Partial Orders on Positive Hilbert Space Operators and Related Preserver Problems

by

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The author hereby declares that the dissertation contains no materials accepted for any other degrees in any other institution and that the dissertation contains no materials previously written or published by another person except where appropriate acknowledgment is made in the form of bibliographic reference.

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Chapter 1

Introduction

1.1 Preserver problems

The general theme of this dissertation revolves around *Preserver Problems*. In loose terms, preserver problems concern the study of transformations (which are called *preservers*) of a structure which leave invariant a given mathematical object. These kind of transformations are found everywhere in mathematics. Just to mention some basic examples: homomorphisms on algebraic structures which are preservers of the operations; isometries on metric spaces, which are distance preservers; monotone maps, which are order preservers. A comprehensive account of the prevailing preserver problems can be found in Molnár's book: *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces* [18].

In general, the goal of a preserver problem is to give a complete, concrete, and explicit form of the preservers in question. Let us illustrate this with the following theorem, due to Frobenius. It concerns the transformations on the matrices which preserve the determinant function. It is generally regarded as the first result about preserver problems.

1.1.1 Theorem (Frobenius, 1897). *Let $n \geq 2$ be a positive integer and denote by $\mathbb{M}_n$ the vector space of all complex $n \times n$ matrices. If a bijective linear map $\phi : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ preserves the determinant, i.e., it satisfies* $\det \phi(A) = \det A$, *for all $A \in \mathbb{M}_n$, then ϕ is of the form*

$$\phi(A) = MAN, \quad A \in \mathbb{M}_n;$$

or

$$\phi(A) = MA^TN, \quad A \in \mathbb{M}_n;$$

where M, N are invertible matrices with $\det(MN) = 1$ and A^T denotes the transpose of the matrix A.

Here the goal is to give an explicit description of the determinant preservers, more precisely, the linear determinant preservers. We have known examples of linear preservers of the determinant function: the transposition operation and multiplication by matrices with determinant 1. The essence of the theorem is that all linear determinant preservers are obtained from these two transformations.

Despite the presence of preserver problems in many areas of mathematics, systematic studies of those problems only appear in linear algebra and functional analysis: namely in matrix theory and operator theory. This is not surprising given that the spaces of matrices and operators in general carry very rich mathematical structures. On the one hand, they are algebraic structures which serve as interesting examples for non commutative algebra. On the other hand, they are geometrical structures too, they can carry several different metrics. These spaces also carry important order structures. All these make the spaces of matrices and operators a very suitable and interesting ground for investigating preserver problems.

Besides their predominance in mathematics, there are also preserver problems that originated from physics. Operator theory serves as a mathematical language in which quantum theory were formulated. In the von Neumann formalism of quantum mechanics, any given physical system is associated with a Hilbert space H and others physical objects are represented by operators (or subset of operators) on H. The concept of symmetry is very important in physics, a symmetry is a transformation that leaves invariant some property of the physical system. This leads to interesting preserver problems on different subsets of the algebra of operators on the Hilbert space H of the system. To illustrate this, we present the celebrated Wigner's theorem.

In a physical system whose associated Hilbert space is H, the (pure) states of the system are represented by rank one projections, and the trace of the product of two rank one projections P, Q, that we denote by $\mathrm{Tr}(PQ)$, is what is called the transition probability. This is a very important quantity in quantum mechanics and a bijective map ϕ on the set of rank one projections which preserves this quantity is called a quantum mechanical symmetry transformation. Wigner's theorem describes the form of all such maps.

1.1.2 Theorem (Wigner's theorem)**.** *Let H be a Hilbert space, and $P_1(H)$ be the set of rank one projections on H. Let $\phi : P_1(H) \longrightarrow P_1(H)$ be a bijective map such that*

$$\mathrm{Tr}(\phi(P)\phi(Q)) = \mathrm{Tr}(PQ), \quad P, Q \in P_1(H),$$

where Tr *is the trace function. Then there exists a unitary or anti-unitary operator* $U : H \longrightarrow H$ *such that*

$$\phi(P) = UPU^*, \quad P \in P_1(H).$$

It is easy to see that the maps of the form $P \longmapsto UPU^*$, where U is unitary or anti-unitary operator, preserve the transition probability, so this theorem is actually an "if and only if" statement. Looking at the end result, this theorem can also be construed as characterisation of the unitary/anti-unitary transformation on the underlying Hilbert space.These maps are of highest importance in preserver problems. In fact these maps are involved in the solutions to several preserver problems (see for example Chapter 2 and Section 3 in the Introduction of [18]).

Whether it is considered in a purely mathematical context, or as a representation of a physical observable (as in quantum mechanics), operators on Hilbert spaces are one of the richest and most interesting objects to study. The set of selfadjoint operators and its subsets come with several functions or operations or relations that may or may not have some physical significance, but even without direct physical meaning, it is an important task in mathematics to study these objects and describe the transformations that leave them invariant. In this thesis, we present our contributions on the study of selfadjoint operators, and more precisely positive operators. Our results are presented in Chapter 2 and 3. The results contribute mainly to the study of two important order relations on the set of positive operators and operator means which are binary operations closely related to the orders.

1.2 Orders on positive operators and their preservers

We are interested in two partial orders on selfadjoint operators, that we present now. We begin with the necessary definitions and notation. In what follows, H is a complex Hilbert space, the inner product on H is denoted by $\langle ., . \rangle$ and the corresponding norm is denoted by $\|.\|$. The unit sphere in H with respect to the Hilbert space norm is denoted by S_H, the kernel and the range of an operator A is denoted by $\ker A$ and $\operatorname{rng} A$ respectively. The symbol $B(H)$ stands for the algebra of all bounded linear operators on H, we also use the following notation for subsets of $B(H)$:

- $B_{sa}(H)$ stands for the space of all selfadjoint elements of $B(H)$;
- $B(H)^+$ stands for the set of selfadjoint operators whose spectra are contained in $[0, +\infty[$;

- $B(H)^{++}$ is the set of invertible elements in $B(H)^+$;
- $\mathcal{E}(H)$ is the effect algebra, it is the subset of $B_{sa}(H)$ which consists of the selfadjoint operators whose spectrum are contained in the positive real unit interval $[0, 1]$;
- the symbol $P(H)$ stands for the set of projection operators on H and $P_1(H)$ stands for the set of all rank-one projections. We denote by P_x the rank one projection on the one dimensional subspace generated by $x \in S_H$, we may also use the notation $x \otimes x$. There is a natural order on $P(H)$ which is induced by the inclusion of the ranges of the projections: $P \leq Q$ when $\operatorname{rng} P \subset \operatorname{rng} Q$.

Let A be a selfadjoint operator on H. Then A is uniquely defined by its spectral measure E_A. The correspondence between A and its spectral measure is given by the following equation

$$\langle Ax, x\rangle = \int_{\mathbb{R}} td\langle E_A(]-\infty, t])x, x\rangle, \quad x \in H.$$

The map $t \in \mathbb{R} \longmapsto E_A(]-\infty, t])$ is known to have the following properties:

- it is monotone increasing with respect to the natural order on the projection operators,
- it is strongly right-continuous,
- for small enough real numbers it is 0 and for large enough real numbers it is the identity operator I.

A map $t \in \mathbb{R} \longrightarrow E_t \in P(H)$ from the set of real numbers into the set of projection operators on H with these properties is called a *resolution of the identity* and it is a folklore result in operator theory that there is a bijective correspondence between selfadjoint operators and resolutions of identity. The map $t \in \mathbb{R} \longmapsto E_A(]-\infty, t])$ is referred to as the resolution of the identity of A (or the spectral resolution of A) and we use the notation $E_t^A = E_A(]-\infty, t])$.

A selfadjoint operator can be thought of as a non commutative random variable, where E_A is the corresponding non commutative probability distribution (measure), the spectral resolution E_t^A is the corresponding non commutative distribution function, and $\langle Ax, x\rangle$ is the expectation value of A in the state represented by the vector $x \in S_H$. Therefore selfadjoint operators can be compared by means of the expectation values or the distribution functions. These yield the partial orders on the set of selfadjoint operators in which we are interested, respectively: the Löwner order or usual order, and the Olson order which is commonly called the spectral order [26]. Precisely, these two orders are defined as follows:

1. A is less than B in the usual order, $A \leq B$, when $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$;

2. A is less than B in the spectral order, $A \preccurlyeq B$, when $E_t^B \leq E_t^A$ for all $t \in \mathbb{R}$.

We first note that the usual order coincide with the range inclusion order on the set of projection operators, so the notation here is consistent. Secondly, an operator $A \in B(H)$ belongs to $B(H)^+$ if and only if $\langle Ax, x\rangle \geq 0$ holds for all $x \in H$, in this case we say that A is positive or positive semidefinite. If A is also invertible, this means that A is positive invertible i.e. $A \in B(H)^{++}$, then we say that A is positive definite.

The set $B(H)^+$ of positive operators is in fact a convex cone in $B(H)$ and the order induce by $B(H)^+$ is the usual order. On the other hand, the spectral order is not a linear order in the sense that it does not respect the translations. There are several other substantial differences among the properties of those two partial orders. For example, by a famous observation due to Kadison which was made in Theorem 6 in [12], the supremum or infimum of two self-adjoint operators with respect to that order exists only in the most trivial case, if the two operators are in fact comparable. This means that the Lwner order makes the set of all self-adjoint operators an anti-lattice. In certain respects, this is a quite inconvenient property. Olson [26] on the other hand proved that the spectral order makes the set of selfadjoint operators a conditionally complete lattice. An other important and big difference is the following. Let us call a monotone increasing real function f operator monotone with respect to some given partial order $\mathcal{R}$ on $B_{sa}(H)$ if, for any $A, B \in B_{sa}(H)$,

$$A\mathcal{R}B \implies f(A)\mathcal{R}f(B).$$

The operator monotone functions with respect to the Lwner order $\leq$ are very special, they have a well-known and deep theory essentially due to Lwner. On the other hand, it is easy to see that every monotone increasing function is operator monotone with respect to the spectral order $\preccurlyeq$. In fact, one can easily prove that the spectral order is the finest partial order among the ones coarser than the Lwner order with respect to which all monotone increasing real functions are operator monotone.

Now let us come back to the topic of preservers. Preservers of these two orders on several subsets of the set of selfadjoint operators have been investigated. We define an order isomorphism between two subsets $S_1, S_2 \subset B_{sa}(H)$ as a bijective map which preserves the order in both direction, i.e., $\phi\colon S_1 \longrightarrow S_2$ is such that

$$A \le B \iff \phi(A) \le \phi(B), \quad A, B \in S_1.$$

In a natural way, we define an order anti-isomorphism as a bijective map which reverses the order in both direction.

The structure of the order isomorphisms of the usual order on $B(H)^+$ and $B_{sa}(H)$ were first studied and described by Molnár [16]. Šemrl [30] also determined the the order isomorphisms of $\mathcal{E}(H)$ and $B(H)^{++}$, and more general operator intervals. An operator interval is a set

$$[A, B] := \{X \in B_{sa}(H);\ A \le X \le B\},$$

$$[A, B) := \{X \in B_{sa}(H);\ A \le X \le B, B - X \in B(H)^{++}\},$$

where $A, B \in B_{sa}(H)$ such that $B - A \in B(H)^{++}$. Similarly, an operator interval open from the right and open from both sides are defined, and the operator intervals which are unbounded are naturally defined. In this interval notation we have

$$\mathcal{E}(H) = [0, I]; \quad] - \infty, +\infty[:= B_{sa}(H); \quad [0, +\infty[:= B(H)^+; \quad (0, +\infty[:= B(H)^{++}.$$

None of the these four operator intervals is isomorphic or anti-isomorphic to another, notice that the non existence of order isomorphism between $B_{sa}(H)$ and $B(H)^{++}$ is non trivial, it was proved in [21]. It is also easy to see that any operator interval is isomorphic or anti-isomorphic to one of the above four operator intervals. We gather the results describing the order isomorphisms of the four basic operator intervals in the following theorem, the proofs can be found in Molnár's papers [16, 17, 21] and Šerml's paper [30].

1.2.1 Theorem. *Let H be a complex Hilbert space with $\dim H \ge 2$. We have the following.*

(1) If $\phi\colon B_{sa}(H) \longrightarrow B_{sa}(H)$ is an order isomorphism, then there exist a linear or conjugate-linear bounded invertible operator $T\colon H \longrightarrow H$ and $T_0 \in B_{sa}(H)$ such that

$$\phi(A) = TAT^* + T_0, \quad A \in B_{sa}(H).$$

(2) If $\phi\colon B(H)^+ \longrightarrow B(H)^+$ is an order isomorphism, then there exist a linear or conjugate-linear bounded invertible operator $T\colon H \longrightarrow H$ such that

$$\phi(A) = TAT^*, \quad A \in B(H)^+.$$

(3) If $\phi\colon B(H)^{++} \longrightarrow B(H)^{++}$ is an order isomorphism, then there exist a linear or conjugate-linear bounded invertible operator $T\colon H \longrightarrow H$ such that

$$\phi(A) = TAT^*, \quad A \in B(H)^{++}.$$

(4) If $\phi\colon \mathcal{E}(H) \longrightarrow \mathcal{E}(H)$ is an order isomorphism, then there exist a linear or conjugate-linear bounded invertible operator $T\colon H \longrightarrow H$, with norm not exceeding 1, and two real numbers $p \in (0,1); q \in]-\infty, 1)$ such that

$$\phi(A) = f_q\left(f_p(TT^*)^{-1/2} f_p(TAT^*) f_p(TT^*)^{-1/2}\right), \quad A \in \mathcal{E}(H),$$

where the function f_t is defined on $[0,1]$ by $f_t(x) = x(tx + (1-t))^{-1}$.

We see that the form of the order isomorphisms of the effect algebra is rather complicated compared to that of the other four intervals, but with the additional constraint that $\phi\left(\frac{1}{2}I\right) = \frac{1}{2}I$ then ϕ has the form $\phi(A) = TAT^*, A \in \mathcal{E}(H)$ (see for example Corollary 4 in [17]).

We point out that a description of the order isomorphisms of the set of selfadjoint operators in a C*-algebra was done by Kadison in [13], with the additional hypothesis of linearity and preservation of the identity operator. The impressive feature of Molnár's result about the order isomorphisms of $B_{sa}(H)$ is the removal of the linearity assumption. Molnár and Šemrl's result presented in the above theorem have been generalised to the more general setting of von Neumann algebra by Mori [25].

The isomorphisms of the spectral order were studied by Molnár and Šemrl in [24] and Bohata [4]. Their description involves a family of highly non-trivial maps that we explain now. A map $S : H \to H$ is said to be semilinear if it is additive and there exists a field isomorphism ρ of $\mathbb{C}$ such that $S(\zeta h) = \rho(\zeta) S(h)$ for $\zeta \in \mathbb{C}$ and $h \in H$. A linear map is obviously semilinear, so is a conjugate linear map. Now let $S : h \longrightarrow H$ be a bijective bounded linear or conjugate linear operator if H is infinite dimensional, or a bijective semilinear operator if H is finite dimensional. For any $A \in B_{sa}(H)$ with spectral measure E_A, the map

$$t \longmapsto I - P_{S(\operatorname{rng} E_A(]t,\infty[))}, \quad t \in \mathbb{R}$$

is a resolution of the identity. Denote the corresponding spectral measure by E_A^S. Define

$$\psi_S(A) = \int_{-\infty}^{+\infty} t \, dE_A^S(]-\infty, t]), \quad A \in B_{sa}(H).$$

It was proved in Proposition 1 in [24] that $\psi_S : B_{sa}(H) \to B_{sa}(H)$ is a spectral order isomorphism. Since ψ_S fixes 0 and I, $\psi_S : \mathcal{E}(H) \to \mathcal{E}(H)$ and $\psi_S : B(H)^+ \to B(H)^+$ are also spectral order isomorphisms. Notice also that if $f : [0,1] \longrightarrow [0,1]$ is bijective increasing then the map $A \longmapsto f(A)$ is a spectral order isomorphisms of $\mathcal{E}(H)$. These two maps make up all the spectral order isomorphisms of $\mathcal{E}(H)$, and the description of the spectral order isomorphisms of $B(H)^+$ and $B_{sa}(H)$ was deduced from that, as it is shown in the following theorem.

1.2.2 Theorem (Molnár-Šemrl[24], Bohata [4])**.** *Let H be a complex Hilbert space with $\dim H \geq 3$. We have the following.*

(1) If $\phi\colon B_{sa}(H) \longrightarrow B_{sa}(H)$ is a spectral order isomorphism, then there exists a bijective increasing function $f : \mathbb{R} \longrightarrow \mathbb{R}$, and a bijective operator $S : H \longrightarrow H$, which is semilinear in the case where $3 \leq \dim H < \infty$ and bounded linear or conjugate linear in the case where H is infinite dimensional, such that

$$\phi(A) = \psi_S(f(A)), \quad A \in B_{sa}(H).$$

(2) If $\phi\colon B(H)^+ \longrightarrow B(H)^+$ is a spectral order isomorphism, then there exists a bijective increasing function $f : [0,+\infty[\longrightarrow [0,+\infty$, and a bijective operator $S : H \longrightarrow H$, which is semilinear in the case where $3 \leq \dim H < \infty$ and bounded linear or conjugate linear in the case where H is infinite dimensional, such that

$$\phi(A) = \psi_S(f(A)), \quad A \in B(H)^+.$$

(3) If $\phi\colon \mathcal{E}(H) \longrightarrow \mathcal{E}(H)$ is a spectral order isomorphism, then there exists a bijective increasing function $f : [0,1] \longrightarrow [0,1]$, and a bijective operator $S : H \longrightarrow H$, which is semilinear in the case where $3 \leq \dim H < \infty$ and bounded linear or conjugate linear in the case where H is infinite dimensional, such that

$$\phi(A) = \psi_S(f(A)), \quad A \in \mathcal{E}(H).$$

Compared to Theorem 1.2.1, we see that a description of the spectral order isomorphisms of $B(H)^{++}$ is still missing. This is one of the contributions we make in this thesis, that we present in Chapter 2.

