1 & 1 & 1 & & \\ 1 & -2 & & & 1 & \\ 1 & & -2 & & & 1 \\ 1 & & & -2 & & \\ & 1 & & & -2 & \\ & & 1 & & & -2 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 & 1 & 1 & & \\ 1 & -2 & & & 1 & \\ 1 & & -\frac{3}{2} & & & 1 \\ 1 & & & -2 & & \\ & 1 & & & -2 & \\ & & 0 & & & -2 \end{pmatrix} \sim$$

$$\begin{pmatrix} -2 & 1 & 1 & 1 & & \\ 1 & -\frac{3}{2} & & & 1 & \\ 1 & & -\frac{3}{2} & & & 1 \\ 1 & & & -2 & & \\ & 0 & & & -2 & \\ & & 0 & & & -2 \end{pmatrix} \sim \begin{pmatrix} -\frac{3}{2} & 1 & 1 & 1 & & \\ 1 & -\frac{3}{2} & & & 1 & \\ 1 & & -\frac{3}{2} & & & 1 \\ 0 & & & -2 & & \\ & 0 & & & -2 & \\ & & 0 & & & -2 \end{pmatrix} \sim$$

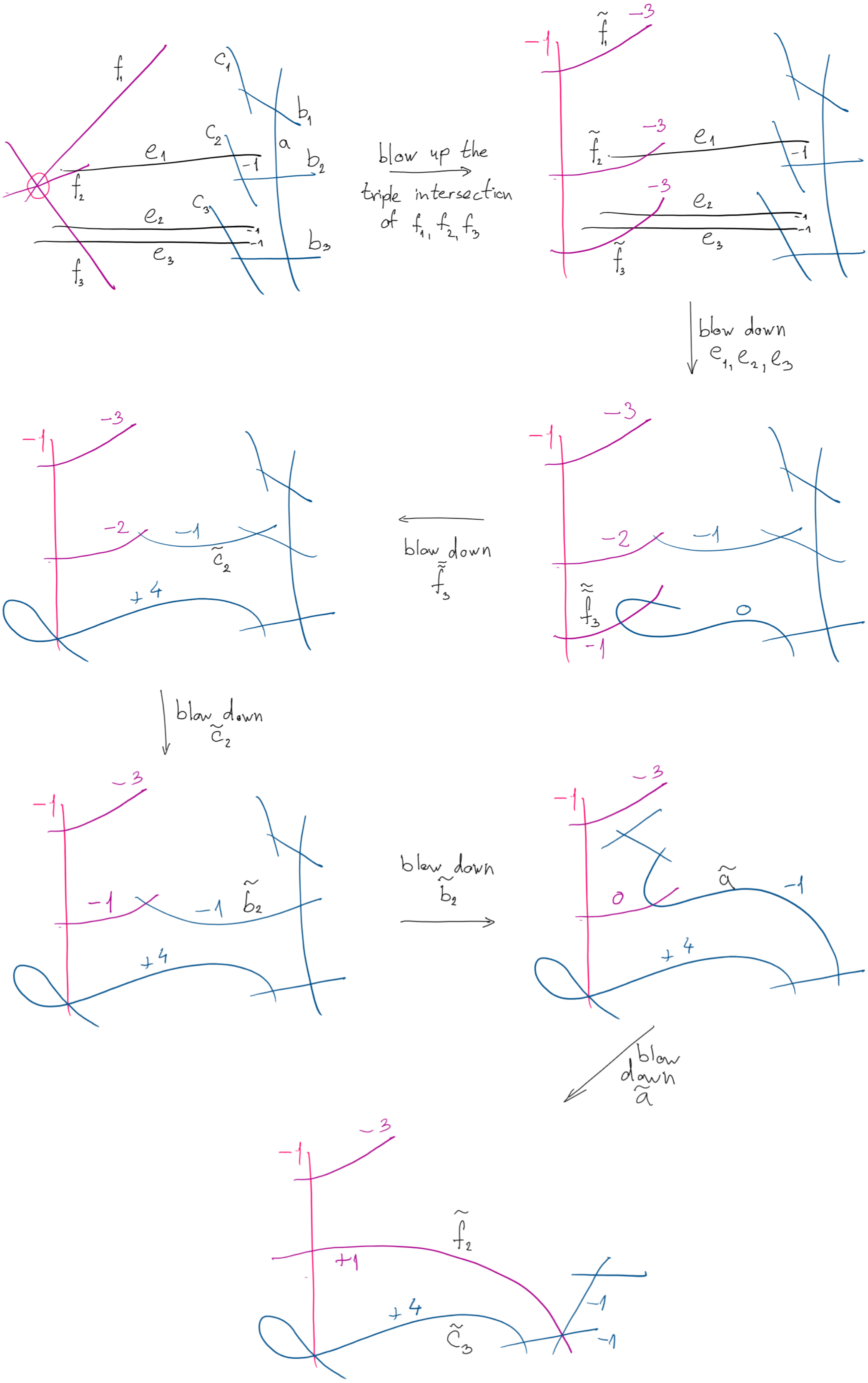
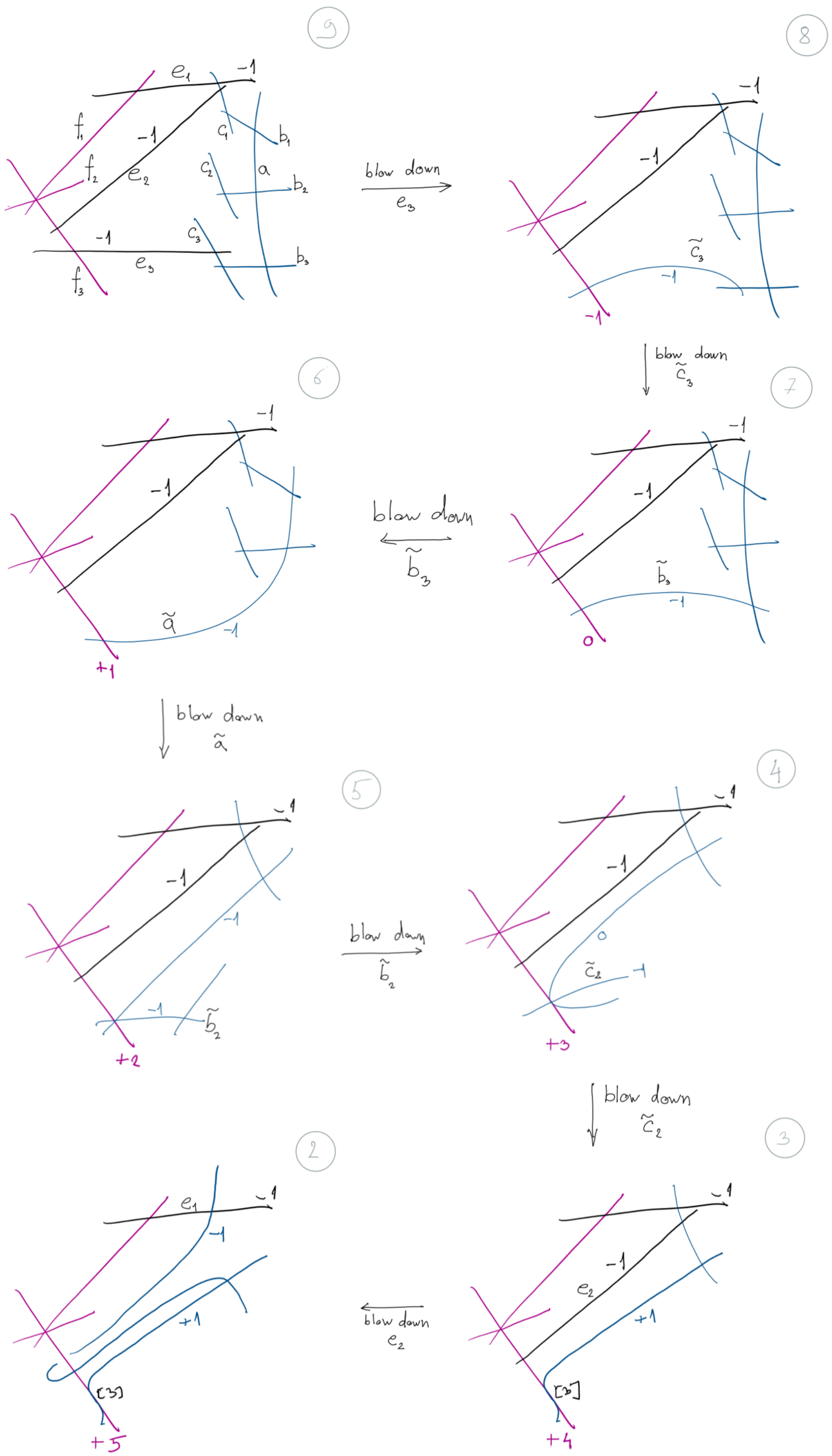