1.3 Outline of the dissertation

The main idea we explore in this thesis is the representation of positive operators as real valued functions, defined on the unit sphere of the underlying Hilbert space. We have several motivations for this investigation.

Firstly, real valued functions inherits the order structure from the real numbers. Given any set of real valued functions, needless any additional data, a partial order can be defined on the set which is just the pointwise order. So on one side, we have the orders between bounded linear operators on a complex Hilbert space which are quite complicated objects, and on the other side, we have the natural order between real valued functions which is easier to handle. Therefore, it is a natural idea to try to represent the latter objects by the help of the former ones in a sufficiently faithful way. In fact this is not a completely new idea, one trivial possibility is to represent the selfadjoint operators by their quadratic forms. Clearly, that is a really faithful representation, the restriction of the quadratic form of an operator on the unit sphere completely determines the operator and, moreover, the usual Lwner order between selfadjoint operators is then transformed into the pointwise order between quadratic forms.

Secondly, this investigation is motivated by another functional representation of positive operators, which was introduced by Busch and Gudder in [5], the so-called the *strength function*, following an idea of Ludwig given in his fundamental treatise [15] on the foundations of quantum mechanics (see the proof of Theorem 5.22, especially, the first sentence on page 228). It was shown in [5] that strength functions faithfully represent effects and respect order in the sense that the Lwner order between positive operators is transformed to the pointwise order between the representing strength functions. It was initially defined for Hilbert space effects in [5], but the definition and the results trivially extends for positive Hilbert space operators.

These lead us to introduce two numerical functions on the unit sphere of a Hilbert space that we associate to positive operators, these functions are a spectral order analogue of the aforementioned two types of numerical functions (quadratic forms and strength functions). In fact, the first one, what we will consider as an analogue of the quadratic form is the local spectral radius of a positive operator. The other one, which we call spectral strength function, is the very natural adaptation of Busch and Gudder's strength function for the case of the spectral order in the place of the Lwner order.

In Chapter 2, we study the four functions mentioned above from various points of view. Namely, we show that the two new functional representations are also faithful representations of positive operators transforming the spectral order between operators to the pointwise order between the representing functions. We investigate which algebraic operations those four representations respect and then obtain that the collections of the representing functions have certain algebraic structures. Furthermore, we give some formulas for the representing functions and explore some of their properties (continuity and range). We point out that there are simple inequalities among those functions and study what their possible equality means for the corresponding operator. We will use the new representations studied in the first part of Chapter 2 to describe the spectral order isomorphisms of $B(H)^{++}$, this is the only result missing in Theorem 1.2.2 on the spectral order isomorphisms of the four basic operator intervals. The contents of Chapter 2 are already published in [31].

In Chapter 3, we use the representing functions to characterise the lattice operations with respect to the spectral order in $B(H)^{+}$. This characterisation is based on a generalisation of the properties of the so called Kubo-Ando means [14] (which are binary operations closely related to the usual order) to the setting of the spectral order. Namely, the Kubo-Ando means are essentially the binary operations which are monotone and satisfy a very important inequality called transformer inequality, we derive a spectral analogues of the transformer inequality and prove that this inequality essentially characterises the lattice operations. The contents of Chapter 3 are from the paper [32].

Chapter 2

Functional representations of positive operators

In this chapter, we consider some faithful representations of positive Hilbert space operators on structures of nonnegative real functions defined on the unit sphere of the Hilbert space in question. Those representations turn order relations between positive operators to order relations between real functions. Two of them turn the usual Lwner order between operators to the pointwise order between functions, another two turn the spectral order between operators to the same, pointwise order between functions. We investigate which algebraic operations those representations preserve, hence which kind of algebraic structure the representing functions have. We study the differences among the different representing functions of the same positive operator.

2.1 Representing functions

Let us introduce the four nonnegative real valued functions on the unit sphere in H, which we associate to an arbitrary positive operator and study in this paper. For any $A \in B(H)^+$ we define

$$w(A, x) = \langle Ax, x \rangle, \quad x \in S_H; \tag{2.1}$$

$$\lambda(A, x) = \sup\{t \geq 0 \,:\, tP_x \leq A\}, \quad x \in S_H; \tag{2.2}$$

$$r(A, x) = \lim_n \|A^n x\|^{1/n}, \quad x \in S_H; \tag{2.3}$$

$$\nu(A, P) = \sup\{t \geq 0 \,:\, tP_x \preccurlyeq A\}, \quad x \in S_H. \tag{2.4}$$

The function $w(A, .)$ is the quadratic form corresponding to $A \in B(H)^+$ restricted to the unit sphere. The function $\lambda(A, .)$ in (2.2) is called the strength function of A, the quantity $\lambda(A, x)$ is said to be the strength of A along the ray represented by the vector $x \in S_H$. In a quite similar way, we can introduce the function $\nu(A, .)$ in (2.4) which we call the spectral strength function of A. Finally, the quantity $r(A, x)$ in (2.3) is called the local spectral radius of A at $x \in S_H$. The function $r(A, .)$ is related to the spectral strength function $\nu(A, .)$ in a way very similar to the function $w(A, .)$ in (2.1) is related to the original strength function $\lambda(A, .)$. We will see this later on.

Let us make some introductory comments on the quantities/functions defined in (2.1) - (2.4). Firstly, it is clear that the $\sup$ in (2.2) can be replaced by $\max$. The same holds for (2.4), too. Indeed, it follows, for example, by using the fact that for positive operators $A, B \in B(H)^+$, we have $A \preceq B$ if and only if $A^n \leq B^n$ holds for all $n \in \mathbb{N}$. This was shown in Theorem 3 in [26], in the next section we will present an alternative proof of that fact. It is an apparent question why the limit in (2.3) exists. The answer is given in the next proposition. Recall that E_A denotes the spectral measure of A.

2.1.1 Proposition. *Let $A \in B(H)^+$ and pick $x \in S_H$. We have*

$$r(A, x) = \lim_n \|A^n x\|^{1/n} = \min\{t \geq 0 \,:\, P_x \leq E_A([0, t])\}. \tag{2.5}$$

Proof. Obviously, for any $A \in B(H)^+$ and $x \in S_H$ we have

$$\|A^n x\|^{1/n} = \langle A^{2n} x, x\rangle^{1/(2n)} \tag{2.6}$$

and here the latter quantity is the $2n$-norm of the identity function on the nonnegative reals with respect to the probability measure $\langle E_A(.)x, x\rangle$. As $p \to \infty$, the p-norm of an essentially bounded measurable function with respect to any probability measure is well-known to converge monotone increasingly to its ∞-norm, i.e., to the essential supremum of the function in question. Therefore, we have that the limit of the sequence in (2.6) exists and equals to

$$\inf\{t \geq 0 \,:\, \langle E_A(]t, \infty[)x, x\rangle = 0\} = \min\{t \geq 0 \,:\, \langle E_A(]t, \infty[)x, x\rangle = 0\}$$

which is the minimum of all such nonnegative real numbers t for which x is in the orthogonal complement of the range of $E_A(]t, \infty[)$, or equivalently, for which $P_x \leq E_A([0, t])$ holds. This completes the proof. ■

Observe that from (2.5) it follows that $\operatorname{rng} E_A([0,t])$ equals the set of all scalar multiples of the unit vectors $x \in S_H$ for which $r(A,x) \le t$. So by definition of $r(A,x)$, we then have that $x \in H$ belongs to $\operatorname{rng} E_A[0,t]$ if and only if

$$\|A^n x\|^{1/n} \le t\|x\|^{1/n}, \quad n = 1, 2, 3...,$$

and $x \in H$ does not belong to $\operatorname{rng} E_A[0,t]$ if and only if

$$\limsup_n \frac{1}{t^n}\|A^n x\| = \infty.$$

These observations will be useful to us in the next chapter.

2.2 Orders among positive operators and their representing functions

In this section we show that the representations

$$A \mapsto w(A,.), \quad A \mapsto \lambda(A,.), \quad A \mapsto r(A,.), \quad A \mapsto \nu(A,.), \tag{2.7}$$

are transformations from the set $B(H)^+$ of all positive operators on H to the set $\mathbb{R}_+^{S_H}$ of all nonnegative real valued functions on S_H which preserve order in both directions, and hence they are all faithful (i.e., injective). Here we consider either the usual Lwner order or the spectral order on $B(H)^+$ on the one hand and the pointwise order among functions on the other hand. The main results of this section is summarized in the following theorem.

2.2.1 Theorem. *Let $A, B \in B(H)^+$. The transformations w and λ determine the usual order, i.e., the following are equivalent:*

(1) $A \le B$;

(2) $w(A,x) \le w(B,x)$ for all $x \in S_H$;

(3) $\lambda(A,x) \le \lambda(B,x)$ for all $x \in S_H$.

The transformations r and ν determine the spectral order, i.e., the following are equivalent:

(1) $A \preccurlyeq B$;

(2) $r(A,x) \le r(B,x)$ for all $x \in S_H$;

(3) $\nu(A, x) \leq \nu(B, x)$ *for all* $x \in S_H$.

Before proving the theorem we present the following consequence. Clearly, as we have already referred to it, $A \preccurlyeq B$ implies that $f(A) \leq f(B)$ holds for any monotone increasing real function f (here $f(A)$ is defined by the Borel function calculus corresponding to A). So in particular, it implies that $A^n \leq B^n$ for positive integers n. As an easy corollary of the previous theorem, we can easily prove the following known fact (see Theorem 3 in [26]).

2.2.2 Corollary. *For any* $A, B \in B(H)^+$ *we have* $A \preccurlyeq B$ *if and only if* $A^n \leq B^n$ *for all* n.

Proof. Indeed, as for the sufficiency, assuming $A^n \leq B^n$ for every positive integer n, we immediately get $r(A, x) \leq r(B, x)$ for every vector $x \in S_H$ and then using the above theorem we can conclude that $A \preccurlyeq B$. ■

Now we prove Theorem 2.2.1, the proof will be split in several propositions.

We start with the quadratic form, it is obvious that for any $A, B \in B(H)^+$ we have $A \leq B$ if and only if $w(A, x) \leq w(B, x)$ holds for all $x \in S_H$. The fact that the same equivalence is true in relation with the strength functions was proved in Theorem 1 in [5]. Let us present an argument which is different from the one given in [5].

2.2.3 Proposition. *For any* $A, B \in B(H)^+$ *we have* $A \leq B$ *if and only if* $\lambda(A, x) \leq \lambda(B, x)$ *holds for all* $x \in S_H$.

Proof. The necessity part of the statement is clear. As for the sufficiency, observe that what we need to prove can be formulated in the following way: if $tP_x \leq A$ implies $tP_x \leq B$ for any nonnegative real number t, then we have $A \leq B$. Adding arbitrary small positive scalar multiple of the identity to B, we can assume that B is invertible. Multiplying by the inverse of the square root of B from both sides, we can further assume that $B = I$. So, we reduce the problem to show the following: if $tP_x \leq A$ holds only for $t \leq 1$, then we necessarily have $A \leq I$, or equivalently, $\|A\| \leq 1$. But this is easy since if $\|A\| > 1$, then there is an element of the spectrum of A which is greater than 1 implying that there is a nonzero spectral projection P of A and a real number t greater than 1 such that $tP \leq A$, a contradiction. ■

As for the characterization of the spectral order by the local numerical radius, we have the following result.

2.2.4 Proposition. *For any two positive operators $A, B \in B(H)^+$, we have $A \preccurlyeq B$ if and only if $r(A,x) \leq r(B,x)$ holds for every $x \in S_H$.*

Proof. Assume that $\lim_n \|A^n x\|^{1/n} \leq \lim_n \|B^n x\|^{1/n}$ holds for every $x \in S_H$. By (2.5), for any $x \in S_H$ we have

$$\min\{s \geq 0 \,:\, P_x \leq E_A([0,s])\} \leq \min\{s \geq 0 \,:\, P_x \leq E_B([0,s])\}.$$

This implies that for each $t \geq 0$, selecting any rank-one subprojection P_x of $E_B([0,t])$, we obtain that P_x is a subprojection of $E_A([0,t])$. Therefore, we have $E_A([0,t]) \geq E_B([0,t])$ for all $t \geq 0$. This gives us that $A \preccurlyeq B$. The converse statement can also be proved easily by employing the formula (2.5) again. ■

The characterization of the spectral order by the spectral strength function is now easy.

2.2.5 Proposition. *For any $A, B \in B(H)^+$ we have $A \preccurlyeq B$ if and only if $\nu(A,x) \leq \nu(B,x)$ holds for every $x \in S_H$.*

Proof. The necessity is obvious. Observe that for any $A \in B(H)^+$, nonnegative real number t and vector $x \in S_H$, by the definition of the spectral order we have

$$tP_x \preccurlyeq A \Longleftrightarrow P_x \leq E_A([t,\infty[) \tag{2.8}$$

which gives us that

$$\nu(A,x) = \max\{t \geq 0 \,:\, P_x \leq E_A([t,\infty[)\}. \tag{2.9}$$

Assuming now that $\nu(A,x) \leq \nu(B,x)$ holds for every $x \in S_H$, we easily obtain that every rank-one subprojection of $E_A([t,\infty[)$ is a subprojection of $E_B([t,\infty[)$. This implies that $E_A([t,\infty[) \leq E_B([t,\infty[)$ holds for any nonnegative real number t which gives $A \preccurlyeq B$. ■

This completes the proof of Theorem 2.2.1.

2.3 Formulae for the strength functions

The numerical range function and the local spectral radius functions are already explicitely defined. In this section we derive formulae for our two types of strength functions.

The formula for the strength functions in equation (2.11) was given by Busch and Gudder in Theorem 4 in [5]. Here we present a much shorter proof using the famous Douglas majorization and factorization lemma. First observe that for invertible $A \in B(H)^+$, nonnegative real number t and unit vector $x \in S_H$, we have $tP_x \leq A$ if and only if $tA^{-1/2}x \otimes A^{-1/2}x \leq I$ which is equivalent to $t\|A^{-1/2}x\|^2 \leq 1$. Hence

$$\lambda(A, P_x) = \|A^{-1/2}x\|^{-2} = \frac{1}{w(A^{-1}, x)}. \tag{2.10}$$

We also recall the following observation which follows directly from Douglas' result in [6]. Assume $A, B \in B(H)^+$. If $A \leq B$, then one can easily define a linear operator C on H by $C(B^{1/2}x) = A^{1/2}x$ for any $x \in H$. This gives a bounded linear operator of norm not greater than 1 on the range of $B^{1/2}$ which can be extended to the closure of this range and then it can be defined zero on the orthogonal complement of that subspace. Hence, we obtain a bounded linear operator C on H with $\|C\| \leq 1$ such that $CB^{1/2} = A^{1/2}$. Conversely, for such an operator $C \in B(H)$ we have $A = (CB^{1/2})^*CB^{1/2} \leq B^{1/2}IB^{1/2} = B$. It follows that for an $A \in B(H)^+$ and unit vector $x \in S_H$ if we have a positive real number t such that $tP_x \leq A$, then $P_x = CA^{1/2} = A^{1/2}C^*$ for some operator $C \in B(H)$ which implies that $x \in \operatorname{rng} A^{1/2}$. Conversely, if $x \in S_H$ and $x = A^{1/2}y$ for some $0 \neq y \in H$, then we have $P_x = A^{1/2}(y \otimes y)A^{1/2} = \|y\|^2 A^{1/2} P_{y/\|y\|} A^{1/2} \leq \|y\|^2 A$. After these simple facts we can prove the following.

2.3.1 Theorem. *For any $A \in B(H)^+$ and $x \in S_H$, the equality*

$$\lambda(A, x) = \begin{cases} \|A^{-1/2}x\|^{-2}, & \textit{if } x \in \operatorname{rng} A^{1/2} \\ 0, & \textit{otherwise} \end{cases} \tag{2.11}$$

holds. Here $A^{-1/2}$ means the inverse of the operator

$$A^{1/2}\big|_{(\ker A^{1/2})^\perp}$$

from its range $\operatorname{rng} A^{1/2}$ onto $(\ker A^{1/2})^\perp = \overline{\operatorname{rng} A^{1/2}}$.

Proof. We have already clarified that $\lambda(A, x) > 0$ holds if and only if $x \in \operatorname{rng} A^{1/2}$. Let now $x \in \operatorname{rng} A^{1/2}$. We have seen above that for a positive real number t, the inequality $tP_x \leq A$ is equivalent to the existence of a bounded linear operator C on H whose norm is not greater than 1 such that $CA^{1/2} = t^{1/2}P_x$ and C vanishes on $(\operatorname{rng} A^{1/2})^\perp = \ker A^{1/2} = \ker A$. Clearly, such a C has rank one with range included in the subspace generated by x. Therefore, C is necessarily of the form $C = x \otimes y$, where $y \in (\ker A^{1/2})^{\perp}$. From

$$x \otimes A^{1/2}y = CA^{1/2} = t^{1/2}P_x$$

we infer $A^{1/2}y = t^{1/2}x$ which is equivalent to $y = t^{1/2}A^{-1/2}x$. Consequently, we have $tP_x \leq A$ if and only if $\|t^{1/2}A^{-1/2}x\| = \|C\| \leq 1$. The largest such t is obviously $\|A^{-1/2}x\|^{-2}$. This completes the proof. ∎

To obtain an explicit formula for the spectral strength function $\nu(A,.)$, we first consider the case of an invertible positive operator A like we did in relation with the usual strength function $\lambda(A,.)$ above.