Figure 2.13: Flowchart of transformations in Lemma 2.13


9
8
blow down
e3
blow down
c̃3
7
6
blow down
b̃3
blow down
ã
5
4
blow down
b̃2
blow down
c̃2
2
3
blow down
e2

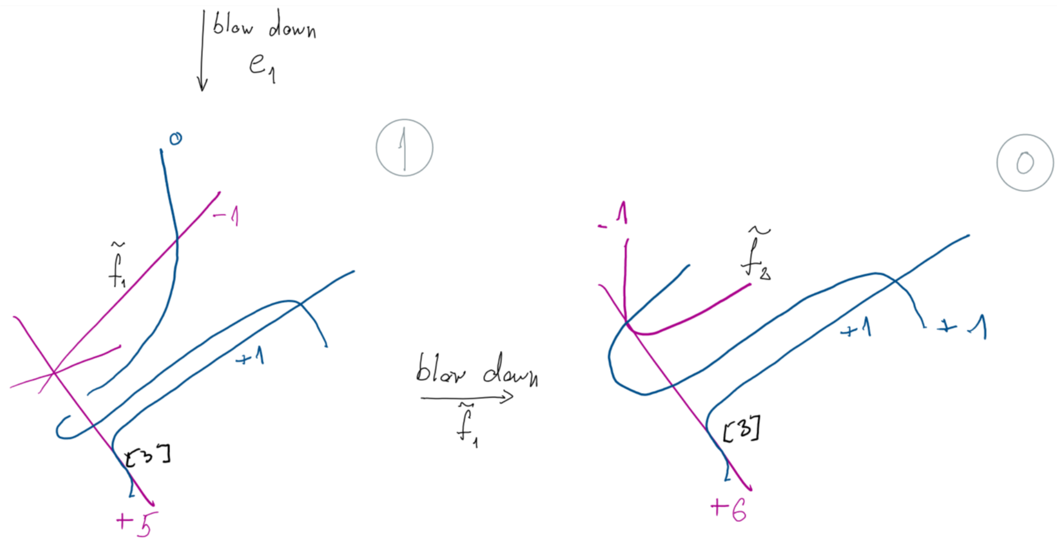


Figure 2.14: Flowchart of transformations in Lemma 2.14

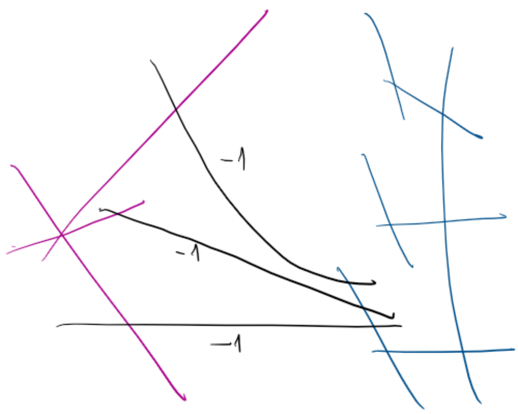


Figure 2.15: Impossible configuration in Lemma 2.16, proof in Figure 2.16

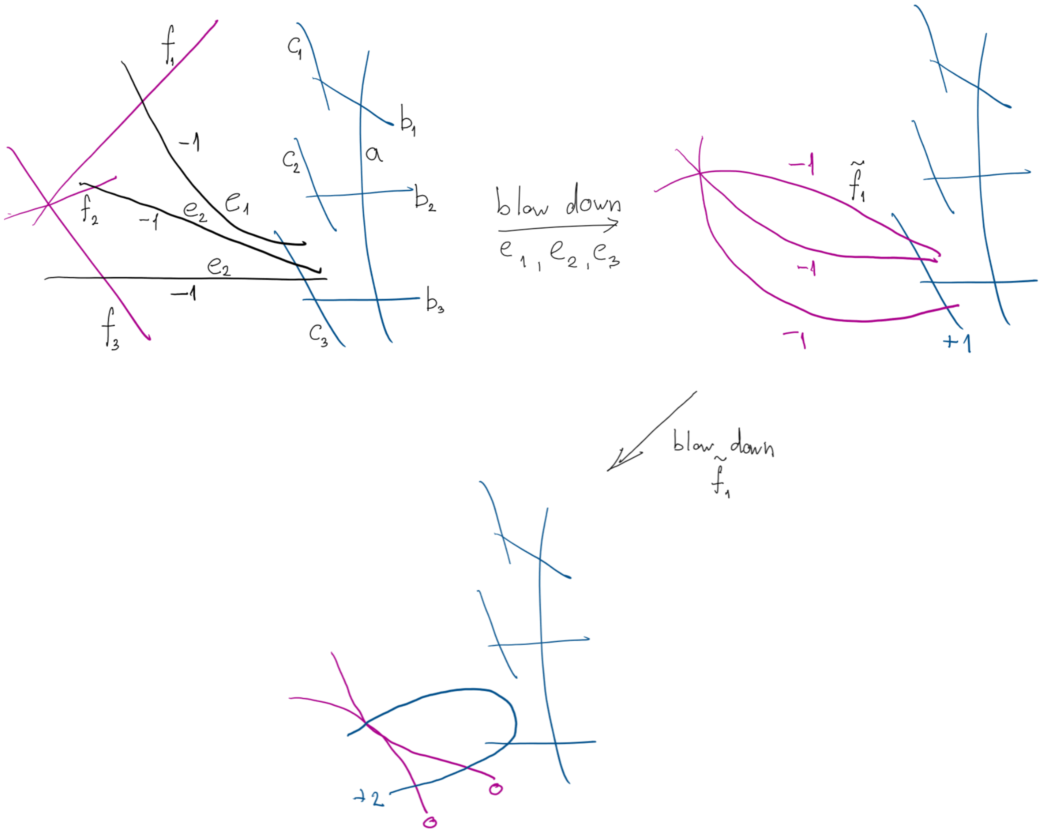


Figure 2.16: Flowchart of transformations in Lemma 2.16

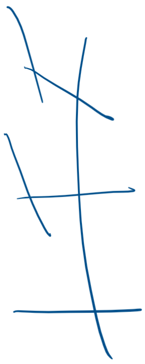


Figure 2.17: $\tilde{E}_6$ fiber without one curve better known as the E_6 fiber

$$\sim \begin{pmatrix} -\frac{1}{6} & 1 & 1 & 1 & & \\ 0 & -\frac{3}{2} & & & 1 & \\ 0 & & -\frac{3}{2} & & & 1 \\ 0 & & & -2 & & \\ & 0 & & & -2 & \\ & & 0 & & & -2 \end{pmatrix}$$

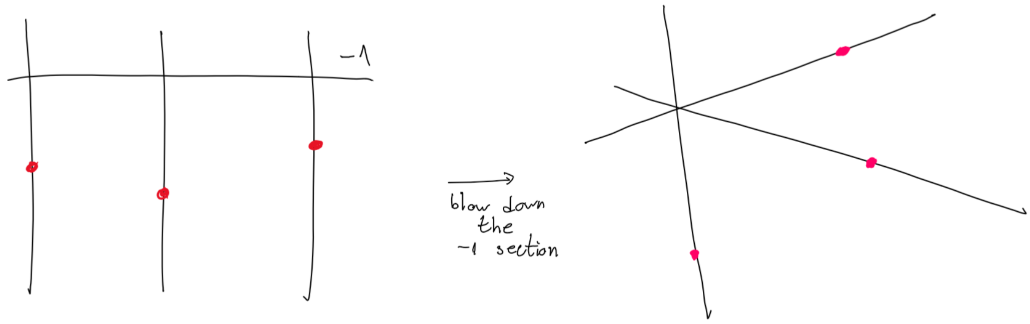
All the eigenvalues of this matrix are negative and therefore the homology classes of the curves giving this intersection form make up a 6 dimensional negative definite subspace in the second homology. This is impossible as the ambient manifold can have at most a 5 dimensional negative subspace due to $b_2^- = 5$. □

Proposition 2.17. *If there is a base point on each of the 3 chosen fibers of the ruling on $\mathbb{F}_1$, and by minimal resolution we get an elliptic fibration with type IV and type $\tilde{E}_6$ fibers, then there are in fact only 3 base points.*

Proof. Any section of the elliptic fibration intersects the $\tilde{E}_6$ fiber in one of the leaves (due to other components having higher multiplicities). Lemma 2.16 shows us that not all 3 sections coming from the initial 3 base points can intersect the same leaf. So they have to intersect at least 2 different leaves of $\tilde{E}_6$. But Proposition 2.9 then claims that actually all 3 sections intersecting different leaves of the type IV fiber, also intersect 3 different leaves of the $\tilde{E}_6$ fiber. Finally, Lemma 2.13 used again for 2 pairs of leaves, tells us that any potential fourth section would provide us with a contradictory configuration. Thus, we only have 3 sections (intersecting all 3 pairs of leaves), and consequently 3 base points in the pencil. □

2.3.2 There is only one such pencil

Proof of Theorem 2.8. We have established that in our case there are exactly 3 base points, one on each of the 3 chosen fibers of the ruling on $\mathbb{F}_1$. After blowing down the -1 section, we examine possible complex curves now in $\mathbb{CP}^2$ which intersect the fibers of the ruling f_1, f_2, f_3 in exactly those 3 base points A_1, A_2, A_3.



After 9 blow-ups we have an elliptic pencil with one singular fiber comprised of f_1, f_2, f_3 and another singular fiber coming from the resolution of the other curve. This means that both these curves were zeroes of certain degree 3 polynomials, and so the other curve defining the pencil is also a degree 3 curve in $\mathbb{CP}^2$. We need to examine which degree 3 curves can intersect 3 lines f_1, f_2, f_3 in exactly those 3 given base points.

There is one important comment to make in the beginning - if a pencil of cubics in $\mathbb{CP}^2$ has a nondegenerate cubic satisfying this intersection condition, there will also be another degenerate cubic satisfying the condition. This is because 3 lines represent a type IV fiber in the elliptic fibration that we get after 9 blow-ups, and this singular fiber cannot be the only singular one in the fibration. This can be explained in various ways, for example it accounts for only $(ab)^2$ out of $(ab)^6$ in monodromy (see Section 2.2 in [SSS07]). Or even simpler, the Euler characteristic of $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ is 12 and the Euler characteristic of the underlying elliptic fibration is the sum of Euler characteristics of its singular fibers. Type IV fiber, topologically, consists of 3 spheres intersecting in one point, and the Euler characteristic of this is 4. This means there must be further singular fibers and we can focus solely on singular cubics passing through 3 base points. There are several options for the other singular cubic, as it can consist of:

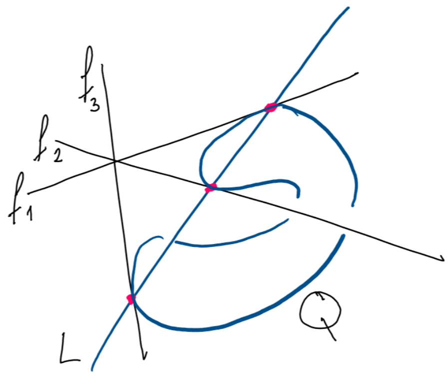
1) 3 lines; 2) A quadric and a line; 3) A degenerate cubic: 1a) cuspidal 1b) fishtail

Case 1: 3 lines

Take any of the 3 lines making up the second singular cubic (let us call them L_1, L_2, L_3) - any two lines in $\mathbb{CP}^2$ intersect in 1 point so our chosen line L_i intersects all of our starting 3 lines f_1, f_2, f_3 exactly in the base points. As this applies to any of the L_1, L_2, L_3, we conclude that in this case the singular cubic is a line of multiplicity 3 passing through the 3 base points, and consequently 3 base points are all on one complex line. This case indeed produces an elliptic fibration with only two singular fibers, a type IV fiber and an $\tilde{E}_6$ fiber, as in Example in Figure 0.2.

Case 2: a quadric and a line

Now we start with a line L and a quadric Q comprising a degenarate degree 3 curve together. Again, as in case 1, the line L has to intersect 3 initial lines f_1, f_2, f_3 in exactly 3 given base points. Any quadric intersects any line in 2 points - these can be either different points or one point of multiplicity 2. Our quadric Q intersects 3 lines f_1, f_2, f_3 in one point (of multiplicity 2) each. These are the 3 base points which are contained also in L and therefore there are at least 3 points in the intersection between L and Q, which is imposible.



Case 3: a degenerate cubic

(a) In case we have a *cuspidal cubic* C with cusp not being one of the base points, this cusp will 'survive' the blow up process and become one of the singular fibers. However we are looking for the situations in which there are only two singular fibers, none of them being the cusp fiber.

Thus we further analyse the case where the cusp of C is one of the base points, ex. A_1 in Figure 2.18. By blowing up this point and two further 'infinitely close' blow-ups, we find two curves: $\tilde{C}$ and a -2 curve $\tilde{e_1}$, intersecting each other in two points. This does not exist as part of the $\tilde{E_6}$ fiber we are aiming for.

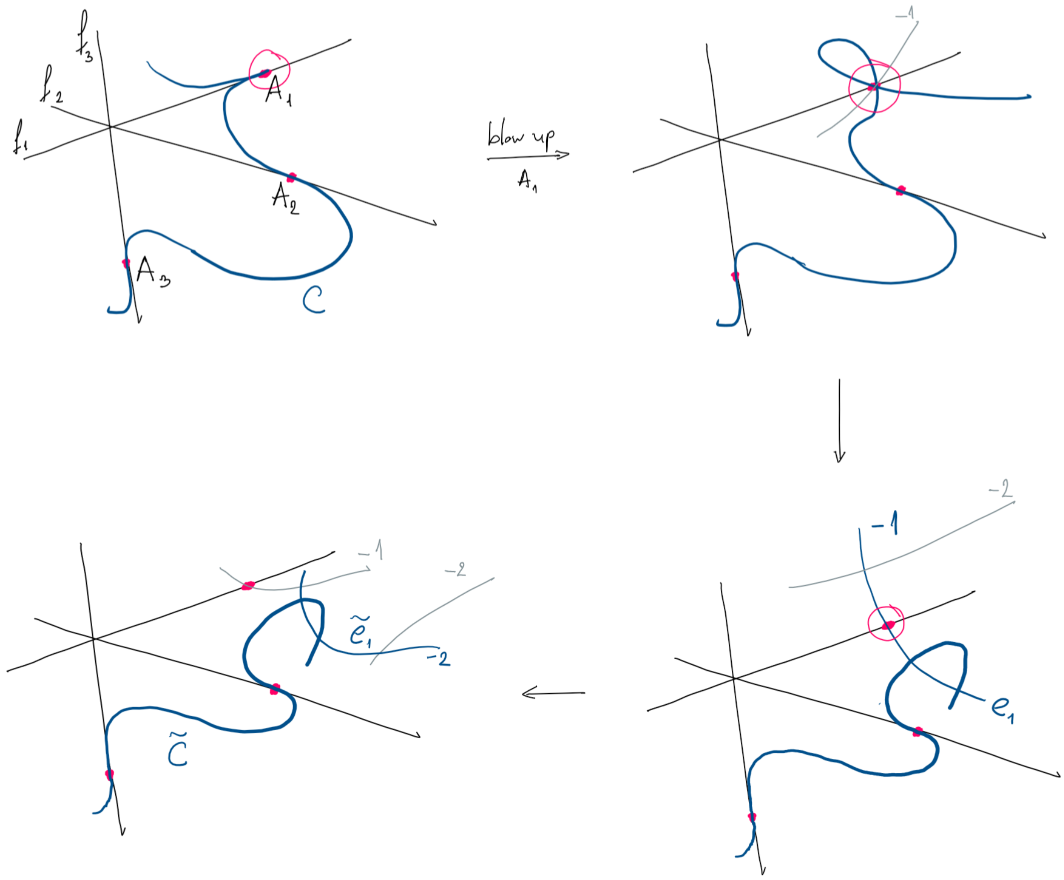


Figure 2.18

(b) In the case of the *fishtail cubic* C with its self-intersection point not one of the base points, the impossibility argument is exactly the same as in the cuspidal case - the fishtail fiber will survive the blow-up procedure and we do not want it to.

When the self-intersection fishtail point is A_1 we follow the flowchart of Figure 2.19 with 3 'infinitely close' blow-ups of A_1. As part of the resolution of the blue curve we get a cycle of three curves: $\tilde{C}$, and two -2 curves $\tilde{e_1}$ and $\tilde{e_2}$, which is again not a part of the $\tilde{E_6}$ fiber.