2.3.2 Proposition. *For any invertible $A \in B(H)^+$ and unit vector $x \in S_H$, we have*

$$\nu(A,x) = \frac{1}{\lim_n \|A^{-n}x\|^{1/n}} = \frac{1}{r(A^{-1},x)}. \tag{2.12}$$

Proof. Indeed, by (2.5) and (2.9) we have

$$\frac{1}{\lim_n \|A^{-n}x\|^{1/n}} = \frac{1}{r(A^{-1},x)} = \max\left\{\frac{1}{t} > 0 \,:\, P_x \leq E_{A^{-1}}([0,t])\right\}$$
$$= \max\left\{\frac{1}{t} > 0 \,:\, P_x \leq E_A([1/t,\infty[)\right\} = \max\{s > 0 \,:\, P_x \leq E_A([s,\infty[)\} = \nu(A,x).$$

∎

Observe that by the formulae (2.10) and (2.12), the function $r(A^{-1},.)$ plays a role in relation with the spectral strength function similar to the role the function $w(A^{-1},.)$ plays in relation with the usual strength function.

Now, the result concerning the formula for $\nu(A,.)$ in the case of general $A \in B(H)^+$ reads as follows.

2.3.3 Theorem. *For any $A \in B(H)^+$ and unit vector $x \in S_H$, we have that $\nu(A,x) > 0$ if and only if x belongs to a closed invariant subspace M of A on which A is invertible and in that case we have*

$$\nu(A,x) = r(A|_M^{-1},x)^{-1}. \tag{2.13}$$

Proof. Assume that $\nu(A,x) > 0$. By (2.8) it follows that $P_x \leq E_A([t,\infty[)$ holds for some positive real number t. Obviously, $E_A([t,\infty[)AE_A([t,\infty[) = AE_A([t,\infty[)$. For the range M of the spectral projection $E_A([t,\infty[)$, it follows that it is an invariant subspace of A which contains x. Since $AE_A([t,\infty[) \geq tE_A([t,\infty[)$, we infer that

$A\big|_M$ is an invertible operator on M. We apply Proposition 2.3.2 to obtain (2.13). Conversely, if the unit vector x belongs to a closed invariant subspace M of A on which A is invertible, then we have that $E_{A|_M}([t,\infty[)$ is the identity on M for some positive real number t implying that $P_x\big|_M \leq E_{A|_M}([t,\infty[)$. The projection P_x is zero on $M^\perp$, hence we obtain $P_x \leq E_A([t,\infty[)$, i.e., by (2.8), $\nu(A,x) \geq t > 0$ and this finishes the proof of the theorem. ■

2.4 Algebraic structures of representing functions

We have seen above that the functional representations in (2.7) are all order isomorphisms (the first two with respect to the Lwner order, the second two with respect to the spectral order) from $B(H)^+$ into the set of all bounded nonnegative real functions on S_H. Moreover, we know that $B(H)^+$ is a semigroup with respect to addition and the representation $A \longmapsto w(A,.)$ is additive, therefore on top of being an order isomorphism, this representation is also a semigroup isomorphism. It is natural to ask whether any of the other three representations is an isomorphism with respect to some algebraic operations too. So we are investigating any type of algebraic structures of the collections of functions appearing in the four functional representations that we are considering.

To begin with, recall that the transformation $A \mapsto w(A,.)$ is additive and positive homogeneous. Therefore, the collection of all functions $w(A,.)$, $A \in B(H)^+$ (with the usual pointwise operations) is a cone.

As for the map $A \mapsto \lambda(A,.)$, it is clearly not additive. In fact, it was proved in Proposition 2 in [22] that the sum of the strength functions of A and B is again a strength function if and only if $A, B \in B(H)^+$ are linearly dependent. However, we will show that the representation $A \mapsto \lambda(A,.)$ preserves the parallel sum and any weighted harmonic mean, it is an isomorphism under each of those operations.

Before formulating our result, we need to clarify the corresponding notions. For nonnegative real numbers t, s, their parallel sum $t : s$ is defined by 0 if one of t, s is zero, otherwise it is defined by

$$t : s = \left(\frac{1}{t} + \frac{1}{s}\right)^{-1}.$$

Furthermore, if α is a given real number satisfying $0 < \alpha < 1$, then the weighted harmonic mean $t!_\alpha s$ is defined by 0 if one of t, s is zero, otherwise it is defined by

$$t!_\alpha s = \left((1-\alpha)\frac{1}{t} + \alpha \frac{1}{s} \right)^{-1}.$$

Naturally, those concepts for nonnegative real functions are defined pointwise.

Weighted harmonic means can also be defined very naturally for positive invertible operators. For $A, B \in B(H)^{++}$ we define $A!_\alpha B$ by

$$A!_\alpha B = \left((1-\alpha)A^{-1} + \alpha B^{-1} \right)^{-1}.$$

But what about non invertible operators? We can follow the Kubo-Ando theory of operator means, see [14]. We elaborate more on the Kubo-Ando means in the next chapter, for now, since we are only concerned with the harmonic mean, we do not judge it necessary to expand about the theory of means here. Instead, to avoid technicalities we do the following. In the general (noninvertible) case, we choose monotone decreasing sequences $(A_n), (B_n)$ of invertible positive operators such that $A_n \to A$ and $B_n \to B$ in the strong operator topology and define $A!_\alpha B$ as the strong limit of the (monotone decreasing) sequence $(A_n !_\alpha B_n)$. Of course, it is not trivial at all to see that this definition is correct, the limit exists, it is a positive operator which does not depend on the particular choice of the sequences $(A_n), (B_n)$. Those properties come from the general theory in [14].

After this we mention that there is a different, quite special way to define the harmonic mean which is due to Ando and can be found in [3]. Namely, for any $A, B \in B(H)^+$, the harmonic mean $A!B = A!_{1/2}B$ can also be defined by

$$A!B = \max \left\{ X \in B(H)^+ : \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \geq \begin{bmatrix} X & X \\ X & X \end{bmatrix} \right\}. \tag{2.14}$$

Indeed, it can be shown that the maximum above exists in the Lwner order. Next, the parallel sum $A : B$ of $A, B \in B(H)^+$ can be defined as the half of the harmonic mean $A!B$. It is well-known that there is a formula for the quadratic form of $A : B$ which reads

$$\langle (A : B)z, z \rangle = \inf\{ \langle Ax, x \rangle + \langle By, y \rangle \, : \, x + y = z \}, \quad z \in H$$

and was proved by Anderson and Trapp in [2].

At this point, let us mention that we will also need the following properties of the weighted harmonic means (which are also satisfied by any Kubo-Ando means): those means are monotone (with respect to the Lwner order) in both of their variables and satisfy the so-called transfer property which tells that the congruence transformations implemented by invertible operators preserve any of those means. Observe that there is in fact no explicit formula for the harmonic mean of two general positive operators.

In what follows we prove the remarkable facts that the transformation $A \mapsto \lambda(A,.)$ respects the operations of parallel sum (which, anyway, is known to make $B(H)^+$ a commutative semigroup) and all weighted harmonic means.

2.4.1 Theorem. *Let $A, B \in B(H)^+$. We have*

$$\lambda(A : B, .) = \lambda(A, .) : \lambda(B, .).$$

Furthermore, for any real number $0 < \alpha < 1$, we have

$$\lambda(A!_\alpha B, .) = \lambda(A, .)!_\alpha \lambda(B, .).$$

Before presenting the proof of this theorem we make a few comments. Firstly, since $\lambda(tA, x) = t\lambda(A, x)$ holds for any nonnegative real number t, $A \in B(H)^+$, $x \in S_H$, it is easy to see that the equality above for the parallel sum or, equivalently, for the harmonic mean implies the equality for any weighted harmonic mean. Indeed, let $0 < \alpha < 1$ be any real number and let $c = (2(1-\alpha))^{-1}, d = (2\alpha)^{-1}$. For any $A, B \in B(H)^+$, we have $A!_\alpha B = (cA)!(dB)$. Indeed, this can be checked easily for invertible A, B and then, for general $A, B \in B(H)^+$, one can consider monotone decreasing sequences $(A_n), (B_n)$ of invertible positive operators such that $A_n \to A, B_n \to B$ in the strong operator topology. In a similar fashion, for arbitrary nonnegative real numbers t, s, we have $t!_\alpha s = (ct)!(ds)$. We conclude, on the one hand, that

$$\lambda(A!_\alpha B, x) = \lambda((cA)!(dB), x), \quad x \in S_H,$$

and, on the other hand, that

$$\lambda(A, x)!_\alpha \lambda(B, x) = (c\lambda(A, x))!(d\lambda(B, x)) = \lambda(cA, x)!\lambda(dB, x)), \quad x \in S_H.$$

Therefore, it is really sufficient to check that the transformation $A \mapsto \lambda(A, .)$ respects the harmonic mean.

Next, recall Theorem 4.2 in [7] which says that for any $A, B \in B(H)^+$, we have

$$\operatorname{rng}(A!B)^{1/2} = \operatorname{rng} A^{1/2} \cap \operatorname{rng} B^{1/2}. \tag{2.15}$$

Let us give a short proof here utilizing ideas behind strength functions and some elements of Kubo-Ando theory. If $x \in \operatorname{rng} A^{1/2} \cap \operatorname{rng} B^{1/2}$ is a unit vector, then we have that $tP_x \le A$, $sP_x \le B$ holds for some positive real numbers t, s. Then for $r = \min\{t, s\}$ we have $rP_x = rP_x!rP_x \le tP_x!sP_x \le A!B$ implying $x \in \operatorname{rng}(A!B)^{1/2}$. Conversely, if $x \in \operatorname{rng}(A!B)^{1/2}$ is a unit vector, then we infer that $rP_x \le A!B$ holds for some positive real number r. It is easy to see that $A!B \le 2A, 2B$. (Indeed, this is obviously true for invertible A, B and then, for general $A, B \in B(H)^+$, one can again consider monotone decreasing sequences $(A_n), (B_n)$ of invertible positive operators such that $A_n \to A, B_n \to B$ in the strong operator topology.) Therefore, we obtain $x \in \operatorname{rng} A^{1/2}, \operatorname{rng} B^{1/2}$ finishing the proof of (2.15).

It then follows from the equality (2.15) that the quantities $\lambda(A{:}B, .)$, $\lambda(A!B, .)$, $\lambda(A, .){:}\lambda(B, .)$, $\lambda(A, .)!\lambda(B, .)$ vanish at exactly the same elements of S_H.

To prove the identity

$$\lambda(A!B, x) = \lambda(A, x)!\lambda(B, x), \quad x \in S_H \tag{2.16}$$

we will need two additional observations. The first one is the very useful identity

$$A!P_x = \frac{2\lambda(A, x)}{1 + \lambda(A, x)} P_x \tag{2.17}$$

which holds for any $A \in B(H)^+$ and $x \in S_H$. It was proved in Lemma 2 in [19]. The second observation is the content of the following simple lemma.

2.4.2 Lemma. *Let $A \in B(H)^+$ and $x \in \operatorname{rng} A$. For the operator $A^{-1} : \operatorname{rng} A \longrightarrow \ker A^{\perp}$ which is the inverse of the restriction of A to $\ker A^{\perp}$ mapping onto $\operatorname{rng} A$ we have $A^{-1}x$ is the shortest element in the preimage of x under A.*

Proof. Notice that the preimage of x is a closed convex set, hence the element of shortest length exists and is unique. Let u be an arbitrary element in the preimage of x, i.e., assume that $Au = x$. We have $A(u - A^{-1}x) = Au - AA^{-1}x = x - x = 0$, hence $u - A^{-1}x \in \ker A$. It follows that $u = (u - A^{-1}x) + A^{-1}x$ and $(u - A^{-1}x)$ and $A^{-1}x$ are orthogonal, consequently we have $\|u\|^2 = \|u - A^{-1}x\|^2 + \|A^{-1}x\|^2 \ge \|A^{-1}x\|^2$, showing that $A^{-1}x$ is the shortest element in the preimage of x under A. ■

We are now in a position to prove Theorem 2.4.1.

Proof of Theorem 2.4.1. By the discussion above, it is sufficient to prove only the equality (2.16) for any $A, B \in B(H)^+$ and $x \in S_H$. Moreover, it is enough to verify that equality for vectors $x \in S_H$ at which no one of the two sides of (2.16) vanishes.

So, for given $A, B \in B(H)^+$, pick a unit vector x from the set $\operatorname{rng}(A!B)^{1/2} = \operatorname{rng} A^{1/2} \cap \operatorname{rng} B^{1/2}$. We have $0 < \lambda(A, x)P_x \le A$ and $0 < \lambda(B, x)P_x \le B$. Denote

$T = (\lambda(A,x)P_x)!(\lambda(B,x)P_x)$ and $s = \lambda(A,x)/\lambda(B,x)$. By the transfer property of means and the formula (2.17), we compute

$$T = \lambda(B,x)\left(\left(\frac{\lambda(A,x)}{\lambda(B,x)}P_x\right)!P_x\right) = \lambda(B,x)((sP_x)!P_x) = \lambda(B,x)\frac{2s}{1+s}P_x$$

and this yields

$$\lambda(T,x) = \lambda(B,x)\frac{2s}{1+s} = \lambda(A,x)!\lambda(B,x).$$

Since, by the monotonicity of means in their variables, we have $T \leq A!B$, it then follows that

$$\lambda(A,x)!\lambda(B,x) \leq \lambda(A!B,x).$$

We need to prove the converse inequality. For this we introduce the following notation. Given a closed subspace $K \subset H$, we denote

$$\Delta(K \oplus K) = \{(u,v) \in K \oplus K : \; u = v\} = \{(u,u) \in H \oplus H : \; u \in K\}.$$

Next, notice that

$$\begin{aligned} \operatorname{rng}\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}^{1/2} &= \operatorname{rng} A^{1/2} \oplus \operatorname{rng} B^{1/2} \supset \operatorname{rng}(A!B)^{1/2} \oplus \operatorname{rng}(A!B)^{1/2} \\ &\supset \Delta(\operatorname{rng}(A!B)^{1/2} \oplus \operatorname{rng}(A!B)^{1/2}). \end{aligned}$$

Since $x \in \operatorname{rng}(A!B)^{1/2}$, it follows that

$$\begin{bmatrix} x \\ x \end{bmatrix} \in \operatorname{rng}\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}^{1/2}.$$

Therefore, we have

$$0 < \lambda\left(\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}, \frac{\sqrt{2}}{2}\begin{bmatrix} x \\ x \end{bmatrix}\right)$$

and, by (2.11), we infer

$$\lambda\left(\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}, \frac{\sqrt{2}}{2}\begin{bmatrix} x \\ x \end{bmatrix}\right) = \left\|\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}^{-1/2}\frac{\sqrt{2}}{2}\begin{bmatrix} x \\ x \end{bmatrix}\right\|^{-2}.$$

We assert that

$$\left\|\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}^{-1/2}\frac{\sqrt{2}}{2}\begin{bmatrix} x \\ x \end{bmatrix}\right\|^{2} = \frac{1}{4}(\|A^{-1/2}x\|^2 + \|B^{-1/2}x\|^2) \tag{2.18}$$

or, equivalently,

$$\lambda\left(\begin{bmatrix}2A & 0\\ 0 & 2B\end{bmatrix}, \frac{\sqrt{2}}{2}\begin{bmatrix}x\\ x\end{bmatrix}\right) = 2(\lambda(A,x)!\lambda(B,x)). \tag{2.19}$$

To show (2.18), let $u, v \in H$ be such that

$$\begin{bmatrix}u\\ v\end{bmatrix} = 2\begin{bmatrix}2A & 0\\ 0 & 2B\end{bmatrix}^{-1/2}\frac{\sqrt{2}}{2}\begin{bmatrix}x\\ x\end{bmatrix}.$$

Then we have

$$\frac{\sqrt{2}}{2}\begin{bmatrix}x\\ x\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2A & 0\\ 0 & 2B\end{bmatrix}^{1/2}\begin{bmatrix}u\\ v\end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix}A^{1/2}u\\ B^{1/2}v\end{bmatrix}.$$

It follows that $A^{1/2}u = x = B^{1/2}v$. By Lemma 2.4.2, $\|u\|^2 \geq \|A^{-1/2}x\|^2$ and $\|v\|^2 \geq \|B^{-1/2}x\|^2$. We then have

$$\left\|\begin{bmatrix}2A & 0\\ 0 & 2B\end{bmatrix}^{-1/2}\frac{\sqrt{2}}{2}\begin{bmatrix}x\\ x\end{bmatrix}\right\|^2 = \left\|\frac{1}{2}\begin{bmatrix}u\\ v\end{bmatrix}\right\|^2 = \frac{1}{4}(\|u\|^2+\|v\|^2) \geq \frac{1}{4}(\|A^{-1/2}x\|^2+\|B^{-1/2}x\|^2).$$

On the other hand, the vector

$$\begin{bmatrix}2A & 0\\ 0 & 2B\end{bmatrix}^{-1/2}\frac{\sqrt{2}}{2}\begin{bmatrix}x\\ x\end{bmatrix}$$

is the element of minimal norm in

$$\text{the preimage of } \frac{\sqrt{2}}{2}\begin{bmatrix}x\\ x\end{bmatrix} \text{ under } \begin{bmatrix}2A & 0\\ 0 & 2B\end{bmatrix}^{1/2}$$

and the vector

$$\frac{1}{2}\begin{bmatrix}A^{-1/2}x\\ B^{-1/2}x\end{bmatrix}$$

is also in this preimage. Therefore,

$$\frac{1}{4}(\|A^{-1/2}x\|^2+\|B^{-1/2}x\|^2) = \left\|\frac{1}{2}\begin{bmatrix}A^{-1/2}x\\ B^{-1/2}x\end{bmatrix}\right\|^2 \geq \left\|\begin{bmatrix}2A & 0\\ 0 & 2B\end{bmatrix}^{-1/2}\frac{\sqrt{2}}{2}\begin{bmatrix}x\\ x\end{bmatrix}\right\|^2.$$

This verifies (2.18).

We also notice that the operators

$$\begin{bmatrix} A!B & A!B \\ A!B & A!B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2(A!B) & 0 \\ 0 & 2(A!B) \end{bmatrix}$$

coincide on the common invariant subspace $\Delta(H \oplus H)$. It easily follows that their strengths along any unit vector from that subspace are the same. Consequently, applying (2.19) for $A!B$ in the place of both A and B, we have

$$\begin{aligned} \lambda\left(\begin{bmatrix} A!B & A!B \\ A!B & A!B \end{bmatrix}, \frac{\sqrt{2}}{2}\begin{bmatrix} x \\ x \end{bmatrix}\right) &= \lambda\left(\begin{bmatrix} 2(A!B) & 0 \\ 0 & 2(A!B) \end{bmatrix}, \frac{\sqrt{2}}{2}\begin{bmatrix} x \\ x \end{bmatrix}\right) \\ &= 2(\lambda(A!B, x)!\lambda(A!B, x)) = 2\lambda(A!B, x). \end{aligned} \tag{2.20}$$

Since

$$\begin{bmatrix} A!B & A!B \\ A!B & A!B \end{bmatrix} \leq \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix},$$

see (2.14), we obtain from (2.19) and (2.20) that

$$2\lambda(A!B, x) \leq 2(\lambda(A, x)!\lambda(B, x))$$

and this finishes the proof. ■

Above we have seen that the map $A \mapsto w(A, .)$ preserves the operation of addition and all weighted arithmetic means while the map $A \mapsto \lambda(A, .)$ preserves the parallel addition and all weighted harmonic means. As for the functional representations

$$A \mapsto r(A, .), \quad A \mapsto \nu(A, .),$$

we can prove that they are isomorphisms under the lattice operations $\vee$ and $\wedge$, respectively. (For positive operators they mean sup and inf with respect to the spectral order, while for real functions they mean the pointwise maximum and minimum, respectively.)

2.4.3 Proposition. *Let $A, B \in B(H)^+$ and $x \in S_H$. We have the following equalities and inequalities:*

$$r(A \vee B, x) = r(A, x) \vee r(B, x), \quad r(A \wedge B, x) \leq r(A, x) \wedge r(B, x)$$

and

$$\nu(A \wedge B, x) = \nu(A, x) \wedge \nu(B, x), \quad \nu(A \vee B, x) \geq \nu(A, x) \vee \nu(B, x).$$

Proof. Recall that, by the results in [26], for the spectral resolutions of $A \vee B$ and $A \wedge B$ we have

$$E_{A\vee B}([0,t]) = E_A([0,t]) \wedge E_B([0,t])$$
$$E_{A\wedge B}([0,t]) = \bigwedge_{\varepsilon>0} E_A([0,t+\varepsilon]) \vee E_B([0,t+\varepsilon])$$

for all real numbers $t \geq 0$. Now, use the equality (2.5). For any nonnegative real number s, we have $r(A \vee B, x) \leq s$ if and only if $P_x \leq E_{A\vee B}([0,s])$ which is equivalent to $P_x \leq E_A([0,s])$ and $P_x \leq E_B([0,s])$. This holds if and only if $r(A,x), r(B,x) \leq s$ which is equivalent to $r(A,x) \vee r(B,x) \leq s$. This gives us the first equality. The first inequality follows from Proposition 2.2.4.

The proofs of the second equality and second inequality are similar, one can use the equality (2.9) and Proposition 2.2.5 there. ■

It follows from the results of this section that the collection of all functions $w(A,.)$ is closed under addition, the collection of all functions $\lambda(A,.)$ is closed under parallel sum, the collection of all functions $r(A,.)$ is closed under the operation $\vee$, and the collection of all functions $\nu(A,.)$ is closed under the operation $\wedge$.

2.5 Continuity of the representing functions

One may like to represent continuous operators by continuous functions. Therefore, in this section we investigate the question when the four different types of representing functions are continuous. Obviously, as for the functions $w(A,.)$, $A \in B(H)^+$, they are all continuous. The situation is very different with strength functions $\lambda(A,.)$.

2.5.1 Proposition. *For any $A \in B(H)^+$, we have that $\lambda(A,.)$ is continuous if and only if A is invertible.*

Proof. The sufficiency is obvious, see the formula (2.11). Assume now that A is not invertible. Then $A^{1/2}$ is also not invertible. It follows that $A^{1/2}$ is not surjective. At the elements of $S_H \cap \operatorname{rng} A^{1/2}$, the function $\lambda(A,.)$ takes positive values while at the elements of S_H which do not belong to $\operatorname{rng} A^{1/2}$, this value is zero. Since every former element can be approximated by a sequence of the latter elements, we obtain that $\lambda(A,.)$ is not continuous. This proves the proposition. ■

As for the spectral strength functions $\nu(A,.)$, the situation is even more extreme. In fact, let $A \in B(H)^+$ have finite spectrum $s_1 < s_2 < ... < s_n$ with corresponding spectral projections $E_{s_1}, ..., E_{s_n}$ (i.e., $E_{s_k} = E_A(\{s_k\})$, $k = 1, ..., n$). Then the spectral strength function of A is easily seen to be the following:

$$\nu(A,x) = s_k \quad \text{if } x \in \operatorname{rng}(E_{s_k} + E_{s_{k+1}} + ... + E_{s_n}) \setminus \operatorname{rng}(E_{s_{k+1}} + ... + E_{s_n}),$$

where $k = 1, ..., n$. Clearly, this function is non-continuous, hence $\nu(A,.)$ can be quite discontinuous even for invertible A. Indeed, we have the following result.

2.5.2 Proposition. *Let $A \in B(H)^+$. We have that $\nu(A,.)$ is continuous if and only if A is a scalar multiple of the identity.*

Proof. Assume that the spectrum of A has at least two elements $0 \le s_1 < s_2$. Select any numbers r, r' such that $s_1 < r < r' < s_2$. Then the range of $E_A([r,\infty[)$ is a proper subspace of H. Choosing a unit vector x from the complement of $\operatorname{rng} E_A([r,\infty[)$, which is a dense subset in H, we have that $\nu(A,x) < r$. On the other hand, for any unit vector x from the closed nonzero subspace $\operatorname{rng} E_A([r',\infty[)$ of H we have $\nu(A,x) \ge r'$. Now, we can easily deduce that $\nu(A,.)$ is not continuous. Therefore, only the scalar multiples of the identity may have continuous spectral strength functions which obviously really do. ■

Applying very similar reasoning and using the formula (2.5) we have the following.

2.5.3 Proposition. *Let $A \in B(H)^+$. The function $r(A,.)$ is continuous if and only if A is a scalar multiple of the identity.*

2.6 The ranges of the representing functions

In this section we investigate the ranges of the functions $w(A,.)$, $\lambda(A,.)$, $r(A,.)$, $\nu(A,.)$ and, especially, explore their relation to the spectrum of A.

For the first function $w(A,.)$, its range is clearly the numerical range $W(A)$ of the operator $A \in B(H)^+$. Let the convex hull of the spectrum $\sigma(A)$ of A be the interval with end points $\alpha \le \beta$, i.e.,

$$\operatorname{conv}(\sigma(A)) = [\alpha, \beta].$$

Clearly, α is the smallest elements of the spectrum of A and β is its largest element. The closure of the numerical range of a normal operator equals the convex hull of its spectrum. Therefore, we have that $W(A)$ contains the open interval $]\alpha, \beta[$ and is contained in the closed interval $[\alpha, \beta]$. We further have that β belongs to $W(A)$ if and only if it is an eigenvalue of A and similar holds for α. This is what we can say about the range of $w(A, .)$.

The relationship between the range $\Lambda(A)$ of the second function $\lambda(A.,)$ above and the numerical range (and the spectrum) of the operator A in question was investigated by Busch and Gudder in [5] (see Theorem 5). They proved the following:

(a) if $\alpha > 0$, then $\alpha \in \Lambda(A)$ if and only if α is an eigenvalue of A;

(b) $\beta \in \Lambda(A)$ if and only if β is an eigenvalue of A;

(c) if $\alpha > 0$, then $]\alpha, \beta[\subset \Lambda(A) = W(A) \subset [\alpha, \beta]$;

(d) if $\alpha = 0$ is an isolated eigenvalue of A, then $\{0\}\cup]\alpha_0, \beta[\subset \Lambda(A) \subset \{0\} \cup [\alpha_0, \beta]$ where $\alpha_0 = \min(\sigma(A) \setminus \{0\})$;

(e) if $\alpha = 0$ is an accumulation point of $\sigma(A)$, then $[0, \beta[\subset \Lambda(A) \subset [0, \beta]$.

As we have seen above, the ranges of $w(A, .)$ and $\lambda(A, .)$ are connected to the convex hull of the spectrum of A. The situation is different with the ranges of the other two functions $r(A, .)$ and $\nu(A, .)$, which are closely connected to the spectrum itself.

2.6.1 Proposition. *Let $A \in B(H)^+$. A real number t belongs to the range of $\nu(A, .)$ if and only if it is an element of $\sigma(A)$ which is either not isolated from the right, or it is isolated from the right and $E(\{t\}) \neq 0$.*

Proof. Assume t is not in the spectrum of A. Then there is a positive number ϵ such that $E_A([t, t+\epsilon[) = 0$. For any rank-one projection P we have that $P \leq E_A([t, \infty[)$ implies that $P \leq E_A([t+\epsilon, \infty[)$. This means that t does not belong to the range of $\nu(A, .)$.

Assume next that t is an element of the spectrum $\sigma(A)$ which is isolated from the right. We have a positive number ϵ such that the intersection of $[t, t+\epsilon[$ with $\sigma(A)$ is the singleton $\{t\}$. If $E_A(\{t\}) \neq 0$, then for any unit vector x in the nonempty set $\operatorname{rng} E_A([t, \infty[) \setminus \operatorname{rng} E_A([t+\epsilon, \infty[)$ we have $\nu(A, x) = t$. In the case where $E_A(\{t\}) = 0$, for every unit vector x in $\operatorname{rng} E_A([t, \infty[)$, we have that $\nu(A, x) \geq t + \epsilon > t$ meaning that t does not belong to the range of $\nu(A, .)$.

Finally, assume that t is an element of the spectrum of A which is not isolated from the right. In that case we can find a sequence (r_n) of elements of the spectrum which is strictly monotone decreasing and converges to t such that the sequence $(E_A([r_n, \infty[))$ is strictly increasingly converging to $E_A(]t, \infty[)$. We can pick a unit vector x from the range of this latter projection which is not in the range of any element of the sequence $(E_A([r_n, \infty[))$. We have $t \leq \nu(A, x) \leq r_n$ for every positive integer n and it implies that $\nu(A, x) = t$. ■

From the above proposition we immediately have the following.

2.6.2 Corollary. *If $A \in B(H)^+$ has finite spectrum or if A is compact, then the range of $\nu(A, .)$ is exactly the spectrum of A.*

Using the formula (2.5), an argument similar to what we have presented in Proposition 2.6.1 can be applied to obtain the next statement.

2.6.3 Proposition. *Let $A \in B(H)^+$. A real number t belongs to the range of $r(A, .)$ if and only if t is an element of $\sigma(A)$ which is either not isolated from the left, or it is isolated from the left and $E(\{t\}) \neq 0$.*

From this we obtain the following.

2.6.4 Corollary. *If $A \in B(H)^+$ has finite spectrum or if A is compact and noninjective, then the range of $r(A, .)$ is exactly the spectrum of A.*

To conclude the section, we can tell that the ranges of $r(A, .)$ and $\nu(A, .)$ provide more information about the spectrum of an operator $A \in B(H)^+$ than the other two functions $w(A, .)$ and $\lambda(A, .)$ do.

2.7 Equality among the various representing functions

Let $A \in B(H)^+$. By the definition of the functions $w(, .), \lambda(A, .), r(A, .), \nu(A, .)$, for any unit vector $x \in H$ we have the following inequalities

$$\nu(A, x) \leq \lambda(A, x) \leq w(A, x) \leq r(A, x). \tag{2.21}$$

Indeed, the first inequality follows from the fact that the spectral order is a coarser relation than the Lwner order. To the second one, observe that $tP_x \leq A$ for a rank-one projection P_x and nonnegative real number t implies that $tP_x \leq P_xAP_x$ and taking trace we have $t \leq w(A, x)$. The last inequality is a consequence of the following:

$$w(A, x) \leq \|Ax\| \leq \sup_n \|A^n x\|^{1/n} = r(A, x).$$

In this section we investigate when, for a given positive operator A, a pair of the above four functions coincide.

2.7.1 Proposition. *Let $A \in B(H)^+$. Then $\nu(A,x) = \lambda(A,x)$ holds for every $x \in S_H$ if and only if A is a positive scalar multiple of a projection.*

Proof. The sufficiency part is easy. To the necessity, we first show that the problem can be reduced to the case of invertible operators. So, let $A \in B(H)^+$ and assume that $\nu(A,.) = \lambda(A,.)$. Then it follows that for any $x \in \operatorname{rng} A^{1/2}$, meaning that $\lambda(A,x) > 0$, we have that $\nu(A,x) > 0$, i.e., there is a positive real number t such that $x \in \operatorname{rng} E_A([t,\infty[)$, see (2.9). We claim that if $0 \in \sigma(A)$, then 0 is an isolated point of the spectrum of A. Indeed, otherwise we would have pairwise disjoint open sets $U_n =]\alpha_n, \beta_n[$ in $]0,\infty[$ such that $\beta_{n+1} < \alpha_n$ and $\beta_n \le 1/n^4$ and for which the projections $E_A(U_n)$ are all nonzero. For any positive integer n, select a unit vector e_n from $\operatorname{rng} E_A(U_n)$. Then the sequence (e_n) is an orthonormal sequence in H. Define $z = \sum_n Ae_n$. The vectors Ae_n are mutually orthogonal, and we have $\|Ae_n\|^2 \le \beta_n^2$. Therefore, $z \in H$ is well-defined.

By Shmul'yan's characterization of the ranges of bounded linear operators (see, e.g., Corollary 2 in [7]) we have that $y \in \operatorname{rng} A^{1/2}$ if and only if there is a positive constant K such that $|\langle y,x\rangle|^2 \le K\langle Ax,x\rangle$ holds for all $x \in H$. Let us consider our z defined above. We compute

$$|\langle z,x\rangle|^2 = |\sum_n \langle Ae_n,x\rangle|^2 \le \left(\sum_n |\langle Ae_n,x\rangle|\right)^2 \le \left(\sum_n \|A^{1/2}e_n\|\|A^{1/2}x\|\right)^2$$
$$\le \left(\sum_n \|A^{1/2}e_n\|\right)^2 \|A^{1/2}x\|^2.$$

We have $\|A^{1/2}e_n\|^2 = \langle Ae_n,e_n\rangle \le \beta_n \le 1/n^4$ which implies that $\|A^{1/2}e_n\| \le 1/n^2$. It follows that $\sum_n \|A^{1/2}e_n\|$ is convergent and we deduce that $z \in \operatorname{rng} A^{1/2}$. On the other hand, it is clear that z is not in the range of any spectral projection $E_A([t,\infty[)$, $t > 0$ and this is a contradiction.

Consequently, we obtain that 0 is an isolated point of the spectrum of A. For any $A \in B(H)^+$, we have $\ker A = \operatorname{rng} E_A(\{0\})$. We obtain that $\ker A^\perp = \operatorname{rng} E_A([t,\infty[)$ for some positive real number t, on which subspace A is bounded from below, i.e., invertible. Therefore, we have that A can be written as $A = 0 \oplus A\big|_{\ker A^\perp}$ where the latter operator is invertible. We see that both $\nu(A,.)$ and $\lambda(A,.)$ are zero on the complement of the set $\ker A^\perp$.

The above way we can reduce the problem to the case of an invertible operator A. In order to complete the proof, we show that for such an A we necessarily have that A a positive scalar multiple of the identity. To verify this, by (2.10) and (2.12), it is sufficient to consider the case where for a given $T \in B(H)^+$ we have

$$r(T, x) = \|T^{1/2}x\|^2, \quad x \in S_H. \tag{2.22}$$

As we have already mentioned in the proof of Proposition 2.1.1, the sequence $(\|T^n x\|^{1/n})$ (whose limit is $r(T, x)$) is monotone increasing in n and its elements are clearly greater than or equal to $\|T^{1/2}x\|^2$. So, from (2.22) we obtain that $\|Tx\| = \langle Tx, x\rangle$, $x \in S_H$ which, by the equality case in the Cauchy-Schwarz inequality, implies that Tx is a scalar multiple of x. This implies that every nonzero vector is an eigenvector of T which gives us that T is a scalar multiple of the identity. The proof is complete. ■

The remaining equality cases in our inequalities (2.21) are more easy to handle.

2.7.2 Proposition. *Let $A \in B(H)^+$. Then $\lambda(A, x) = w(A, x)$ holds for every $x \in S_H$ if and only if A is a scalar multiple of the identity.*

Proof. We consider only the necessity part of the statement, the sufficiency is obvious. Assume $\lambda(A, x) = w(A, x)$ holds for every unit vector $x \in H$. Then we have that $\lambda(A, .)$ is continuous and hence, by Proposition 2.5.1, A is invertible. Moreover, using the formula (2.10), we have

$$1 = \|A^{-1/2}x\|^2 \|A^{1/2}x\|^2$$

for all unit vectors $x \in H$. This means that

$$\|x\|^2 = \|A^{-1/2}x\| \|A^{1/2}x\|, \quad x \in H.$$

Replacing x by $A^{1/2}x$, we have

$$\langle Ax, x\rangle = \|x\| \|Ax\|, \quad x \in H.$$

Since this means that there is equality in the Cauchy-Schwarz inequality, we obtain just as in the proof the previous proposition that Ax is a scalar multiple of x for any $x \in H$ and then conclude that A is a scalar multiple of the identity. ■

2.7.3 Proposition. *Let $A \in B(H)^+$. Then we have $w(A, x) = r(A, x)$ for every $x \in S_H$ if and only if A is a scalar multiple of the identity.*

Proof. Assume that $w(A, x) = r(A, x)$ holds for every $x \in S_H$. We have $w(A, x) \le \|Ax\| \le r(A, x)$, $x \in S_H$ and it then follows that

$$\langle Ax, x\rangle = \|Ax\| \|x\|, \quad x \in H.$$

We can complete the proof just as in the case of the previous proposition. ■

2.8 Spectral order isomorphisms of the positive definite cone

In this section we use the properties of the local spectral radius function to prove a characterization of the scalar elements of $B(H)^+$. This will allow us to give a proof of another characterization of the same type of operators expressed by the spectral order. We will apply that characterization to determine the structure of all spectral order isomorphisms of the positive definite cone $B(H)^{++}$ of $B(H)$ (i.e., the set of all invertible positive operators on H).