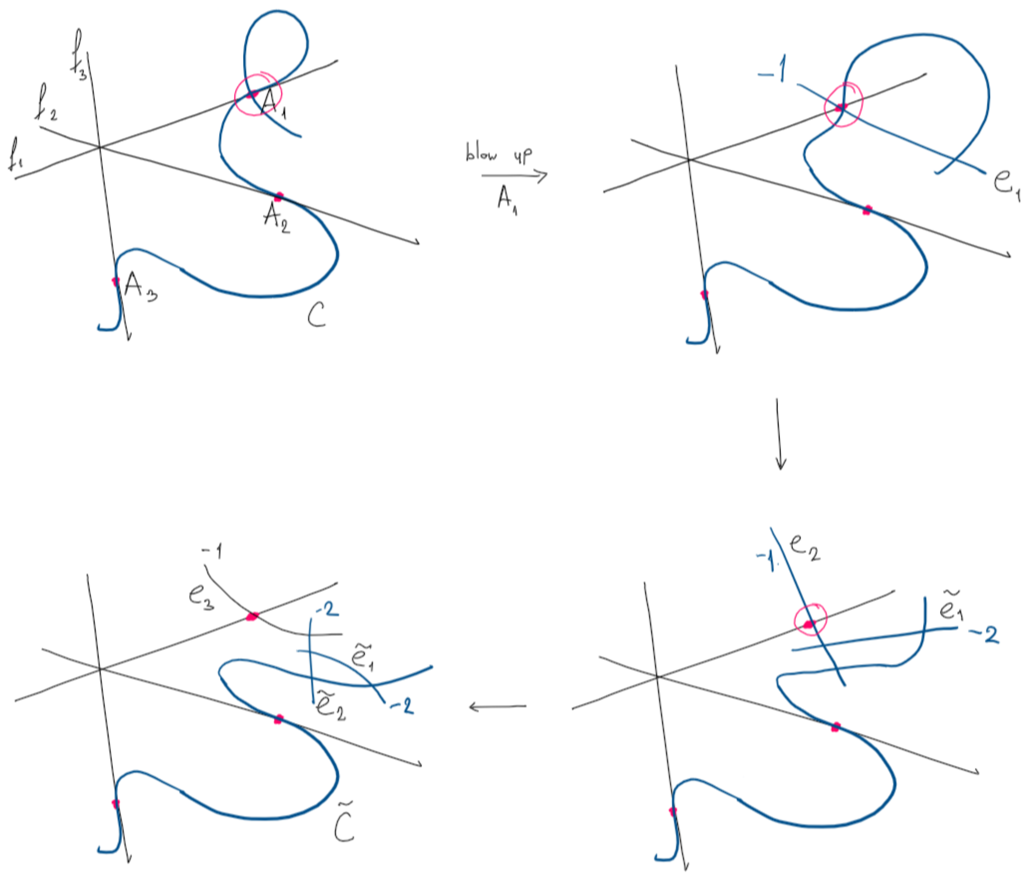


Figure 2.19

In conclusion, the discussion of this subsection and Proposition 2.17 prove the Theorem 2.8. Subsequently, the proof of Theorem 2.1 is also complete. □

Part III

Sliceness in 4-manifolds

3 Sliceness in 4-manifolds

In this final part of the thesis, we prove that all knots with unknotting number at most 21 are smoothly slice in the $K3$ surface and we also prove a more general statement for 4-manifolds that contain a plumbing of spheres. The strategy is based on a flexible method to remove double points of immersed surfaces in 4-manifolds by tubing over neighbourhoods of embedded trees.

3.1 Introduction

A knot $K \subset \mathbb{S}^3$ is called (smoothly) *slice in* X if it bounds a properly embedded smooth disc $D \subset X^\circ := X - \operatorname{Int} \mathbb{B}^4$. We prove the following theorem:

Theorem 3.1. *Every knot K in $\mathbb{S}^3$ with unknotting number $u(K) \leq 21$ is smoothly slice in $K3$.*

Let $\mathcal{S}(X)$ denote the set of knots slice in X. Depending on the 4-manifold, $\mathcal{S}(X)$ can coincide with the set of classically slice knots $\mathcal{S}(\mathbb{S}^4)$ (e.g. for $X = \mathbb{S}^4$, $\mathbb{S}^1 \times \mathbb{S}^3$, or $\mathbb{T}^4$), with the set of all knots (e.g. for $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, cf. [Nor69; Suz69]), or can differ from both of them. As far as we know, $\mathbb{CP}^2$ (together with $\overline{\mathbb{CP}^2}$) is the only known example of a simply connected 4-manifold with $\mathcal{S}(\mathbb{S}^4) \subsetneq \mathcal{S}(X) \subsetneq \{\text{knots}\}$. The case of the simplest simply connected 4-manifold which is not homeomorphic to a connected sum of $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, and $\mathbb{S}^2 \times \mathbb{S}^2$, namely the $K3$ surface, is still open.

The question whether $\mathcal{S}(K3) \neq \{\text{knots}\}$ was raised in [MMP20, Question 6.1]. Prior to our work, it was shown in [MMP20, Corollary 2.8] that every knot with unknotting number ≤ 2 is slice in $K3$. This result was later strengthened in unpublished works of Mukherjee and Stipsicz-Szabó to unknotting number ≤ 4 and ≤ 9, respectively. Our new bound in Theorem 3.1 follows from the following more general theorem and from the existence of a plumbing tree of 22 spheres smoothly embedded in $K3$.

Theorem 3.2 (The Main Theorem III)**.** *If there is a plumbing tree of n smooth (resp. locally flat) spheres $S = S_1 \cup \ldots \cup S_n$ embedded into a 4-manifold X^4, then any knot $K \subset \mathbb{S}^3$ with 4-dimensional clasp number $c_4(K) \leq n-1$ (resp. $c_4^{\mathrm{top}}(K) \leq n-1$) is smoothly (resp. topologically) slice in X^4.*

The smooth (resp. topological) 4-dimensional clasp number $c_4(K)$ (resp. $c_4^{\mathrm{top}}(K)$) appearing in the statement above is the minimum number of self-intersections of a smooth (resp. locally flat) normally immersed disc in $\mathbb{B}^4$ with boundary K. The inequalities $c_4^{\mathrm{top}}(K) \leq c_4(K) \leq u(K)$ hold for every knot K. For more details see Section 3.3.1.

The $K3$ surface is a very natural example to consider. On one hand it has a simple Morse theoretical description as it is geometrically simply connected. On the other hand it has a rich geometric structure (it is symplectic, and in fact a Kähler surface) and already displays all the exotic complications of dimension 4 having infinitely many exotic copies. Because of this, understanding $\mathcal{S}(K3)$ could hint at the more general behaviour of 4-manifolds with non-trivial Seiberg-Witten invariants.

Furthermore, understanding $\mathcal{S}(K3)$ can shed light on whether sliceness detects exotic pairs, i.e. if there exists homeomorphic 4-manifolds whose smooth types are distinguished by their sets of slice knots. This question is [MMP20, Question 6.2], and is motivated by the hope that sliceness could be used to disprove the 4-dimensional Poincaré conjecture [Fre+10].

While the question is still open, it was recently shown that *H-sliceness* (another generalisation of the notion of classical sliceness to all 4-manifolds) does indeed detect the exotic pair given by $K3\#\overline{\mathbb{CP}^2}$ and $3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$ [MMP20, Corollary 1.5]. Note that every knot is slice in $3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$. Then, an example of a knot that is not slice in $K3$ would be a great step towards showing that the above exotic pair is detected by sliceness too. On the other hand, if every knot is slice in $K3$, then there is no hope to distinguish this exotic pair by sliceness.

We mentioned earlier that our proof is based on the existence of a plumbing tree of 22 spheres smoothly embedded in the $K3$ surface [FM97, Proposition 1]. Such a plumbing tree was used by Finashin-Mikhalkin to build a (-86)-framed sphere in $K3$ [FM97, Theorem 1]. Their result was recently expanded by Stipsicz-Szabó [SS21, Theorem 1.1], who exhibited plumbing trees of spheres in all elliptic surfaces $E(n)$, and used them to produce very negative spheres in $E(n)$. $E(2) = K3$ recovers the result of Finashin-Mikhalkin, and using the plumbing trees from [SS21] we can in fact prove the following result for $E(n)$:

Corollary 3.3. *For $n \geq 2$, every knot K with 4-dimensional clasp number*

$$c_4(K) \leq 11 \cdot n - \left\lceil \frac{n}{5} \right\rceil$$

is smoothly slice in $E(n)$.

For $n = 1$, $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$, so every knot is slice in $E(1)$. For $n = 2$ we recover Theorem 3.1 (with $u(K) \leq 21$ replaced by the more general $c_4(K) \leq 21$).