In Proposition 2.4.3, we have proved the inequality

$$r(A \wedge B, x) \leq r(A, x) \wedge r(B, x),$$

for all $A, B \in B(H)^+$ and $x \in S_H$. Here we do not have equality in general. To see this, consider, for example, two nonzero projections P, Q on H whose ranges have trivial intersection. Then we have $r(P \wedge Q, x) = 0$ and $r(P, x) \wedge r(Q, x) = 1$ for every $x \in S_H$ which is neither in the kernel of P, nor in the kernel of Q.

Interestingly, as we show below, for a given $A \in B(H)^+$, the equality $r(A \wedge B, x) = r(A, x) \wedge r(B, x)$ holds for all $B \in B(H)^+$ and $x \in S_H$ if and only if A is a scalar multiple of the identity.

2.8.1 Lemma. *Let $A \in B(H)^+$. Then $A = aI$ holds with some real number $a \geq 0$ if and only if for every $B \in B(H)^+$ and $x \in S_H$ we have $r(A \wedge B, x) = r(A, x) \wedge r(B, x)$.*

Proof. If $A = aI$, then $r(A, x) = a$ for any $x \in S_H$. The spectral resolutions of A and $A \wedge B$ are the following

$$E_A([0, t]) = \begin{cases} 0, & 0 \leq t < a \\ I, & a \leq t, \end{cases}$$

$$E_{A \wedge B}([0, t]) = E_A([0, t]) \vee E_B([0, t]) = \begin{cases} E_B([0, t]), & 0 \leq t < a \\ I, & a \leq t, \end{cases}$$

and we see that $r(A \wedge B, x) = \min\{t \geq 0 \, : \, P_x \leq E_{A \wedge B}([0, t])\}$ equals the minimum of $r(B, x)$ and $\lambda = r(A, x)$.

Assume now that $A \in B(H)^+$ is not scalar and, without loss of generality, assume further that $\|A\| = 1$. For a unit vector $x \in S_H$, we have $r(A, x) = 1$ if and only if x does not belong to the range of $E_A([0, 1 - 1/n])$, $n \in \mathbb{N}$. This is equivalent to the fact that x is in the complement of the union of the ranges of $E_A([0, 1-1/n])$, $n \in \mathbb{N}$. Clearly, these ranges are nowhere dense sets, hence their union is of first category. Therefore, the set of all $x \in S_H$ for which $r(A, x) = 1$ holds is of second category. Let B be any nonzero subprojection of a nonzero spectral projection $E_A([0, s])$, where $s < 1$ (the existence of such a spectral projection follows from the assumption that $A \neq I$). Then we have $\|A \wedge B\| \leq s$. Since the orthogonal complement of the range of B is of first category, we have an $x \in S_H$ which is not in this orthogonal complement (and hence $r(B, x) = 1$ holds) and satisfies $r(A, x) = 1$. We have $r(A \wedge B, x) \leq \|A \wedge B\| \leq s < 1 = r(A, x) \wedge r(B, x)$. This completes the proof of the statement. ■

In the paper [4], Bohata gave a very nice characterization of central elements in AW^* algebras, see Proposition 3.8 [4]. In what follows we present a proof of that characterization in the setting of $B(H)$ using the local spectral radius function and the previous lemma.

2.8.2 Theorem. *For an operator $T \in B_{sa}(H)$ we have that $T \wedge (A \vee B) = (T \wedge A) \vee (T \wedge B)$ holds for any $A, B \in B_{sa}(H)$ if and only if T is a scalar multiple of the identity.*

Proof. Assume that $T \in B_{sa}(H)$ is a scalar multiple of the identity and let $A, B \in B_{sa}(H)$ be arbitrary. For a large enough $\mu \in \mathbb{R}$, the operators $T' = T + \mu I, A' = A + \mu I, B' = B + \mu I$ all belong to $B(H)^+$. Since translation by μI is a spectral order isomorphism of $B_{sa}(H)$, for $L = T \wedge (A \vee B)$ and $R = (T \wedge A) \vee (T \wedge B)$ we have $L + \mu I = T' \wedge (A' \vee B')$ and $R + \mu I = (T' \wedge A') \vee (T' \wedge B')$. Applying the trivial part of the previous lemma and Proposition 2.4.3, for any $x \in S_H$ we have

$$\begin{aligned} r(L + \mu I, x) &= \min\{r(T', x),\ r(A' \vee B', x)\} \\ &= \min\{r(T', x),\ \max\{r(A', x), r(B', x)\}\} \\ &= \max\{\min\{r(T', x), r(A', x)\},\ \min\{r(T', x), r(B', x)\}\} \\ &= \max\{r(T' \wedge A', x),\ r(T' \wedge B', x)\}\} \\ &= r(R + \mu I, x). \end{aligned}$$

By Proposition 2.2.4, we obtain that $L + \mu I = R + \mu I$, i.e., that $L = R$.

Assume now that $T \in B_{sa}(H)$ is not scalar. Again, using a translation and maybe also a positive scalar multiplication (both of those transformations are spectral order isomorphisms of $B_{sa}(H)$), we can assume that T is positive with norm $\|T\| = 1$. Similarly to the proof of the previous lemma, we choose a projection $B = P_y$ and select a vector $x \in S_H$ not orthogonal to y such that $r(T \wedge B, x) < 1 = r(T, x)$. Setting $A = I - P_x$, we have $A \vee B = I$. Clearly, $T \preccurlyeq I = A \vee B$ and we have $L = T \wedge (A \vee B) = T$. On the other hand, the spectral resolution of $T \wedge A$ is

$$E_{T \wedge A}([0, t]) = E_T([0, t]) \vee E_A([0, t]), \quad t \geq 0.$$

We have $E_A([0,t]) = P_x$ for any $0 \leq t < 1$. By (2.5), it follows that $r(T \wedge A, x) = 0$ and hence we obtain that

$$r(R, x) = \max\{r(T \wedge B, x), r(T \wedge A, x)\} = r(T \wedge B, x) < 1 = r(T, x).$$

This means that $R \neq T = L$ and the proof of the theorem is complete. ■

We remark that employing translations, one can see that the arguments above can easily be modified to yield the same sort of characterization of scalar multiples of the identity in the case where the collection $B_{sa}(H)$ in Theorem 2.8.2 is replaced by the positive semidefinite cone $B(H)^+$ or the positive definite cone $B(H)^{++}$.

The above characterization in the setting of AW^*-algebras played an important role in [4] in the descriptions of spectral order isomorphisms between the selfadjoint parts, the positive semidefinite cones, and the effect algebras of AW^*-factors of Type I. As we already mentioned in the Introduction, Molnár and Šemrl [24] and Bohata [4] gave the structure of spectral order isomorphisms of the effect algebra $\mathcal{E}(H)$, the spectral order isomorphisms of the positive semidifinite cone $B(H)^+$ and the spectral order isomorphisms of the whole space $B_{sa}(H)$. It is a remarkable fact about these spectral order isomorphisms that they all have the same form (in contrast with the isomorphisms of with the respect to the usual order). Below we point out that similar result holds also for the positive definite cone $B(H)^{++}$ which observation we will need in the last section of this chapter.

Before the formulation of our result let us recall the different maps that are going to be involved. For an operator $S : H \to H$ which is bijective bounded linear or conjugate linear if H is infinite dimensional, or bijective semilinear if H is finite dimensional, recall the definition of ψ_S: given $A \in B_{sa}(H)$ with spectral measure E_A, the map

$$t \longmapsto I - P_{S(\operatorname{rng} E_A(]t,\infty[))}, \quad t \in \mathbb{R}$$

is a resolution of the identity that we denote by E_A^S, and we define

$$\psi_S(A) = \int_{-\infty}^{+\infty} \lambda \, dE_A^S(\lambda), \quad A \in B_{sa}(H). \tag{2.23}$$

It was proved in Proposition 1 in [24] that $\psi_S : B_{sa}(H) \to B_{sa}(H)$ is a spectral order isomorphism, i.e., a bijective map such that

$$A \preccurlyeq B \iff \psi_S(A) \preccurlyeq \psi_S(B), \quad A, B \in B_{sa}(H).$$

We also have that $\psi_S([0,I]) = [0,I]$, $\psi_S(B(H)^+) = B(H)^+$, $\psi_S(B(H)^{++}) = B(H)^{++}$. As shown in Theorem 1.2.2 in Chapter 1, we know that all the spectral order isomorphisms of $[0,I]$, $B(H)^+$, $B_{sa}(H)$ are of the form $A \mapsto \psi_S(f(A))$, where S is an operator as above and f is a strictly increasing bijective function of the intervals $[0,1], [0,\infty[,]-\infty,\infty[$ in the respective cases.

As a contribution to this line of research we prove that the same structural result holds for the spectral order isomorphisms of the positive definite cone $B(H)^{++}$, too. This is the content of the following result.

2.8.3 Theorem. *Let $\phi : B(H)^{++} \to B(H)^{++}$ be a spectral order isomorphism. Then there is a strictly increasing bijective continuous function $f :]0,\infty[\to]0,\infty[$ and an additive bijection $S : H \to H$ which is semilinear in the case where $3 \leq \dim H < \infty$ and it is bounded linear or conjugate linear in the case where H is infinite dimensional such that*

$$\phi(A) = \psi_S(f(A)), \quad A \in B(H)^{++}. \tag{2.24}$$

Proof. By the remark after the proof of Theorem 2.8.2, ϕ maps scalar operators to scalar operators and hence it gives a strictly monotone increasing bijection f of the set of positive real numbers such that $\phi(tI) = f(t)I$ holds for all positive number t. The map $\phi(f^{-1}(.))$ is clearly a spectral order isomorphism of $B(H)^{++}$ which fixes the scalar multiples of the identity. Therefore, we can assume that already the original isomorphism ϕ has this property.

We can follow the idea of reducing the problem to the case of the effect algebra just as in the proof of Theorem 4 in [24] (see also Section 4 in [4]). Pick any two positive numbers, $\alpha < \beta$. Restricting ϕ to the interval $[\alpha I, \beta I]$, we obtain a spectral order isomorphism of this interval. But using the bijective increasing affine function $g_{\alpha,\beta} : [\alpha,\beta] \to [0,1]$, we have that $g_{\alpha,\beta}(\phi(g_{\alpha,\beta}^{-1}(.)))$ is a spectral order isomorphism of the effect algebra $[0,I]$. From the structure of the spectral order isomorphisms of $[0,I]$ presented in Theorem 1.2.2, we have a corresponding operator $S_{\alpha,\beta}$ (bijective semilinear in the finite dimensional case, bijective bounded linear or conjugate linear in the infinite dimensional case) such that

$$g_{\alpha,\beta}^{-1}(\psi_{S_{\alpha,\beta}}(g_{\alpha,\beta}(A))) = \phi(A), \quad A \in [\alpha I, \beta I].$$

But it can easily be verified that all the maps ψ_S commute with strictly increasing affine functions, hence we in fact have $\psi_{S_{\alpha,\beta}}\big|_{[\alpha I, \beta I]} = \phi\big|_{[\alpha I, \beta I]}$. Letting α decrease and β increase, one can see that $\psi_{S_{\alpha,\beta}}$ are all necessarily equal. This completes the proof. ■

Let us point out an interesting fact. As we see above, the forms of the spectral order isomorphisms of the effect algebra, the positive definite and semidefinite cones, and the whole space of self-adjoint operators are all the same. This is different with the usual Lwner order. There is an other difference related to the order isomorphisms that we should point out. For the usual order, there are only five non-order isomorphic classes of operator intervals which are represented by the particular intervals

$$[0, I], \quad [0, \infty[, \quad]-\infty, 0], \quad]0, \infty[, \quad]-\infty, \infty[. \tag{2.25}$$

The order isomorphisms of these special intervals are given in Theorem 1.2.1 in Chapter 1. As for the operator intervals and their isomorphisms with respect to the spectral order, we believe that the situation is much more complicated, we have plenty of isomorphism classes. This is definitely an interesting question that deserves to be investigated further.

We conclude this section with the next simple result which describes the intersection of the isomorphism groups of the two different orders (usual order and spectral order) for the case of the positive definite cone. Similar assertion holds for the positive semidefinite cone, the effect algebra and also the space of all self-adjoint operators. In the case of the positive definite cone, it says that the intersection of those two groups consists exactly of the positive scalar multiples of unitary-antiunitary congruence transformations.

2.8.4 Proposition. *Assume T is a bounded invertible linear or conjugate linear operator on H. The transformation $A \mapsto TAT^*$ is a spectral order isomorphism of $B(H)^{++}$ if and only if T is a scalar multiple of a unitary or antinuitary operator.*

Proof. We deal only with the linear case. Consider the polar decomposition $T = U|T|$ of T. Clearly, unitary congruence transformations are spectral order isomorphisms, hence it follows that we can assume that T is positive and we claim to prove that then T is necessarily a positive scalar multiple of the identity.

For every rank-one projection P_x on H and for arbitrary small positive number ϵ, we have $P_x + \epsilon I \preccurlyeq (1+\epsilon)I$ implying that $T(P_x + \epsilon I)T \preccurlyeq (1+\epsilon)T^2$. It follows easily from Corollary 2.2.2 and then letting ϵ tend to 0 that $(TP_xT)^2 \leq T^4$. Multiplying by the inverse of T from both sides, we have $P_xT^2P_x \leq T^2$. We deduce that $\|Tx\|^2 x \otimes x \leq T^2$ for every unit vector $x \in H$ which yields that $w(T^2, x) \leq \lambda(T^2, x)$. Since the converse inequality always holds (see (2.21)), applying Proposition 2.7.2 we have that T^2 is a scalar multiple of the identity. This immediately gives the statement. ∎

2.9 Metrics corresponding to representing functions and their isometries

In this last section we discuss the relations of our functions in (2.7) to certain metrics on the positive semidefinite or definite cone in $B(H)$.

The most natural metric on spaces of bounded real or complex valued functions is the one which comes from supremum norm. Let us denote it by $\|.\|$ and hope it does not cause confusion with the notation of the operator norm. It is clear that

$$\|w(A,.) - w(B,.)\| = \|A - B\|, \quad A, B \in B(H)^+.$$

Hence, the supremum norm distance for the functions $w(A,.)$, $A \in B(H)^+$ reproduces the operator norm distance on $B(H)^+$. Clearly, with this metric, $B(H)^+$ is a complete metric space.

In [22], Molnár introduced the Busch-Gudder metric d_{BG} on $B(H)^+$ which he defined as the metric induced by the supremum norm distance of the strength functions

$$d_{BG}(A,B) = \|\lambda(A,.) - \lambda(B,.)\|, \quad A, B \in B(H)^+.$$

In [22], Molnár proved the completeness of d_{BG} in the finite dimensional case (and left the infinite dimensional case as an open problem). In the same paper we determined the surjective isometries of $B(H)^+$ with respect to d_{BG} and obtained that they are exactly the unitary or antiunitary congruence transformations implying that the isometry groups of $B(H)^+$ with respect to the operator norm metric and the Busch-Gudder metric coincide.

Let us now consider the positive definite cone $B(H)^{++}$ and recall the important metric on it called the Thompson metric (or Thompson part metric). In fact, this sort of distance can be defined in a more general setting. If $\mathcal{A}$ is a C^*-algebra, the definition of the Thompson metric d_T on its positive definite cone $\mathcal{A}^{++}$ reads as follows

$$d_T(A,B) = \log \max\{M(A/B), M(B/A)\}, \quad A, B \in \mathcal{A}^{++},$$

where $M(X/Y) = \inf\{t > 0 \,:\, X \leq tY\}$ for any $X, Y \in \mathcal{A}^{++}$. It is easy to see that d_T can also be rewritten as

$$d_T(A,B) = \left\|\log\left(A^{-1/2} B A^{-1/2}\right)\right\|, \quad A, B \in \mathcal{A}^{++}. \tag{2.26}$$

Here $\|.\|$ denotes the C^*-norm on $\mathcal{A}$. The most important property of this metric which makes it so useful is completeness. Hence, such important tools as Banach fixed point theorem can be used. This has deep applications e.g., in the theory of operator means.

Observe the following. For any $A, B \in B(H)^{++}$ and positive real number t, the inequality $A \leq tB$ is equivalent to $w(A, .) \leq tw(B, .)$ and also to $\lambda(A, .) \leq t\lambda(B, .)$. It follows easily that we have

$$d_T(A, B) = \| \log w(A, .) - \log w(B, .)\| = \| \log \lambda(A, .) - \log \lambda(B, .)\|.$$

(Although the supremum norm distance for the functions $w(A, .)$, $A \in B(H)^+$ induces the usual operator norm distance on $B(H)^+$ which is certainly different from the metric what the supremum norm distance for the functions $\lambda(A, .)$, $A \in B(H)^+$ induces (namely, the Busch-Gudder metric), for the logarithms of those functions the induced metrics are the same, the Thompson metric on $B(H)^{++}$.)