Our technique to prove Theorem 3.2 (which we outline in Section 3.2 below) is quite flexible and amenable to different applications. Recall that, given a knot $K \subset \mathbb{S}^3$ and a closed 4-manifold X, the *slice genus in* X of K is defined as

$$g_X(K) = \min\left\{g(\Sigma) \mid \Sigma \overset{\text{sm}}{\hookrightarrow} X^\circ, \partial\Sigma = K\right\}.$$

It follows from a classical lemma of Norman [Nor69, Lemma 1] that for a closed 4-manifold X with a 0-framed sphere the function g_X is bounded. We re-interpret this result using our technique.

Theorem 3.4 ([Nor69])**.** *If X^4 contains a smoothly embedded 0-framed sphere S with a geometrically dual surface S^*, then for every knot K*

$$g_X(K) \leq g(S^*).$$

We remark that if X is closed, and the homology class $[S]$ is primitive (not a multiple of another class), a geometrically dual surface always exists.

Note that both $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ have a 0-framed sphere with a geometrically dual sphere. Then, as a corollary of Theorem 3.4 we recover the following classical results of Norman and Suzuki, which we already mentioned above.

Corollary 3.5 ([Nor69; Suz69])**.** *Every knot is smoothly slice in $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.*

Remark 3.6. *The careful reader will have noted that we stated Theorems 3.1 and 3.4 and Corollary 3.3 only in the smooth case. This is because by Freedman's classification theorem every closed, indefinite, simply connected 4-manifold contains a topological $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ summand, so every knot is topologically slice in it.*

3.1.1 Organisation

In Section 3.2 we give a short outline of the proof of Theorem 3.2, which is the main result of this Part III. Section 3.3 is the technical part: we introduce locally bipartitioned trees and explain how they can be used to remove double points of normally immersed surfaces. In Section 3.4 we prove Theorems 3.2 and 3.4 and their corollaries.

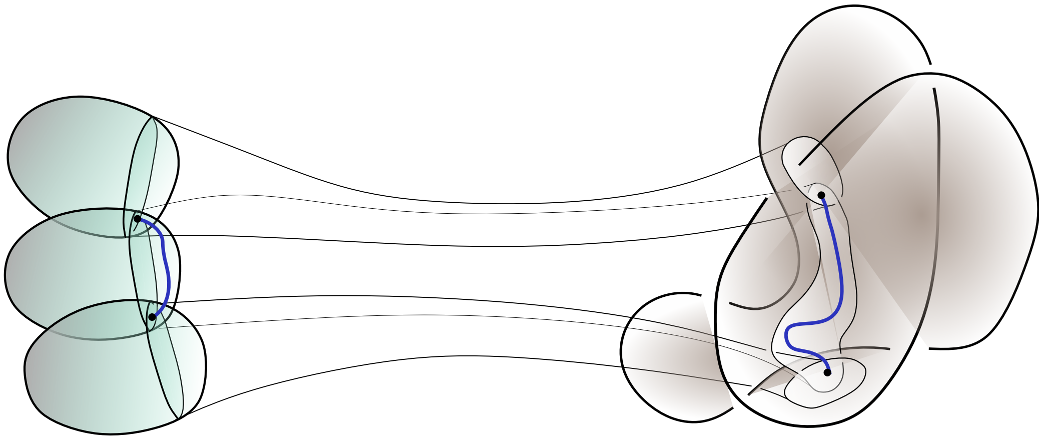


Figure 3.1: Sketch of the argument: for simplicity, the embedded tree consists of an edge connecting 2 vertices. We remove neighbourhoods of this tree from both the sphere plumbing (left) and the immersed disk (right), and get 3-chain links as new boundary components. We connect these via 3 tubes to get rid of all self-intersections.

3.1.2 Conventions

Given a link L in $\mathbb{S}^3$, $m(L)$ denotes its mirror image, and if L is oriented, $-L$ denotes the mirror image with orientation reversed on each component. We denote the number of components of L by $|L|$.

3.2 Outline of the proof of Theorem 3.2

We now outline the proof of Theorem 3.2. We remark that an independent proof of Theorem 3.1, which we now deduce from Theorem 3.2, will appear in a future version of this thesis.

The key ingredient of our proof is the existence of a plumbing tree $S = S_1 \cup \ldots \cup S_n$ of n spheres smoothly embedded in X°. Given a knot with $c_4 \leq n-1$, there is an immersed disc D with $n-1$ double points in $\mathbb{B}^4$, and hence in a collar neighbourhood of $\partial(X^\circ) = \mathbb{S}^3$. The bulk of our technical work lies in finding two 4-balls $\mathbb{B}_1$ and $\mathbb{B}_2$ in X°, one containing all the double points of D and the other all the double points of S, such that the links $D \cap \partial\mathbb{B}_1$ and $S \cap \partial\mathbb{B}_2$ are mirror of each other. Once that is done, we remove $D \cap \mathbb{B}_1$ and $S \cap \mathbb{B}_2$ from the surfaces and tube what is left of them to obtain a smoothly embedded surface (with no double points).

In our argument, the balls $\mathbb{B}_1$ and $\mathbb{B}_2$ are chosen as regular neighbourhoods of the same tree embedded in D and S respectively, so that the vertices of the embedded trees are exactly the double points. We will need to endow the trees with *local bipartitions* (cf. Definition 3.8) to keep track of the two local sheets of the surface near the double points (cf. Definition 3.12).

We show that given a plumbing tree of spheres there is always a suitable locally bipartitioned tree embedded therein (cf. Lemma 3.14), and that the same plumbing tree can be re-embedded also in the immersed disc D (cf. Lemma 3.15). To conclude, we note that a locally bipartitioned tree embedded in a normally immersed surface Σ completely determines the link $\Sigma \cap \partial\mathbb{B}$ (cf. Lemma 3.13), and therefore we can tube and eliminate all the double points.

3.3 Immersed surfaces and locally bipartitioned trees

3.3.1 Normal immersions

Recall that a (smooth or locally flat) immersion $i\colon \widetilde{\Sigma}^2 \to X^4$ is called *normal* if $i(\widetilde{\Sigma}) \cap \partial X = i(\partial\widetilde{\Sigma})$, i is transverse to ∂X, and all self-intersections of $i(\widetilde{\Sigma})$ are transverse double points in $\operatorname{Int} X$. In such a case, we call $\Sigma := i(\widetilde{\Sigma})$ a *normally immersed surface*. We denote the set of double points by $\mathcal{D}(\Sigma) \subset \Sigma$.

Following Shibuya [Shi74], we define the *4-dimensional clasp number* of a knot $K \subset \mathbb{S}^3$ as

$$c_4(K) := \min\left\{ |\mathcal{D}(\Delta)| \;\middle|\; \Delta \subset \mathbb{B}^4 \text{ normally immersed disc}, \partial\Delta = K \right\}.$$

Here we assume that the disc be *smoothly* immersed. Of course there is an analogous locally flat definition, denoted c_4^{top} in [FP22], but we focus on the smooth version c_4 (see Remark 3.6).

There are inequalities

$$g_4(K) \leq c_4(K) \leq u(K),$$

where g_4 is the (smooth) slice genus (in $\mathbb{B}^4$), and u is the unknotting number. Any non-trivial slice knot K gives an example where $c_4(K) = 0$ and $u(K) \neq 0$. As for the other inequality, there are examples K_n with $g_4(K_n) = n$ and $c_4(K_n) \geq 2n$ [FP22; JZ20], but it is unknown whether $c_4 \leq 2g_4$ for all knots. For relations of c_4 with the slicing number and the concordance unknotting number we refer the reader to [OS16].

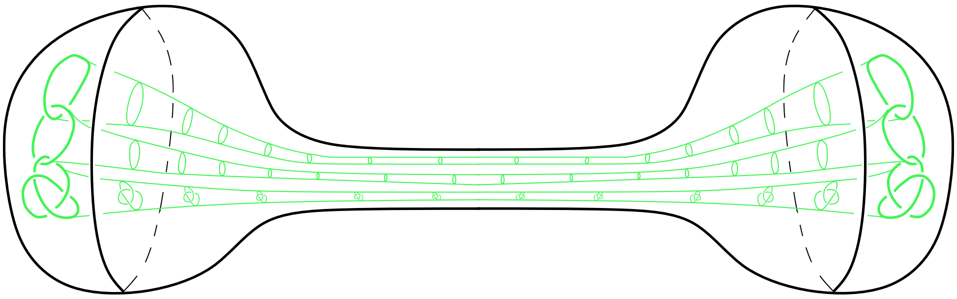


Figure 3.2: Sketch of the tubing operation.