We can define a spectral order variant of the Thompson metric by replacing the usual Lwner order $\leq$ by $\preccurlyeq$ in the definition of $M(X/Y)$ above. What we get we denote by d_{sT} and call it the spectral Thompson metric. Apparently, by Proposition 2.2.4 and Proposition 2.2.5 , for any positive number t, we have $A \preccurlyeq tB$ exactly when $r(A, .) \leq tr(B, .)$ which is equivalent to $\nu(A, .) \leq t\nu(B, .)$. Therefore, we conclude

$$\begin{aligned} d_{sT}(A, B) &= \| \log r(A, .) - \log r(B, .)\| \\ &= \| \log \nu(A, .) - \log \nu(B, .)\|, \quad A, B \in B(H)^{++}. \end{aligned} \tag{2.27}$$

Since the spectral order as a relation is coarser than the usual Lwner order, it follows that

$$d_T(A, B) \leq d_{sT}(A, B), \quad A, B \in B(H)^{++}.$$

2.9.1 Proposition. *The metric d_{sT} is a complete metric on $B(H)^{++}$.*

Proof. Let (A_n) be a Cauchy sequence in $B(H)^{++}$ with respect to the metric d_{sT}. Then, by (2.9), it is also a Cauchy sequence with respect to the Thompson metric. Hence, it converges to some $A \in B(H)^{++}$ with respect to that metric and, as an easy consequence of the formula (2.26), also with respect to the operator norm distance. On the other hand, for any $\epsilon > 0$ we have a positive integer n_0 such that for all integers $n, m \geq n_0$ we have $A_n \preccurlyeq (1 + \epsilon)A_m$. This implies that $A_n^k \leq (1+\epsilon)^k A_m^k$ for all $n, m \geq n_0$ and positive integer k. It follows that $A^k \leq (1+\epsilon)^k A_m^k$, $A_n^k \leq (1 + \epsilon)^k A^k$ for all $n, m \geq n_0$ and positive integer k implying $A \preccurlyeq (1 + \epsilon)A_m$, $A_n \preccurlyeq (1 + \epsilon)A$ for all $n, m \geq n_0$. Consequently, we have that (A_n) converges to A in the metric d_{sT}. ■

Therefore, the spectral Thompson metric also has the nice property that it makes the positive definite cone $B(H)^{++}$ a complete metric space (a property that the metric of the operator norm lacks). We are curious if one can provide useful applications of the spectral Thompson metric based on this property. Furthermore, let us point out a nice property what d_{sT} has but d_T does not. In Theorem 15 in [23] we proved that, in the setting of C^*-algebras, the Thompson metric has nontrivial dilations (homotheties) only if the underlying algebra is commutative which is a rather strange thing. With the spectral Thompson metric the situation is more "natural", it has dilations with arbitrary ratios: $d_{sT}(A^p, B^p) = pd_{sT}(A, B)$ holds for all invertible positive elements A, B and any positive real number p.

Symmetries play an important role in most parts of mathematics. For that general reason, when it comes to metric spaces, we are interested in the descriptions of the corresponding isometry groups. In the paper [20], Molnár determined the surjective Thompson isometries of $B(H)^{++}$ (and in [11] we extended that result to the setting of positive definite cones in general C^*-algebras). The result in [20] says that the surjective Thompson isometries of $B(H)^{++}$ are exactly the maps $A \mapsto TAT^*$, $A \mapsto TA^{-1}T^*$, where T is an invertible bounded either linear or conjugate linear operator on H. In other words, one can say that the group of Thompson isometries of $B(H)^{++}$ consists exactly of the homogeneous Lwner order isomorphisms and the antihomogeneous Lwner order antiisomorphisms (see the discussion after (2.25)). Antihomogeneity here means homogeneity of order -1.

It would be interesting to know the structure of all surjective spectral Thompson isometries of $B(H)^{++}$. We leave this as an open problem and formulate the conjecture that, in analogy with the case of the usual Thompson metric, the group of the surjective spectral Thompson isometries consists exactly of the homogeneous spectral order isomorphisms and the antihomogeneous spectral order antiisomorphisms. To express this more explicitly, we conjecture that those isometries are exactly the transformations ψ_S (see (2.23)) and their compositions with the inverse operation. Just as before, in the finite dimensional case, S is a bijective semilinear operator, while in the infinite dimensional case, it is an invertible bounded either linear or conjugate linear operator on H.

Although we do not know the isometry group of the spectral Thompson metric yet, still we know its intersection with the isometry group of the usual Thompson metric. Indeed, we finish this section with the following analogue of Proposition 2.8.4.

2.9.2 Proposition. *The intersection of the groups of all Thompson isometries and all spectral Thompson isometries of $B(H)^{++}$ consists exactly of the transformations $A \mapsto$*

$cUAU^*$ and $A \mapsto cUA^{-1}U^*$, where U is a unitary or antiunitary operator on H and c is a positive real number.

Proof. We know the structure of all Thompson isometries of $B(H)^{++}$. We also know that the unitary or antiunitary congruence transformations as well as the inverse operation are isometries with respect to both the usual Thompson metric and the spectral Thompson metric. Therefore, as in the proof of Proposition 2.8.4, we only need to show that if T is an invertible positive operator on H and the transformation $A \mapsto TAT$ is a spectral Thompson isometry, then T is necessarily a positive scalar multiple of the identity.

Pick any rank one projection P_x on H, where $x \in S_H$. Clearly, $d_{sT}(I + P_x, I) = \log 2$. Therefore, we have $d_{sT}(T(I + P_x)T, T^2) = \log 2$. This implies that $T(I + P_x)T \preccurlyeq 2T^2$. In particular, we have $(T(I + P_x)T)^2 \leq 4T^4$ from which we infer $(I + P_x)T^2(I + P_x) \leq 4T^2$. Multiplying this inequality by P_x from both sides we obtain the equality $4P_xT^2P_x = 4P_xT^2P_x$. It is easy to see that this implies that $(I + P_x)T^2(I + P_x)x = 4T^2x$ which gives $P_xT^2x = T^2x$, i.e., $\langle T^2x, x\rangle x = T^2x$. Since this holds for all $x \in S_H$, just as in the last part of the proof of Proposition 2.7.1, we deduce that T^2 is a positive scalar multiple of the identity and then we arrive at the desired conclusion. ■

Chapter 3

Means of Kubo-Ando type for spectral order

We have already encoutered some examples of operator means earlier in this thesis. Means are very important objects in mathematics. Means of positive numbers are of central importance in mathematics, and the theory of means have been long studied. The three most fundamental means are the arithmetic, the harmonic and the geometric means. The arithmetic and the harmonic means (see [1]) can be naturally defined for positive operators, at least for positive definite operators. Let H be a complex Hilbert space and $A, B \in B(H)^{++}$ (the set of positive definite operators on H), the arithmetic mean and the harmonic mean of A and B are respectively defined by

$$A \nabla B = \frac{1}{2}(A + B), \quad A!B = 2(A^{-1} + B^{-1})^{-1}.$$

The definition of a geometric mean for positive definite operators is not as straightforward. Pusz and Woronowitz [29] introduced the following definition (see also Ando [3]),

$$A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Ando proved in [3] Corollary I.2.1 and Corollary I.3.1 that the above harmonic and geometric means for positive definite operators extend to positive operators and they share the following properties, which are also obviously satisfied by the arithmetic mean. If σ is the arithmetic mean ∇ or the harmonic mean $!$ or the geometric mean $\#$, then

(C0) $I\sigma I = I$;

(C1) if $A \leq C$ and $B \leq D$, then $A\sigma B \leq C\sigma D$;

(C2) $S(A\sigma B)S \leq (SAS)\sigma(SBS)$;

(C3) if $A_n \downarrow A$ and $B_n \downarrow B$, then we have $A_n\sigma B_n \downarrow A\sigma B$ (here, $A_n \downarrow A$ means that $A_n \geq A_{n+1}$, for $n \geq 0$, and that A_n converges strongly to A),

where $A, B, C, D, S \in B(H)^+$(the set of positive operators on the Hilbert space H), I is the identity operator, and $\leq$ denotes the usual order. Based on these common properties of the three most fundamental means, Kubo and Ando developped a theory of means for positive operators in [14], they did an axiomatic study of binary operations on positive operators which satisfy (C0) to (C3), these operations are now well known as Kubo-Ando means.

The Kubo-Ando means are closely related to the usual order, we immediately see that from (C1), (C2) and (C3) above. It turns out that Kubo-Ando means are also preserved by the isomorphisms of $B(H)^+$ with respect to the usual order. In fact, if σ is such a mean and S is a bijective linear or conjugate linear operator on H then we have

$$S(A\sigma B)S^* = (SAS^*)\sigma(SBS^*), \tag{3.1}$$

for all $A, B \in B(H)^+$ (See Theorem 3.5 in [14]). By the structure theorem for order isomorphisms of the set of positive operators formulated in Theorem 1.2.1, the maps $A \longmapsto SAS^*$, where S is a bijective linear or conjugate linear operator on H, are exactly the isomorphisms of $B(H)^+$ with respect to the usual order.

Perhaps the most important fact about Kubo-Ando means is the following. A countinuous real valued function f is said to be operator monotone on $[0, \infty[$ if

$$A \leq B \implies f(A) \leq f(B), \quad A, B \in B(H)^+, \tag{3.2}$$

and Kubo and Ando proved that to each Kubo-Ando mean σ there is a operator monotone function f_σ associated (with $f_\sigma(1) = 1$). This correspondence is bijective, and we have

$$A\sigma B = A^{1/2} f_\sigma(A^{-1/2} B A^{-1/2}) A^{1/2}, \quad A, B \in B(H)^{++}.$$

The arithmetic, the harmonic and the geometric mean correspond respectively to the operator monotone functions

$$\frac{1+t}{2}, \quad \frac{2t}{1+t}, \quad \sqrt{t}, \quad t \in [0, +\infty[.$$

There are several authors who investigated generalisations and extensions of the Kubo-Ando theory. For example, multivariable extensions are presented in [27, 28], the notion of solidarity which generalises Kubo-Ando means is also studied in [8]. The present paper arose from the desire to investigate operator means of Kubo-Ando type but in relation to the spectral order.

In this chapter, we will see that the lattice operations on $B(H)^+$, that we will denote by $A \vee B$ and $A \wedge B$ (respectively, these are the supremum and the infimum of the set $\{A, B\}$ with respect to the spectral order, these are also called the join and the meet of A and B), present similar behavours to the arithmetic and harmonic means. We also prove that they satisfy a spectral order analogue of the properties (C0) to (C3) (see Proposition 3.2.1). It is clear what are the spectral order analogues of property (C0), (C1) and (C3), one only needs to put the spectral order in the place of the usual order. As for (C2), we derive a transformer inequality for the spectral order and prove that it is satisfied by the lattice operations. Then lastly, we prove the main result Theorem 3.2.2, which gives a characterisation of the lattice operations as the only operations, in regard of the spectral order, similar to the original Kubo-Ando means.

3.1 Similarity between the lattice operations and the arithmetic and harmonic means

Recall that the spectral order induces a lattice structure on $B(H)^+$, we argue that the join $\vee$ and meet $\wedge$ operations present some interesting connections to the arithmetic and harmonic means.

Firstly, in terms of formula, we know that the meet operation can be obtained as limit of arithmetic means and the join operation as limit of harmonic means. More precisely we have the following results (whose proofs can be found, for example, in [9]). Consider $A, B \in B(H)^+$, we have that

$$A \wedge B = \lim_{p \longrightarrow +\infty} (A^p \,!\, B^p)^{1/p}, \quad \text{and}\ , \quad A \vee B = \lim_{p \longrightarrow +\infty} (A^p \,\triangledown\, B^p)^{1/p}. \tag{3.3}$$

The limits are taken with respect to the strong operator topology.

Secondly, we have the following interesting similarity between the harmonic mean and the join operation. Ando proved in [3] Theorem I.3 that

$$A\,!\,B = \max\left\{X \geq 0 : \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \geq \begin{bmatrix} X & X \\ X & X \end{bmatrix}\right\}. \tag{3.4}$$

Since $B(H)^+$ with the usual order is not a lattice (see [12]), even the existence of a supremum here is not trivial. If we switch from the usual order in equation (3.4) to the spectral order, then it is clear that the set

$$\left\{X \succcurlyeq 0 : \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \succcurlyeq \begin{bmatrix} X & X \\ X & X \end{bmatrix}\right\}$$

is bounded from above, hence the corresponding supremum with respect to the spectral order is guaranteed to exist. We prove that the supremum is actually a maximum, and it yields something familiar which is $A \wedge B$. For positive operators $A, B \in B(H)^+$, let us denote

$$\mathbb{H}_\infty(A,B) := \sup\left\{X \succcurlyeq 0 : \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \succcurlyeq \begin{bmatrix} X & X \\ X & X \end{bmatrix}\right\}.$$

3.1.1 Proposition. *For $A, B \in B(H)^+$, $\mathbb{H}_\infty(A,B)$ is equal to $A \wedge B$, and it is indeed a max, not just a sup, i.e.,*

$$A \wedge B = \max\left\{X \succcurlyeq 0 : \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \succcurlyeq \begin{bmatrix} X & X \\ X & X \end{bmatrix}\right\}.$$

Proof. We first prove that

$$\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \succcurlyeq \begin{bmatrix} A \wedge B & A \wedge B \\ A \wedge B & A \wedge B \end{bmatrix}, \tag{3.5}$$

hence $A \wedge B \preccurlyeq \mathbb{H}_\infty(A,B)$. From the remark following Proposition 2.1.1 in Chapter 2, recall that for a positive operator $T \in B(H)^+$ and $t \geq 0$, we have

$$\operatorname{rng} E_T(]-\infty, t]) = \left\{x \in H \;:\; \|T^n x\| \leq t^n \|x\|, n \geq 1\right\}.$$

If

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \operatorname{rng} E_{\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}}(]-\infty, t])$$

then $x \in \operatorname{rng} E_A(]-\infty, t/2])$ and $y \in \operatorname{rng} E_B(]-\infty, t/2])$, hence $\|A^n x\| \leq (t/2)^n \|x\|$ and $\|B^n y\| \leq (t/2)^n \|y\|$, for $n \geq 1$. It follows that

$$\begin{aligned}\left\| \begin{bmatrix} A\wedge B & A\wedge B \\ A\wedge B & A\wedge B \end{bmatrix}^n \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 &= 2\|2^{n-1}(A\wedge B)^n(x+y)\|^2 \\ &\leq 4\|2^{n-1}(A\wedge B)^n x\|^2 + 4\|2^{n-1}(A\wedge B)^n y\|^2 \\ &\leq 4\|2^{n-1}A^n x\|^2 + 4\|2^{n-1}B^n y\|^2 \\ &= \|2^n A^n x\|^2 + \|2^n B^n y\|^2 \\ &\leq t^{2n}(\|x\|^2 + \|y\|^2) = t^{2n} \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 .\end{aligned}$$

This shows that

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \operatorname{rng} E_{\begin{bmatrix} A\wedge B & A\wedge B \\ A\wedge B & A\wedge B \end{bmatrix}}(]-\infty, t]).$$

It follows that we have

$$\operatorname{rng} E_{\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}}(]-\infty, t]) \subset \operatorname{rng} E_{\begin{bmatrix} A\wedge B & A\wedge B \\ A\wedge B & A\wedge B \end{bmatrix}}(]-\infty, t]),$$

for all $t \geq 0$, which proves the inequality (3.5).

Now we prove that $\mathbb{H}_\infty(A, B) \preccurlyeq A \wedge B$. If X is not less that $A \wedge B$, say X is not less that A, then there exists $t \geq 0$ and a unit vector x such that $x \in \operatorname{rng} E_A(]-\infty, t])$ but $x \notin \operatorname{rng} E_X(]-\infty, t])$. According to the remark following Proposition 2.1.1, we see that this is equivalent to

$$\|A^n x\| \leq t^n \|x\|, \quad \text{and} , \quad \limsup_n \frac{1}{t^n}\|X^n x\| = \infty.$$

Now we compute

$$\left\| \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}^n \begin{bmatrix} x \\ 0 \end{bmatrix} \right\| = \|2^n A^n x\| \leq (2t)^n \|x\|,$$

showing that

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \in \operatorname{rng} E_{\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}}(]-\infty, 2t]).$$

But

$$\frac{1}{2^n t^n}\left\|\begin{bmatrix} X & X \\ X & X \end{bmatrix}^n \begin{bmatrix} x \\ 0 \end{bmatrix}\right\| = \frac{\sqrt{2}}{2^n t^n}\|2^{n-1}X^n x\| = \frac{\sqrt{2}}{2t^n}\|X^n x\|,$$

so that

$$\limsup_n \frac{1}{2^n t^n}\left\|\begin{bmatrix} X & X \\ X & X \end{bmatrix}^n \begin{bmatrix} x \\ 0 \end{bmatrix}\right\| = \infty,$$

therefore

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \notin \operatorname{rng} E_{\begin{bmatrix} X & X \\ X & X \end{bmatrix}}]-\infty, 2t].$$

It then follows that X is not less that $\mathbb{H}_\infty(A, B)$. We then proved that $X \preccurlyeq \mathbb{H}_\infty(A, B)$ implies that $X \preccurlyeq A \wedge B$ so that $\mathbb{H}_\infty(A, B) \preccurlyeq A \wedge B$. ■

Let us also point out an other connection between $\vee, \wedge$ and the arithmetic and harmonic means. We studied the following representations of positive operators by positive real valued functions: for any $A \in B(H)^+$ we define

$$w(A, x) = \langle Ax, x\rangle, \quad x \in S_H;$$

$$\lambda(A, x) = \sup\{t \geq 0 \,:\, tP_x \leq A\}, \quad x \in S_H.$$

We considered the maps $A \longmapsto w(A, .)$ and $A \longmapsto \lambda(A, .)$. These are faithful representations of $B(H)^+$ into the set of nonnegative real valued function on S_H, and they determine the usual order on $B(H)^+$ in the following way: for $A, B \in B(H)^+$, we have that $A \leq B$ if and only if $w(A, .) \leq w(B, .)$, if and only if $\lambda(A, .) \leq \lambda(B, .)$. Besides the order determining property of these representations, we also proved that w and λ preserve the arithmetic mean and the harmonic mean respectively, i.e. for all $A, B \in B(H)^+$, we have

$$w(A\nabla B, \cdot) = w(A, \cdot)\nabla w(B, \cdot), \quad \text{and} \quad \lambda(A!B, \cdot) = \lambda(A, \cdot)!\lambda(B, \cdot). \tag{3.6}$$

Then we also defined the following functions:

$$r(A, x) = \lim_n \|A^n x\|^{1/n}, \quad x \in S_H;$$

$$\nu(A, P) = \sup\{t \geq 0 \,:\, tP_x \preccurlyeq A\}, \quad x \in S_H.$$

The function $r(A, .)$ is the local spectral radius function of A, and the function $\nu(A, .)$ is the spectral order analogue of the strength function, which we call the spectral strength function of A. We showed that the representations $A \longmapsto r(A, .)$ and $A \longmapsto \nu(A, .)$ are also faithful representations of $B(H)^+$ into the set of non-negative real valued functions on S_H and they determine the spectral order in the sense that $A \preccurlyeq B$ if and only if $r(A, .) \leq r(B, .)$, if and only if $\nu(A, .) \leq \nu(B, .)$. Interestingly, in analogy with the previous case, the representations r and ν also preserve the join and the meet operations, respectively, i.e. for all $A, B \in B(H)^+$, we have

$$r(A \vee B, .) = r(A, .) \vee r(B, .), \quad \text{and} \quad \nu(A \wedge B, .) = \nu(A, .) \wedge \nu(B, .). \tag{3.7}$$

We clearly see in equations (3.6) and (3.7) the analogy between the arithmetic mean and the meet operation, the harmonic mean and the join operation. These are more evidences supporting the claim that the meet and join operations play analogous roles to the arithmetic and harmonic means.