3.3.2 Tubing self-intersections

A standard technique in 4-dimensional topology is to remove double points of normally immersed surfaces by tubing. Given two double points of an immersed surface, one can remove a small 4-ball centred at each of them: since the surface intersects the boundary 3-spheres in Hopf links, these can be tubed to create a new surface with two fewer double points.

The main idea of this chapter is to perform a tubing operation over more complex links.

Lemma 3.7. *Let $\Sigma^2 \subset X^4$ be a normally immersed surface in a connected 4-manifold, and let $\mathbb{B}_1$ and $\mathbb{B}_2 \subset X$ be two closed 4-balls with boundary $\mathbb{S}_1$ and $\mathbb{S}_2$, respectively. If the links $L_1 := \Sigma \cap \mathbb{S}_1$ and $L_2 := \Sigma \cap \mathbb{S}_2$ are mirror of each other, we can eliminate all the self-intersections of Σ in $\mathbb{B}_1 \cup \mathbb{B}_2$ and build a new surface Σ' by tubing. If Σ is oriented, and $L_1 = -L_2$ with the induced orientations, then Σ' inherits a natural orientation too.*

Proof. Pick an arc $\gamma \subset X \setminus (\mathbb{B}_1 \cup \mathbb{B}_2 \cup \Sigma)$ connecting $\mathbb{S}_1$ and $\mathbb{S}_2$. The arc can be chosen to connect the two sphere boundaries of $X \setminus (\mathbb{B}_1 \cup \mathbb{B}_2)$ and avoid Σ by a transversality argument. If $\mathcal{N}(\gamma)$ denotes a small regular neighbourhood of γ, then $\mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathcal{N}(\gamma)$ is a 4-ball $\mathbb{B}_3$, and the link $\Sigma \cap \partial\mathbb{B}_3$ is $L_1 \sqcup L_2$.

If $L_1 = m(L_2)$, then $L_1 \sqcup L_2$ bounds $|L_1|$ disjoint annuli in $\mathbb{B}_3$, which can be glued to $\Sigma \setminus (\mathbb{B}_1 \cup \mathbb{B}_2)$ to obtain Σ'. The usual tubing involving two points connects 2 Hopf links via two disjoint tubes, whereas we use multiple tubes. See Figure 3.2 for an illustration.

If Σ is oriented and $L_1 = -L_2$, then the annuli can be oriented and the result of the gluing is oriented too. □

Note that the tubing operation decreases the Euler characteristic of the surface: with the notation of Lemma 3.7, $\chi(\Sigma') = \chi(\Sigma) - 2|L_1|$.

We will apply Lemma 3.7 to pairs of 4-balls arising as regular neighbourhoods of trees embedded in a normally immersed surface. The next subsection is devoted to defining and explaining the objects we consider.

3.3.3 Locally bipartitioned trees

Given a graph Γ, we denote the set of its vertices by $V(\Gamma)$ and the set of its edges by $E(\Gamma)$. Given a vertex $v \in V(\Gamma)$, we denote the set of vertices adjacent to v by $E(v)$.

Definition 3.8. *A* locally bipartitioned tree $\left(T, \{\Pi_v\}_{v \in V(T)}\right)$ *is given by*

- *a finite tree T and,*
- *for each vertex $v \in V(T)$, a set $\Pi_v = \{A_v, B_v\}$ which gives a bipartition of $E(v)$ (i.e., $E(v) = A_v \sqcup B_v$).*

Given $v \in V(T)$ and $e \in E(v)$, we let $\pi_v(e) \in \Pi_v$ denote the element of the bipartition containing the edge e.

Remark 3.9. *A tree T together with a bicolouring $\varphi\colon E(T) \to \{0, 1\}$ of its edges naturally induces a locally bipartitioned tree, by considering for each $v \in V(T)$ the partition of $E(v)$ defined by the colour of the edge.*

Vice versa, given a locally bipartitioned tree $(T, \{\Pi_v\})$, it is always possible to find a bicolouring of T that induces the local bipartitions Π_v. (This follows by induction, using the fact that each non-empty tree has a leaf.)

In all statements and proofs we follow the locally bipartitioned perspective, but in the figures for simplicity we always represent locally bipartitioned trees by bicolourings, and we use red and blue for the two colours.

Given a locally bipartitioned tree $(T, \{\Pi_v\})$, in Definition 3.10, we will associate a link $L(T, \{\Pi_v\})$ in $\mathbb{S}^3$ with it. Its geometric meaning is the following: given a *suitable embedding* of $(T, \{\Pi_v\})$ into an immersed surface $\Sigma \subset X^4$ (as defined in Definition 3.12), $L(T, \{\Pi_v\})$ is the link defined by intersecting Σ with the boundary of a regular neighbourhood of T (see Lemma 3.13).

Definition 3.10. *Given a locally bipartitioned tree $(T, \{\Pi_v\})$, its* associated link *is the unoriented link $L(T, \{\Pi_v\})$ in $\mathbb{S}^3$ defined in two steps as follows:*

1. *for each vertex $v \in V(T)$, take a Hopf link with the two components labelled by the two elements of Π_v;*

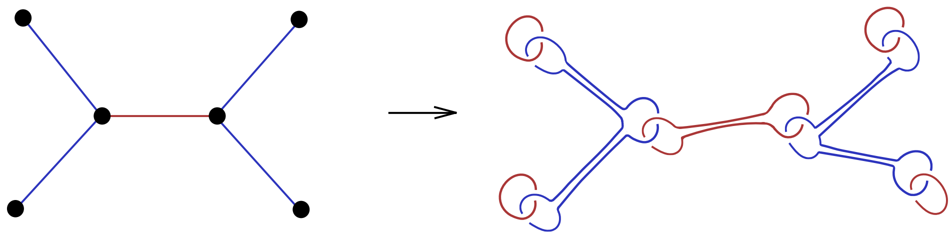


Figure 3.3: An example of a locally bipartitioned tree and its associated link in $\mathbb{S}^3$.

2. *for each edge $e \in E(T)$, connecting vertices v and w, connect sum the two Hopf links associated with v and w at the components labelled $\pi_v(e)$ and $\pi_w(e)$.*

See Figure 3.3 for an illustration of a locally bipartitioned tree and its associated link.

Remark 3.11. *Let $(T, \{\Pi_v\})$ be any locally bipartitioned tree. As an (unoriented) link, $L(T, \{\Pi_v\})$ is isotopic to its mirror image. This can be easily checked for the Hopf link, and for the general case it follows from the commutativity of the connected sum operation.*

Definition 3.12. *Let $(T, \{\Pi_v\})$ be a locally bipartitioned tree and $\Sigma^2 \subset X^4$ be a normally immersed surface, and let $\mathcal{D}(\Sigma)$ denote the set of double points of Σ. A* suitable embedding *of $(T, \{\Pi_v\})$ into Σ is an embedding*

$$f : T \to \Sigma$$

such that

- *$f^{-1}(\mathcal{D}(\Sigma)) = V(T)$, and*
- *for each vertex $v \in V(T)$, any two edges in $E(v)$ from the same element of the bipartition Π_v map into the same local component of Σ, whereas any two edges from the two different elements of the bipartition Π_v map into two different local components of Σ.*

See Figure 3.4 for an illustration of a suitable embedding.

Lemma 3.13. *Let $\Sigma^2 \subset X^4$ be a normally immersed surface, and let f be a suitable embedding of a locally bipartitioned tree $(T, \{\Pi_v\})$ into Σ. Then the link of the embedding, i.e., the intersection of Σ with $\partial\mathcal{N}(f(T)) \cong \mathbb{S}^3$ is exactly the associated link $L(T, \{\Pi_v\})$.*

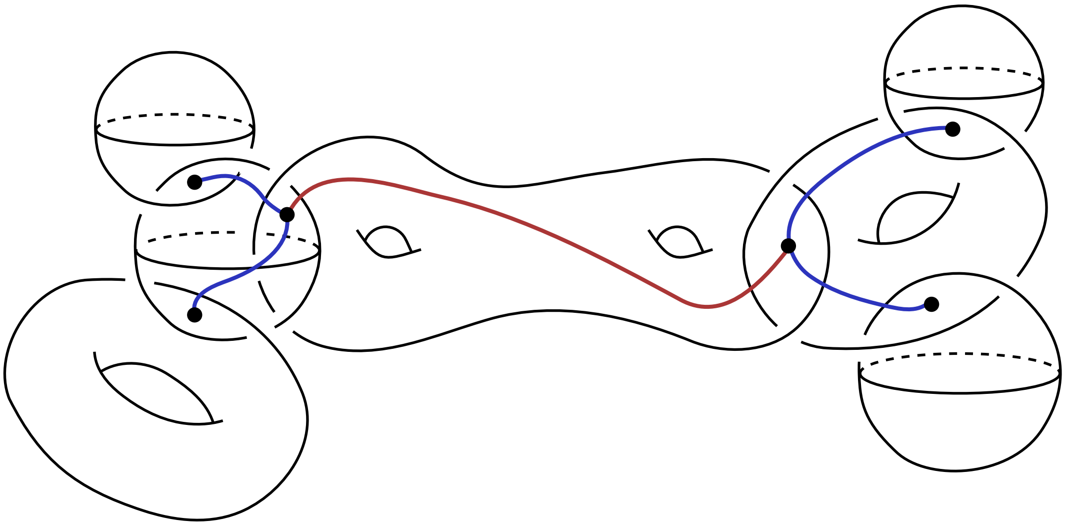


Figure 3.4: Sketch of a surface plumbing and a suitable embedding of the locally bipartitioned tree from Figure 3.3.