3.2 Characterisation of the meet and join operations

In the first part of this section, we will prove that $\vee$ and $\wedge$ satisfy a spectral order version of the conditions (C0) to (C3). Looking at the conditions (C0) to (C3), we see that conditions (C0),(C1) have natural analogues in the setting of the spectral order. At this point it is not clear what should be the correct analogue of condition (C2), the so called *transformer inequality*

(C2) $S(A\sigma B)S \leq (SAS)\sigma(SBS)$; for all $A, B, S \in B(H)^+$.

This inequality expresses the relationship of the given operation σ with the order $\leq$ and the maps $\mathcal{C}_S : A \longmapsto SAS$, where $S \in B(H)^+$. We then need to see what is the analogue of the maps $\mathcal{C}_S$ for the spectral order. We observe that, when $S \in B(H)^+$ is invertible, the map $\mathcal{C}_S$ is an isomorphism of $B(H)^+$ with respect to the usual order. In fact, as formulated in Theorem 1.2.1 in Chapter 1, these are all the order isomorphisms of $B(H)^+$ with respect to the usual order, up to unitary or anti-unitary equivalence. So the maps $\mathcal{C}_S$ can be construed as weakened order isomorphisms, in the sense that their structure is similar to the order isomorphisms, they are still monotone increasing, but they are no longer required to be bijective. We then need to know the form of the spectral order isomorphisms of $B(H)^+$, and then, naturally we replace the congruence maps with some kind of weakened order isomorphisms of $B(H)^+$ with respect to the spectral order. Molnár and Šemrl [24] and Bohata [4] gave the exact structure of a spectral order isomorphism of $B(H)^+$, that we presented in Theorem 1.2.2.

Recall that the spectral order isomorphisms of $B(H)^+$ are the maps $A \longmapsto \psi_S(f(A))$ where $S : H \longrightarrow H$ is a bounded bijective linear or conjugate linear map if the dimension of H is infinite and a bijective semilinear map if the dimension of H is finite, and f is a nondecreasing continuous bijective map on $[0, +\infty[$. The operator $\psi_S(A)$ is defined as the operator whose spectral resolution is given by

$$t \longmapsto I - P_{S(\operatorname{rng} E_A(]t,\infty[))}, \quad t \in \mathbb{R},$$

When $S \in B(H)^{++}$, it was proved in [24] Proposition 2, that the sequence $(SA^nS)^{1/n}_{n\geq 1}$ is increasing with respect to the usual order and converges to $\psi_S(A)$ in the strong operator topology,

$$\psi_S(A) = \lim_{n \longrightarrow +\infty} (SA^nS)^{1/n}, \quad A \in B(H)^+. \tag{3.8}$$

An examination of the proof reveals that S need not to be invertible for these to hold. Therefore the equality in (3.8) is still true for all $S \in B(H)^+$, and hence ψ_S is also well defined for $S \in B(H)^+$. This naturally suggests that in our quest of a spectral order analogue of the transformer inequality, the congruence maps should be replaced by the maps $A \longmapsto \psi_S(f(A)) = \lim_n (Sf(A)^nS)^{1/n}$, where we no longer require $S \in B(H)^+$ to be invertible and where f is no longer required to be bijective but only continuous increasing with $f(0) = 0$.

3.2.1 Proposition. *Let $\hat{\sigma}$ be the join operation $\vee$ or the meet operation $\wedge$ on $B(H)^+$, then $\hat{\sigma}$ satisfies the following properties:*

(C'0) normalisation: $I\hat{\sigma}I = I$.

(C'1) monotonicity: $A \preccurlyeq C$ and $B \preccurlyeq D$ implies that $A\hat{\sigma}B \preccurlyeq C\hat{\sigma}D$,

(C'2) transformer inequality: $\psi_S(f(A\hat{\sigma}B)) \preccurlyeq \psi_S(f(A))\hat{\sigma}\psi_S(f(B))$ where $S \in B(H)^+$ and $f : [0, +\infty[\longrightarrow [0, +\infty[$ is a continuous increasing function with $f(0) = 0$.

(C'3) upper continuity: if a_n and b_n, $n \in \mathbb{N}$, are decreasing positive numbers which converges to a and b respectively, then $(a_nI)\hat{\sigma}(b_nI) \downarrow (aI)\hat{\sigma}(bI)$.

Before presenting the proof of the Proposition, we make the following remark. We know that for the arithmetic mean, we always have equality in the transformer inequality. Interestingly, we also have the same behaviour of the join $\vee$ operation, that is

$$\psi_S(f(A \vee B)) = \psi_S(f(A)) \vee \psi_S(f(B)), \tag{3.9}$$

for all $S \in B(H)^+$ and for all nondecreasing continuous function $f : [0, +\infty[\longrightarrow [0, +\infty[$ with $f(0) = 0$.

Proof. It is clear that the lattice operations satisfy the properties (C0'),(C1') and (C'3). We only need to prove (C'2). We start by recalling Hansen's inequality from [10], for $A, S \in B(H)^+$ with $\|S\| \leq 1$, and a function f operator monotone on $[0, +\infty[$, we have

$$Sf(A)S \leq f(SAS). \tag{3.10}$$

Now, let $A, B, S \in B(H)^+$, with $\|S\| \leq 1$ and $f : [0, +\infty[\longrightarrow [0, +\infty[$. We have seen that the sequence $(SA^nS)^{1/n}, n \geq 1,$ is increasing and converges to $\psi_S(A)$ in the strong operator topology. Since multiplication is continuous with respect to the strong operator topology on bounded subsets, given an integer $m \geq 1$ and taking into account equation (3.8), we have that $(SA^nS)^{m/n}$ strongly converges to $\psi_S(A)^m$ as $n \longrightarrow \infty$. Therefore if $A \preccurlyeq B$, then $SA^nS \leq SB^nS$ for all $n \geq 1$, and for $n \geq m$, the function $f(t) = t^{m/n}$ is operator monotone, hence $(SA^nS)^{m/n} \leq (SB^nS)^{m/n}$. This implies that $\psi_S(A)^m \leq \psi_S(B)^m$, this holds for arbitrary $m \geq 1$, therefore $\psi_S(A) \preccurlyeq \psi_S(B)$. Recall also from [26] Corollary 1 that the map $X \in B(H)^+ \longmapsto f(X)$ is monotone increasing (with respect to the spectral order) when f is continuous increasing on $[0, +\infty[$. Therefore, the map $B(H)^+ \ni X \longmapsto \psi_S(f(X))$ is also monotone increasing. We then have

$$\psi_S(f(A \wedge B)) \preccurlyeq \psi_S(f(A)) \wedge \psi_S(f(B)),$$

and

$$\psi_S(f(A \vee B)) \succcurlyeq \psi_S(f(A)) \vee \psi_S(f(B)).$$

It now remains to prove that $\psi_S(f(A \vee B)) \preccurlyeq \psi_S(f(A)) \vee \psi_S(f(B))$. Let $R \succcurlyeq \psi_S(A), \psi_S(B)$, and let $k \leq l \leq m \leq n \in \mathbb{N}$, we have $R^m \geq \psi_S(A)^m \geq (SA^nS)^{m/n}$. Since the function $t \longmapsto f(t) = t^{m/n}$ is operator monotone on $[0, +\infty[$, by Hansen inequality, $(SA^nS)^{m/n} \geq SA^mS$, the same holds for B. It follows that

$$R^m \geq S\frac{A^m + B^m}{2}S.$$

By operator monotonicity of the function $t \longmapsto t^{k/m}$, we have

$$R^k \geq \left(S\frac{A^m + B^m}{2}S\right)^{k/m} = \left(S\frac{A^m + B^m}{2}S\right)^{k/l \times l/m},$$

and applying Hansen inequality to the operator monotone function $t \longmapsto t^{l/m}$, we obtain

$$R^k \geq \left(S \left(\frac{A^m + B^m}{2} \right)^{l/m} S \right)^{k/l}, \quad k \leq l \leq m \in \mathbb{N}.$$

The sequence

$$S \left(\frac{A^m + B^m}{2} \right)^{l/m} S$$

is known to converge to $S(A \vee B)^l S$ as $m \longrightarrow \infty$, so that $R^k \geq \left(S(A \vee B)^l S\right)^{k/l}$, and the sequence $\left(S(A \vee B)^l S\right)^{k/l}$ converges to $(\psi_S(A \vee B))^k$ as $l \longrightarrow \infty$, we then have $R^k \geq (\psi_S(A \vee B))^k$. This holds for arbitrary k therefore we conclude that $R \succcurlyeq \psi_S(A \vee B)$, which proves that $\psi_S(A) \vee \psi_S(B) \succcurlyeq \psi_S(A \vee B)$. Recall that ψ_S is monotone (with respect to the spectral order) so we always have $\psi_S(A) \vee \psi_S(B) \preccurlyeq \psi_S(A \vee B)$, therefore

$$\psi_S(A \vee B) = \psi_S(A) \vee \psi_S(B). \tag{3.11}$$

Next we prove that $f(A \vee B) = f(A) \vee f(B)$. We can always consider f to be defined on $\mathbb{R}$, continuous and nondecreasing, by setting $f(t) = 0$ for $t < 0$. Define the function

$$f^*(s) = \sup\{t \in \mathbb{R};\ f(t) \leq s\}.$$

Given a positive operator $T \in B(H)^+$, if $f^*(s) < +\infty$ then $E_{f(T)}(]-\infty, s]) = E_T(]-\infty, f^*(s)])$. In this case, we have successively

$$\begin{aligned} E_{f(A \vee B)}(]-\infty, s]) &= E_{A \vee B}(]-\infty, f^*(s)]) \\ &= E_A(]-\infty, f^*(s)]) \wedge E_B(]-\infty, f^*(s)]) \\ &= E_{f(A)}(]-\infty, s]) \wedge E_{f(B)}(]-\infty, s]) \\ &= E_{f(A) \vee f(B)}(]-\infty, s]). \end{aligned}$$

If $f^*(s) = +\infty$, this means that $f(t) \leq s$ for all $t \in \mathbb{R}$, then $\|f(A \vee B)\|, \|f(A) \vee f(B)\| \leq s$ so that $E_{f(A \vee B)}(]-\infty, s]) = E_{f(A) \vee f(B)}(]-\infty, s]) = I$. It then follows that

$$f(A \vee B) = f(A) \vee f(B). \tag{3.12}$$

Combining equations (3.11) and (3.12), we finally get (3.9). ∎

Now our goal is to give a complete list of all the binary operations on $B(H)^+$ which satisfy the same properties as the meet and the join operations given in Proposition 3.2.1. We easily see that the left mean $A\hat{\sigma}B = A$ and the right mean $A\hat{\sigma}B = B$ are examples of such operations, besides the meet and the join. Next, we prove the following theorem which basically says that the conditions (C′0) to (C′3) characterise the four binary operations mentioned above.

3.2.2 Theorem. *Let $\hat{\sigma}$ be a binary operation on $B(H)^+$ that satisfies the conditions (C′0), (C′1), (C′2) and (C′3) of Proposition 3.2.1. Then the operators $0\hat{\sigma}I$ and $I\hat{\sigma}0$ can only be 0 or I. We have the following cases:*

1. *if $0\hat{\sigma}I = I\hat{\sigma}0 = I$, then $A\hat{\sigma}B = A \vee B$ for all $A, B \in B(H)^+$.*
2. *if $0\hat{\sigma}I = I\hat{\sigma}0 = 0$, then $A\hat{\sigma}B = A \wedge B$ for all $A, B \in B(H)^+$.*
3. *if $0\hat{\sigma}I = I$ and $I\hat{\sigma}0 = 0$, then $A\hat{\sigma}B = B$ for all $A, B \in B(H)^+$.*
4. *if $0\hat{\sigma}I = 0$ and $I\hat{\sigma}0 = I$, then $A\hat{\sigma}B = A$ for all $A, B \in B(H)^+$.*

We make some remarks before presenting the proof.

Firstly, the binary operations on $B(H)^+$ which satisfy the conditions of Theorem 3.2.2 are in some sense the "Kubo-Ando means" with respect to the spectral order. It is a desirable property for such an operation $\hat{\sigma}$ to be positive homogenuous, i.e.

$$\alpha(A\hat{\sigma}B) = (\alpha A)\hat{\sigma}(\alpha B), \quad A, B \in B(H)^+, \alpha > 0. \tag{3.13}$$

For the original Kubo-Ando means, the positive homogeneity follows from the transformer inequality. This is also the case for the spectral order analogues. In fact we have the stronger property that if h is a bijective increasing function on $[0, +\infty[$ then

$$h(A\hat{\sigma}B) = h(A)\hat{\sigma}h(B), \quad A, B \in B(H)^+. \tag{3.14}$$

Then the positive homogeneity property in equation (3.13) is obtained by taking the function $h(t) = \alpha t$. To prove the identity (3.14), consider a bijective increasing function h on $[0, +\infty[$, by the transformer inequality (C′2), we have

$$h(A\hat{\sigma}B) = \psi_I(h(A\hat{\sigma}B)) \preccurlyeq \psi_I(h(A))\hat{\sigma}\psi_I(h(B)) = h(A)\hat{\sigma}h(B), \quad A, B \in B(H)^+.$$

Since the inverse h^{-1} of h is also bijective increasing on $[0, +\infty[$, we can apply this inequality where we replace h by h^{-1} and A, B by $h(A), h(B)$, we then have

$$h^{-1}(h(A)\hat{\sigma}h(B)) \preccurlyeq h^{-1}(h(A))\hat{\sigma}h^{-1}(h(B)) = A\hat{\sigma}B = h^{-1}(h(A\hat{\sigma}B)), \quad A, B \in B(H)^+.$$

Given that the map $X \longmapsto h^{-1}(X)$ is a spectral order isomorphism of $B(H)^+$, we have that $h(A)\hat{\sigma}h(B) \preccurlyeq h(A\hat{\sigma}B)$, for all $A, B \in B(H)^+$. Hence, given a bijective increasing function h on $[0, +\infty[$, we have (3.14).

Secondly, our proof makes heavy use of the properties of the spectral strength function ν and the local spectral radius function r. Let us recall the properties that will be useful for us.

Let us start with the representation ν. It has the order determining property, i.e., for $A, B \in B(H)^+$, we have $A \preccurlyeq B$ if and only if $\nu(A, x) \leq \nu(B, x)$ for all $x \in S_H$. We always have that $\nu(A, x)P_x \preccurlyeq A$, in fact by definition, $\nu(A, x)P_x \preccurlyeq A$ is a maximal rank one operator below A and A can be recovered from those rank one operators. It is easy to prove that

$$A = \bigvee_{x \in S_H} \nu(A, x)P_x. \tag{3.15}$$

Indeed, if $\nu(A, x)P_x \preccurlyeq B$ for all $x \in S_H$ then $\nu(A, x) \leq \nu(B, x)$ by definition, hence $A \preccurlyeq B$ which shows that A is the least upper bound in the previous equation.

The representation r also has the order determining property, $A \preccurlyeq B$ if and only if $r(A, x) \leq r(B, x)$ for all $x \in S_H$. Although there is no direct way to recover the operator from its local spectral radius function $r(A, .)$ as it is the case for the spectral strength function $\nu(A, .)$, there are useful information one can extract from this function. For example, the kernel of A can be computed using $r(A, .)$, it is easy to see that $x \in \ker A \cap S_H$ if and only $r(A, x) = 0$. Indeed, if $Ax = 0$ then $\|A^n x\|^{1/n} = 0$ for all $n \geq 1$ hence $r(A, x) = 0$, and conversely if $r(A, x) = 0$ then $\|Ax\| \leq r(A, x) = 0$ hence $Ax = 0$. We then see that

$$\ker A = \operatorname{span}\{x \in S_H \ : \ r(A, x) = 0\}. \tag{3.16}$$

In general, we have $\nu(A, x) \leq r(A, x)$ for all $x \in S_H$. The fact that we have the reverse inequality for all $x \in S_H$ means that the operator A is a multiple of the identity operator (see Section 2.7 in Chapter 2).

In the proof of Theorem 3.2.2, we will also need the following two Propositions.