Proof. For simpler notation, in the proof of this lemma we will identify the tree T with the image of the embedding f, so embedded vertices $f(v)$ will be denoted by just v, and embedded arcs $f(e)$ will be denoted by e.

A small neighbourhood of a tree is a 4-ball $\mathbb{B}_0$, and we only need to identify the link we get by intersecting Σ with the boundary 3-sphere $\partial\mathbb{B}_0$. We visualise the whole 4-ball neighbourhood $\mathbb{B}_0$ of the tree T as the union of 4-balls $\mathbb{B}_v$ centred at each vertex v, and 4-dimensional solid tubes joining two 4-balls $\mathbb{B}_v$, one for each edge.

Near the vertex v the surface Σ locally looks like two planes intersecting transversely, hence $\Sigma \cap \partial\mathbb{B}_v$ is a Hopf link, with its two components lying on the two different local sheets of Σ.

Now consider two vertices v_1 and v_2 that are connected by an edge $e \subset \Sigma$. The first bullet point in Definition 3.12 ensures that e does not intersect any double points except v_1 and v_2. The intersection of Σ with the boundary of a small neighbourhood $\mathcal{N}(e)$ of e in $X \setminus (\mathbb{B}_{v_1} \cup \mathbb{B}_{v_2})$ consists of two arcs parallel to the edge, and two smaller arcs, one in each $\partial\mathbb{B}_{v_i}$, shared with the Hopf component at v_i with label $\pi_{v_i}(e)$. Thus, removing $\mathcal{N}(e)$ results into connect summing the two Hopf link along the components labelled $\pi_{v_1}(e)$ and $\pi_{v_2}(e)$. $\square$

3.3.4 From sphere plumbings to locally bipartitioned trees

In the next lemma we show how to find a large locally bipartitioned tree suitably embedded into a plumbing tree of surfaces.

Lemma 3.14. *Let $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n$ be a plumbing tree of n closed surfaces in a 4-manifold X^4. Then there exists a locally bipartitioned tree $(T, \{\Pi_v\})$ with exactly $n-1$ vertices and a suitable embedding $f\colon T \to \Sigma$.*

Moreover, given any such $(T, \{\Pi_v\})$ and f:

- *$f(T)$ contains all the $n-1$ double points of Σ;*
- *for each $i = 1, \ldots, n$, $\Sigma_i \cap f(T)$ is contractible.*

Proof. Leaving the local bipartitions aside for a moment, the fact that there is a tree T with $n-1$ vertices and an embedding f such that $f^{-1}(\mathcal{D}(\Sigma)) = V(T)$ follows by a simple induction argument on n, using the fact that any plumbing tree has a leaf.

Then, once we have such a tree and such an embedding, we choose as local bipartitions Π_v exactly the ones induced by the local sheets of Σ, so that the embedding f is automatically suitable.

Now suppose that we are given a locally bipartitioned tree $(T, \{\Pi_v\})$ with exactly $n-1$ vertices and a suitable embedding $f\colon T \to \Sigma$. Then the first bullet point follows from the fact that T has exactly $n-1$ vertices, Σ has exactly $n-1$ double points, and f is one-to-one. Finally, the second bullet point follows from the fact that Σ is a plumbing *tree*, so if two double points on Σ_i are connected by a path in $\Sigma \cap f(T)$, that path intersects Σ_i in a connected subset of $f(T)$, which is itself contractible. □

3.3.5 Locally bipartitioned trees in immersed connected surfaces

With a variation of the previous argument, we can show the following result.

Lemma 3.15. *Let $i\colon \widetilde{\Sigma} \to X^4$ be a normal immersion of a connected surface with m self-intersections, and let $\Sigma = i(\widetilde{\Sigma})$. Then any locally bipartitioned tree $(T, \{\Pi_v\})$ with $\ell \leq m$ vertices can be suitably embedded in Σ.*

Proof. The proof is by induction on ℓ. The base cases $\ell = 1$ and $\ell = 2$ are straightforward.

For $\ell > 2$ start by choosing a leaf v of T, and let e be the edge connecting it to some vertex w. We define T' by setting $V(T') = V(T) - \{v\}$, and $E(T') = E(T) - \{e\}$. Then

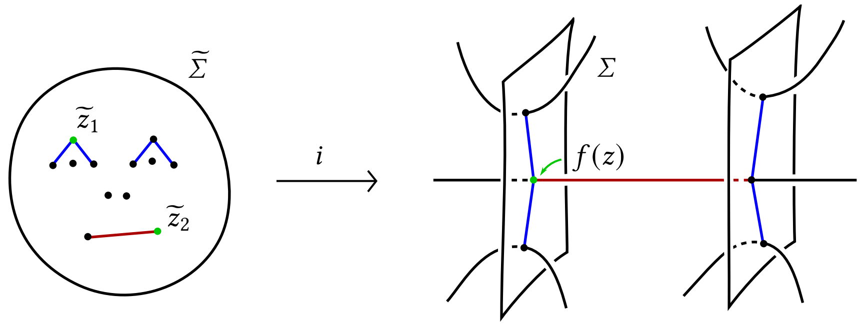


Figure 3.5: The figure shows a normal immersion i of a connected surface $\widetilde{\Sigma}$. The image Σ is represented by the black lines and planes on the right. The tree T from Figure 3.3 is suitably embedded in Σ. On the left we drew the pre-image of T through i. The pre-image of a vertex $f(z)$ is a two-point set $\{\widetilde{z_1}, \widetilde{z_2}\} \subset \widetilde{\Sigma}$, which we identify with $\Pi_z = \{A_z, B_z\}$.

$(T', \{\Pi_v\})$ satisfies the induction hypothesis so we can find a suitable embedding $f' \colon T' \to \Sigma$; note that the number of self-intersections of Σ in the image of f' is strictly less than m. We now wish to define $f \colon T \to \Sigma$ by setting $f|_{T'} = f'$ and so that the remaining vertex v maps to a self-intersection of Σ not in the image of f'. The only subtle point is how to define $f(e)$ so that the bipartition of $E(w)$ coming from the local sheets of Σ is exactly Π_w.

If we let $i \colon \widetilde{\Sigma} \to X^4$ be the normal immersion, then for any vertex $z \in V(T')$ we can identify its two preimages $\{\widetilde{z_1}, \widetilde{z_2}\} = i^{-1}(f'(z)) \subset \widetilde{\Sigma}$ with the partition $\Pi_z = \{A_z, B_z\}$, so that the lift $i^{-1}(f'(e'))$ of an edge $e' \in E(z)$ has $\pi_z(e')$ as one of its endpoints. (The careful reader will note that for this point we need $\ell > 2$.) See Figure 3.5 for an illustration.

From here one can see that the preimage of the embedded tree $i^{-1}(f'(T'))$ consists of a disjoint union of contractible components embedded in $\operatorname{Int} \widetilde{\Sigma}$. (In fact, upon choosing a compatible bicolouring of T', every maximal monochromatic subtree of T' will give rise to a connected tree embedded in $\widetilde{\Sigma}$. See Figure 3.5.)

Choose an arc γ connecting the preimage $\pi_w(e) \in i^{-1}(f'(w))$ to any of the preimages of v. By the previous paragraph, we can choose it so that its interior avoids $i^{-1}(f'(T'))$, and of course also the preimages of the double points. Thus, we can define an embedding f by setting $f(e) = i(\gamma)$. The choice of the starting point of γ (namely $\pi_w(e)$) was made so that the bipartition of $E(w)$ coming from the local sheets of Σ is exactly Π_w, and therefore f defines a suitable embedding of $(T, \{\Pi_v\})$ into Σ. □

3.4 Applications

Theorem 3.2. *If there is a plumbing tree of n smooth (resp. locally flat) spheres $S = S_1 \cup \ldots \cup S_n$ embedded into a 4-manifold X^4, then any knot $K \subset \mathbb{S}^3$ with 4-dimensional clasp number $c_4(K) \leq n-1$ (resp. $c_4^{\text{top}}(K) \leq n-1$) is smoothly (resp. topologically) slice in X^4.*

Proof. The knot K bounds a normally immersed disk D with exactly $n-1$ self-intersections in $\mathbb{S}^3 \times [0,1]$ (if necessary, add extra self-intersections), and thus also in a collar neighbourhood of $\partial(X \setminus \mathbb{B}^4)$. We can assume that the disk $D \subset X \setminus \mathbb{B}^4$ is disjoint from the plumbing tree S.