3.2.3 Proposition. *Let $x \in S_H$ and let $A \in B(H)^+$, we have $\psi_{P_x}(A) = r(A, x)P_x$.*

Proof. We first compute $P_xA^{2k}P_xy$ for an arbitrary $y \in H$

$$\begin{aligned} P_xA^{2k}P_xy &= \langle y, x\rangle\, P_xA^{2k}x \\ &= \langle y, x\rangle \left\langle A^{2k}x, x\right\rangle x \\ &= \|A^kx\|^2 \langle y, x\rangle\, x \\ &= \|A^kx\|^2 P_xy. \end{aligned}$$

We see that the subsequence $(P_xA^{2k}P_x)_k^{1/2k}$ of the converging sequence $(P_xA^nP_x)_n^{1/n}$ is converging to $\lim \|A^kx\|^{1/k}P_x = r(A,x)P_x$. Hence $(P_xA^nP_x)_n^{1/n}$ admits the same limit and therefore $\psi_{P_x}(A) = \lim_n\,(P_xA^nP_x)^{1/n} = r(A,x)P_x$, in the strong operator topology. ■

3.2.4 Proposition. *Let $\hat{\sigma}$ be a binary operation on $B(H)^+$ which satisfies the conditions (C′0), (C′1), (C′2) and (C′3) of Proposition 3.2.1 and let $s, t \geq 0$. The operator $(sI)\hat{\sigma}(tI)$ is a multiple of I, and for a rank one projection $P \in P_1(H)$, $(sP)\hat{\sigma}(tP)$ is a multiple of P. There is a function*

$$f_{\hat{\sigma}} : [0, +\infty[\times[0, +\infty[\longrightarrow [0, +\infty[,$$

such that, for $s, t \geq 0$, we have $sI\hat{\sigma}tI = f_{\hat{\sigma}}(s,t)I$ and if $P \in P_1(H)$ then $(sP)\hat{\sigma}(tP) = f_{\hat{\sigma}}(s,t)P$.

Proof. Let $x \in S_H$, and $r, s > 0$, taking into account the above proposition and using the transformer inequality, we have

$$r(sI\hat{\sigma}tI, x)P_x = \psi_{P_x}(sI\hat{\sigma}tI) \preccurlyeq \psi_{P_x}(sI)\hat{\sigma}\psi_{P_x}(tI) = r(sI,x)P_x\hat{\sigma}r(tI,x)P_x.$$

It is clear that $r(sI,x)P_x\hat{\sigma}r(tI,x)P_x = sP_x\hat{\sigma}tP_x$, and by the monotonicity property, we have $sP_x\hat{\sigma}tP_x \preccurlyeq sI\hat{\sigma}tI$. Therefore we obtain that $r(sI\hat{\sigma}tI, x)P_x \preccurlyeq sI\hat{\sigma}tI$ for all $x \in S_H$. This means that $r(sI\hat{\sigma}tI, x) \leq \nu(sI\hat{\sigma}tI, x)$, for all $x \in S_H$, which shows that $sI\hat{\sigma}tI$ is a scalar multiple of the identity.

Now consider a rank one projection $P = P_x$. By the transformer inequality, we have

$$r(sP\hat{\sigma}tP, y)P_y = \psi_{P_y}(sP\hat{\sigma}tP) \preccurlyeq \psi_{P_y}(sP)\hat{\sigma}\psi_{P_y}(tP) = r(sP,y)P_y\hat{\sigma}r(tP,y)P_y.$$

Let $y \in S_H$ such that $\langle y, x\rangle = 0$, then $r(P,y) = 0$ and from the above equation, we conclude that $r(sP\hat{\sigma}tP, y) = 0$. According to equation (3.16), this means that the orthogonal complement of x is contained in $\ker(sP\hat{\sigma}tP)$ and hence the range of $sP\hat{\sigma}tP$ is a subspace of $\mathbb{C}x = \operatorname{rng} P$. This shows that $sP\hat{\sigma}tP$ is an operator, with rank at most one, whose range is contained in the range of P, it is then necessarily a multiple of P.

Let $f_{\hat{\sigma}}$ and $g_{\hat{\sigma}}$ be the functions such that $sI\hat{\sigma}tI = f_{\hat{\sigma}}(s,t)I$ and $sP_x\hat{\sigma}tP_x = g_{\hat{\sigma}}(s,t)P_x$. We want to prove that these functions are equal. On one hand, we have

$$f_{\hat{\sigma}}(s,t)P_x = r(sI\hat{\sigma}tI, x)P_x \preccurlyeq r(sI,x)P_x\hat{\sigma}r(tI,x)P_x = sP_x\hat{\sigma}tP_x = g_{\hat{\sigma}}(s,t)P_x.$$

On the other hand we have

$$g_{\hat{\sigma}}(s,t)P_x = sP_x\hat{\sigma}tP_x \preccurlyeq sI\hat{\sigma}tI = f_{\hat{\sigma}}(s,t)I.$$

Taking the norms in these equalities gives $f_{\hat{\sigma}}(s,t) = g_{\hat{\sigma}}(s,t)$. ∎

We will also need the following fact: for any $A \in B(H)^+$, we have $A\hat{\sigma}A = A$, as we now show. Recall that, since $I\hat{\sigma}I = I$, then $f_{\hat{\sigma}}(1,1) = 1$ so we have $P_x\hat{\sigma}P_x = f_{\hat{\sigma}}(1,1)P_x = P_x$ for all $x \in S_H$. Given $A \in B(H)^+$ and $x \in S_H$, we have

$$\begin{aligned} \nu(A,x)P_x &= \nu(A,x)(P_x\hat{\sigma}P_x) \\ &= \nu(A,x)P_x\hat{\sigma}\nu(A,x)P_x \\ &\preccurlyeq A\hat{\sigma}A, \end{aligned}$$

hence, by definition of the spectral strength function we have $\nu(A,x) \leq \nu(A\hat{\sigma}A, x)$. By the transformer inequality, we also have

$$r(A\hat{\sigma}A, x)P_x \preccurlyeq r(A,x)P_x\hat{\sigma}r(A,x)P_x = r(A,x)(P_x\hat{\sigma}P_x) = r(A,x)P_x.$$

Comparing the spectral strength functions of A and $A\hat{\sigma}A$ we see that $A \preccurlyeq A\hat{\sigma}A$ and comparing the local spectral radius functions we get that $A\hat{\sigma}A \preccurlyeq A$. Hence $A\hat{\sigma}A = A$. In particular, we have $0\hat{\sigma}0 = 0$.

Now we are ready to prove Theorem 3.2.2.

Proof of Theorem 3.2.2. Let $f_{\hat{\sigma}}$ be the function corresponding to $\hat{\sigma}$ from Proposition 3.2.4, let $j(t) = f_{\hat{\sigma}}(1,t)$ and $b(t) = f_{\hat{\sigma}}(t,1)$. We are going to prove that $j(0)$ and $b(0)$ can only be equal to 0 or 1. This means that the operators $0\hat{\sigma}I = b(0)I$ and $I\hat{\sigma}0 = j(0)I$ only take the value 0 or I.

We know from the upper continuity (C'3) of $\hat{\sigma}$ that j and b are right continuous. Thanks to the positive homogeneity of $\hat{\sigma}$ (see equation (3.13)) we have that $f_{\hat{\sigma}}$ is also positive homogenous so that we have for $t > 0$

$$j(t) = f_{\hat{\sigma}}(1,t) = t f_{\hat{\sigma}}(1/t, 1) = t b(1/t),$$

and

$$b(t) = f_{\hat{\sigma}}(t,1) = t f_{\hat{\sigma}}(1, 1/t) = t j(1/t).$$

From these identities and the right continuity of j and b, we see that j and b are also left continuous on $]0, +\infty[$.

Now, let h be a bijective increasing function on $[0, +\infty[$. From equation (3.14), we have

$$h(b(0))I = h(b(0)I) = h(0\hat{\sigma}I) = h(0)\hat{\sigma}h(I) = 0\hat{\sigma}h(1)I,$$

and by homogeneity of $\hat{\sigma}$, we have

$$0\hat{\sigma}h(1)I = h(1)\ (0\hat{\sigma}I) = h(1)\ (b(0)I).$$

This means that $\frac{h(b(0))}{h(1)} = b(0)$, i.e., $b(0)$ is a fixed point of the function $t \longmapsto \frac{h(t)}{h(1)}$. And this is true for an arbitrary h so $b(0)$ is a common fixed point to all functions $t \longmapsto h(t)/h(1)$ where h is an increasing bijection on $[0, +\infty[$. Therefore the only possible values for $b(0)$ are 0 or 1. Similarly, the only possible values for $j(0)$ are 0 or 1.

Case 1: In case where $j(0) = b(0) = 1$, this means that $0\hat{\sigma}I = I\hat{\sigma}0 = I$, we have that $A\hat{\sigma}B = A \vee B$, for all $A, B \in B(H)^+$.

Let $x \in S_H$, and $A, B \in B(H)^+$. Since $0\hat{\sigma}I = I\hat{\sigma}0 = I$, from Proposition 3.2.4, we have that $0\hat{\sigma}P_x = P_x\hat{\sigma}0 = P_x$. Multiplying by $\nu(A,x)$ and using the homogeneity of $\hat{\sigma}$, we get $0\hat{\sigma}\nu(A,x)P_x = \nu(A,x)P_x\hat{\sigma}0 = \nu(A,x)P_x$. We then have

$$\begin{aligned} A = \bigvee_{x \in S_H} \nu(A,x)P_x &= \bigvee_{x \in S_H} \Big(\nu(A,x)P_x\hat{\sigma}0\Big) \\ &\preccurlyeq \left(\bigvee_{x \in S_H} \nu(A,x)P_x\right)\hat{\sigma}0 = A\hat{\sigma}0. \end{aligned}$$

On the other hand, by monotonicity, we have $A\hat{\sigma}0 \preccurlyeq A\hat{\sigma}A$ and per the remark in the paragraph preceding this proof $A\hat{\sigma}A = A$, so that $A = A\hat{\sigma}0$. By a similar argument, we can prove that $B = 0\hat{\sigma}B$. It then follows that $A \vee B = (A\hat{\sigma}0) \vee (0\hat{\sigma}B)$, as $A\hat{\sigma}0 \preccurlyeq A\hat{\sigma}B$ and $0\hat{\sigma}B \preccurlyeq A\hat{\sigma}B$ we have $(A\hat{\sigma}0) \vee (0\hat{\sigma}B) \preccurlyeq A\hat{\sigma}B$, so $A \vee B \preccurlyeq A\hat{\sigma}B$. On the other hand, it is clear that $A\hat{\sigma}B \preccurlyeq (A \vee B)\hat{\sigma}(A \vee B) = A \vee B$, So finally we proved that $A\hat{\sigma}B = A \vee B$.

Case 2: In case where $j(0) = b(0) = 0$, this means that $0\hat{\sigma}I = I\hat{\sigma}0 = 0$, we have that $A\hat{\sigma}B = A \wedge B$, for all $A, B \in B(H)^+$. We only need to prove this for $0 \preccurlyeq A, B \preccurlyeq I$, and use the homogeneity property of $\hat{\sigma}$ to extend it to all $A, B \in B(H)^+$.

We begin by showing that the restriction of the functions b and j on the unit interval $[0,1]$ is the identity function. To see this, let h be a bijective increasing map on $[0,1]$. Such function is the restriction of some bijective increasing function on $[0,+\infty[$, that we will again call h, with $h(0) = 0$ and $h(1) = 1$. We then have

$$b(h(t))I = (h(t)I)\hat{\sigma}I = (h(t)I)\hat{\sigma}(h(1)I) = h(tI)\hat{\sigma}h(I),$$

using (3.14), we obtain

$$h(tI)\hat{\sigma}h(I) = h(tI\hat{\sigma}I) = h(b(t)I) = h(b(t))I.$$

Hence $b(h(t)) = h(b(t))$ for all $t \in [0,1]$, for all increasing bijection h on $[0,1]$. There are only three possibilities, b is the constant function equal to 1 on the unit interval, or b is the constant function 0, or b is the identity function. Since $b(0) = 0$ and $b(1) = f_{\hat{\sigma}}(1,1) = 1$, b can only be the identity function. Similarly, $j(t) = t$ for all $t \in [0,1]$.

Next we prove that for $A, B \preccurlyeq I$ we have $A\hat{\sigma}I = A$ and $I\hat{\sigma}B = B$. Let $x \in S_H$, since $A \preccurlyeq I$, we have $0 \le \nu(A,x) \le 1$ hence $\nu(A,x) = b(\nu(A,x))$, so we have

$$\nu(A,x)P_x = b(\nu(A,x))P_x = f_{\hat{\sigma}}(\nu(A,x),1)P_x = \nu(A,x)P_x\hat{\sigma}P_x.$$

By definition, $\nu(A,x)P_x \preccurlyeq A$, so from the monotonicity of $\hat{\sigma}$ we then have

$$\nu(A,x)P_x = \nu(A,x)P_x\hat{\sigma}P_x \preccurlyeq A\hat{\sigma}I.$$

Hence $\nu(A,x) \le \nu(A\hat{\sigma}I,x)$. This holds for arbitrary $x \in S_H$, therefore, it follows from the order determining property of the representation ν that $A \preccurlyeq A\hat{\sigma}I$. On the other hand, by virtue of Proposition 3.2.3 and the transformer inequality, we have we

$$\begin{aligned} r(A\hat{\sigma}I,x)P_x &= \psi_{P_x}(A\hat{\sigma}I) \\ &\preccurlyeq \psi_{P_x}(A)\hat{\sigma}\psi_{P_x}(I) = r(A,x)P_x\hat{\sigma}r(I,x)P_x = r(A,x)P_x\hat{\sigma}P_x. \end{aligned}$$

We also have that $0 \le r(A,x) \le 1$ hence $r(A,x) = b(r(A,x))$, so that we have $r(A,x)P_x\hat{\sigma}P_x = b(r(A,x))P_x = r(A,x)P_x$. It then follows that $r(A\hat{\sigma}I,x) \le r(A,x)$ for all $x \in S_H$, i.e. $A\hat{\sigma}I \preccurlyeq A$. We then have the equality $A = A\hat{\sigma}I$ and similarly $B = I\hat{\sigma}B$.

By monotony of $\hat{\sigma}$, since $A, B \preccurlyeq I$, we see that $A\hat{\sigma}B \preccurlyeq A\hat{\sigma}I$ and $A\hat{\sigma}B \preccurlyeq I\hat{\sigma}B$ so that $A\hat{\sigma}B \preccurlyeq (A\hat{\sigma}I) \wedge (I\hat{\sigma}B)$. It follows that

$$A \wedge B = (A \wedge B)\hat{\sigma}(A \wedge B) \preccurlyeq A\hat{\sigma}B \preccurlyeq (A\hat{\sigma}I) \wedge (I\hat{\sigma}B) = A \wedge B.$$

We then proved that for $A, B \preccurlyeq I$, we have $A\hat{\sigma}B = A \wedge B$. Since both $\hat{\sigma}$ and $\wedge$ are homogeneous, by a scaling argument, we see that $A\hat{\sigma}B = A \wedge B$ still holds in general for $A, B \in B(H)^+$.

Case 3: In case where $b(0) = 1$ and $j(0) = 0$, this means that $0\hat{\sigma}I = I$ and $I\hat{\sigma}0 = 0$, we have that $A\hat{\sigma}B = B$, for all $A, B \in B(H)^+$. As in the previous case, we only need to prove the equality $A\hat{\sigma}B = B$ for $0 \preccurlyeq A, B \preccurlyeq I$, and use the homogeneity property of $\hat{\sigma}$ to extend it to all $A, B \in B(H)^+$.

From the proof of the previous case, there are only three possibilities for the restrictions of b and j on $[0, 1]$: the constant function equal to 1 , or the constant function 0, or the identity function. Since $b(0) = 1$ and $b(1) = f_{\hat{\sigma}}(1, 1) = 1$, we see that $b(t) = 1$ for all $t \in [0, 1]$. And since $j(0) = 0$ and $j(1) = f_{\hat{\sigma}}(1, 1) = 1$, we see that $j(t) = t$ for all $t \in [0, 1]$. Since j is the identity function on the $[0, 1]$, we can follow the argument in the previous case to show that if $B \preccurlyeq I$ then we have $I\hat{\sigma}B = B$. We now show that we also have $0\hat{\sigma}B = B$. It is clear by monotonicity of $\hat{\sigma}$ that $0\hat{\sigma}B \preccurlyeq B\hat{\sigma}B = B$, the converse inequality follows from the following. Let $x \in S_H$, notice that $0\hat{\sigma}P_x = f_{\hat{\sigma}}(0, 1)P_x = b(0)P_x = P_x$, we then have

$$\nu(B, x)P_x = \nu(B, x)(0\hat{\sigma}P_x) = 0\hat{\sigma}\nu(B, x)P_x \preccurlyeq 0\hat{\sigma}B,$$

from which we conclude that $\nu(B, x) \leq \nu(0\hat{\sigma}B, x)$ for all $x \in S_H$ and therefore $B \preccurlyeq 0\hat{\sigma}B$, so we get the desired equality. Therefore, for $A, B \preccurlyeq I$ we have

$$B = 0\hat{\sigma}B \preccurlyeq A\hat{\sigma}B \preccurlyeq I\hat{\sigma}B = B,$$

so that $A\hat{\sigma}B = B$ when $0 \preccurlyeq A, B \preccurlyeq I$, and again by scaling argument, this holds for all $A, B \in B(H)^+$.

Case 4: In case where $b(0) = 0$ and $j(0) = 1$, this means that $0\hat{\sigma}I = 0$ and $I\hat{\sigma}0 = I$, we have that $A\hat{\sigma}B = A$, for all $A, B \in B(H)^+$.

This case is proved by reasoning similar to Case 3.

This completes the proof of Theorem 3.2.2. ■

We end this chapter with the following remark concerning the result we obtain in Theorem 3.2.2. Certainly, one of the most fundamental result of the classical Kubo-Ando theory of mean is the bijective correspondence between means and operator monotone functions. This suggest that the analogous theory in the setting of the spectral order would yield even more means, since there are more operator monotone functions (every continuous increasing function on $[0, +\infty[$ is operator monotone with respect to the spectral order). Interestingly, the result we proved here says the complete opposite. Mainly, this is due to the structure of the spectral order isomorphisms, which contains the maps induced by functional calculus with respect to bijective increasing functions. This rises the question about the kind of operations one would get if the functional calculus maps in the new transformer inequality (C'2) are removed. We leave this question for latter investigation.

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