We use Lemma 3.14 to find a suitable embedding f_S of a locally bipartitioned tree $(T, \{\Pi_v\})$ with $n-1$ vertices into S. By Lemma 3.15, the same tree can be suitably embedded into the disc D.

By Lemma 3.13, the links we get on the boundaries of small neighbourhoods of these two embeddings of $(T, \{\Pi_v\})$ are the same. Finally, as these links are isotopic to their mirror images, we can apply Lemma 3.7 and tube the self-intersections of D with the plumbing intersections, thus removing all of them. Note that the second bullet point of Lemma 3.14 implies that after removing a small ball about $f_S(T)$ what is left of each sphere S_i is a disc; therefore, we do not add any genus to the immersed disk D with the tubing procedure. This way, after tubing, we get a properly embedded disk $D' \subset X \setminus \mathbb{B}^4$ with boundary K. □

We now restate and prove Corollary 3.3 from the introduction. Theorem 3.1 from the introduction follows immediately by recalling that $E(2) = K3$ and $c_4 \leq u$.

Corollary 3.16. *For $n \geq 2$, every knot K with 4-dimensional clasp number*

$$c_4(K) \leq 11 \cdot n - \left\lceil \frac{n}{5} \right\rceil$$

is (smoothly) slice in $E(n)$.

Proof. By [SS21, Proof of Theorem 1.1], $E(n)$ contains a plumbing tree of $11 \cdot n - \left\lceil \frac{n}{5} \right\rceil + 1$ smooth spheres. Thus, we can apply Theorem 3.2 to conclude. □

The previous result was an application of Theorem 3.2 to a family of symplectic 4-manifolds, which always have non-vanishing Seiberg-Witten invariants. By contrast, the next result is for 4-manifolds with a 0-framed sphere, whose Seiberg-Witten invariant is always vanishing. This result follows from [Nor69, Lemma 1], but we give here a proof using the techniques that we just developed.

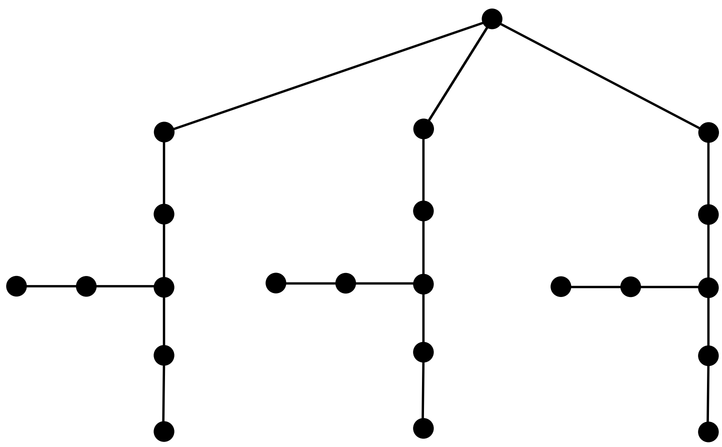


Figure 3.6: A plumbing tree of 22 spheres in $E(2) = K3$. This is the example in [FM97, Figure 3], which can be realised by taking three $\widetilde{E}_6$ fibers and a section.

Recall that every closed, connected, embedded surface Σ in a closed 4-manifold X admits a geometrically dual surface Σ^* if its homology class is primitive. To find a dual surface, first take an algebraically dual immersed surface (which exists by Poincaré duality), and resolve its intersections to make it embedded. As the algebraic intersection between this embedded dual surface and Σ is $+1$, pair the positive and negative intersections except for one. Finally, add tubes for each pair to cancel intersections with Σ until there is only one left.

Theorem 3.4. *If X^4 contains a smoothly embedded 0-framed sphere S with a geometrically dual surface S^*, then for every knot K*

$$g_X(K) \leq g(S^*).$$

Proof. The proof is very similar to the proof of Theorem 3.2, but in this case we exhibit an arbitrarily large plumbing of surfaces, one of which does not have to be a sphere.

Take any knot $K \subset \mathbb{S}^3 = \partial(X^4 \setminus \mathbb{B}^4)$, with unknotting number $n \in \mathbb{N}$. We use the fact that the sphere S is 0-framed so that we are able to take parallel push-offs S_i where $S = S_0$. Define a plumbing tree of $n+1$ surfaces by taking n parallel copies S_i, and one copy of the dual surface S^*. By Lemma 3.14, we can find a suitable embedding of a locally bipartitioned tree $(T, \{\Pi_v\})$ with n vertices into the plumbing. (In this case there is a simple one, namely a uniformly coloured linear tree T with n vertices and $n-1$ edges obtained by choosing an arc in the dual surface S^* that intersects all the n parallel copies of S.)

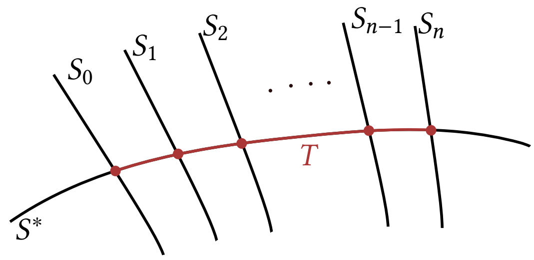


Figure 3.7: The figure shows the push-offs S_i of and dual S^*

By Lemma 3.15, we can suitably re-embed $(T, \{\Pi_v\})$ into an immersed disk D with n self-intersections contained in a collar neighbourhood of $\partial(X \setminus \mathbb{B}^4)$.

Finally, using Lemma 3.7, we tube the self-intersections of D with the plumbing intersections, removing all of them. The only difference with the proof of Theorem 3.2 is that in this case the tubing operation does add genus, because we tube with the surface S^*. Since each surface in the plumbing tree is attached via a single tube to D, we can orient each of them so that the result will be oriented. Thus, we have constructed a properly embedded surface in $X \setminus \mathbb{B}^4$ with genus $g(S^*)$ and boundary K. □

4 Conclusions & Outlook

The project in Part II deals with the case of $\tilde{E}_6$ fiber but we can get 4-dimensional moduli spaces in some other irregular cases as well. In the future we will be examining the appearance of $\tilde{E}_7$ fiber when bundles are of rank 4 with structure group $\mathbb{GL}(4, \mathbb{C})$, and the $\tilde{E}_8$ fiber when bundles are of rank 6 with structure group $\mathbb{GL}(6, \mathbb{C})$. In these cases some conditions are known, [Osh08] provides a combinatorial approach where Proposition 8.1 gives conditions on how the defining curve of the pencil should interact with the fibers contained in the other defining curve. However, for now it is unclear if there are curves in $\mathbb{F}_1$ of such nature, if they appear in the Higgs bundle perspective, and let alone what are all the parameters that lead to those special fibers.

Regarding project in Part III, I believe the technique of removing multiple singularities itself is pretty interesting regardless of the sliceness in 4-manifolds context. In ongoing work, I use an upgrade of our technique to give improvements on something called *the minimal genus function* of a 4-manifold. For a given class $a \in H_2(X, \mathbb{Z})$, there is always an embedded surface which represents it. The minimal genus function assigns to each class a the minimal genus among all smoothly embedded surfaces that represent the homology class a:

$$g_X(a) = \min\Big\{ g(\Sigma) \mid \Sigma \overset{\text{sm}}{\hookrightarrow} X, [\Sigma] = a \Big\}$$

Determining this function is deeply related to exotic structures and it is in general very hard and known only in very special cases - for instance, even for $\mathbb{CP}^2$, it was known as the *Thom conjecture*, the proof needed sophisticated gauge theory and many years to appear [KM94].

I am trying to understand where the technique applies best by studying various 4-manifolds, but to demonstrate the usefulness my claim is that the bounds in case of $\mathbb{CP}^2 \# \mathbb{CP}^2$ are exactly the same as found independently by authors in [Mar+22].

Long term, my dream is to combine perspectives from all three parts of the thesis, and more, and construct an exotic definite 4-manifold.

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