Non-isolated surface singularities

PhD thesis

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To Ági and to all who struggle for a spacious unity

Abstract

This thesis includes two results about non-isolated complex surface singularities.

First, it was recently proved that for finitely determined germs $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, the number $C(\Phi)$ of Whitney umbrella points and the number $T(\Phi)$ of triple values of a stable deformation are topological invariants. The proof uses the fact that the combination $C(\Phi) - 3T(\Phi)$ is topological since it equals the linking invariant of the associated immersion $S^3 \looparrowright S^5$ introduced by Ekholm and Szűcs. We provide a new, direct proof for this equality. We also clarify the relation between various definitions of the linking invariant.

Second, we know that the Milnor fibre boundary of an isolated complex surface singularity has a graph manifold structure. We define a family of non-isolated toric surface singularities by introducing gaps to the semigroups corresponding to cyclic quotient singularites. Then we give a singular Milnor fibration for them via one-parameter toric deformations. We describe the Milnor fibre boundaries as graph manifolds.

We also develop the theoretical background and language needed for these results including theory of analytic singularities, deformations, map germs and affine toric varieties.

“Here, we need a bit of an aid. We agreed that what it all begins with, i.e. that Something Is, is not two things but one. If it Is, then it is Something. And if it is Something, then that already Is. The – we shall say – ‘existence’ and the ‘substance’ is an unbroken unit. One. Instead of words, it is more appropriate to mark it with a single letter. Let us call it epsilon. Lowercase handwritten Greek Epsilon: ε.

(We know that not only are words less than epsilon, such that they never cover it – which is alright – but, which is worse, they are also more: they place it somewhere in their arrangement of concepts, they endow it with an interpretation: they drown that which exists in the swamp of their non-existence.

Yet you use language, that is the way you want to express your epsilon. Why? Firstly, because there is nothing else, for now. Secondly, poets have long discovered ways, modes of abuse, to at least approach epsilon. Of course, it would be yet better to try it with pointing, miming, butt-kicking, screaming, patting. Soon. We will try differently once we abolish all national languages. (Until then, we must have a go at using them for their own abolition. It is rather like the case of Baron Münchausen. But it shall be done.)

Now, you would like an approximation of the (Something – Is) we just named epsilon that is both accurate and certain. You observed with regret that these two wishes are (partially) incompatible. For the epsilon it holds that ε:

> The more certainly you have it, the more inaccurate it is.
> The more accurate you try to make it, the less certainly you have it.
> With the highest accuracy, its existence may vanish. (It becomes clear, but there is no such thing.)
> With the highest certainty of existence, we may lose what it is (this Something that Is to such a high degree).
> If we denote the accuracy of epsilon with (a variable) π, pi, and the certainty of having it with (another variable) ρ, rho, then their product (sum, power) cannot exceed a constant quantity. This illustrates the peculiar situation above:

that they can only be increased at each other’s expense:

$$\pi \cdot \rho \leq \varepsilon\text{”}$$

Buda by Géza Ottlik

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Contents

Introduction

Singularities are present in many contexts in mathematics and beyond. They are exciting points where something special or unusual happens. Secret truths are hidden in them, this makes them so engaging.

In geometry, a singularity of a geometric space (curve or surface for example) is a point where the space is not smooth. We can take an apple as an everyday example. A nice apple's surface is though curved, it is generally smooth: there is well defined tangent plane at each point. On its two ends however, something unusual happens. If we move on the surface, we have to turn abruptly when reaching these points. They are the singularities of the surface of the apple.

Greek mathematicians already worked with singular curves in the Hellenistic period (the first couple centuries B.C.). The first such curve may have been the cissoid ('ivy-shape') of Diocles, a curve used to solve classical problem of duplicating the cube by constructing two line segments from the curve with a ratio of $\sqrt[3]{2}$ (cf. [BK13, §1.4]).

Then, in the 17th century, the Cartesian revolution connected algebra and geometry. With analytic geometry, we can express geometric spaces in two different ways: with equations and with parametrizations. Both methods are crucial in the current thesis. This language also enables us to define singularities more properly. A singular point of a geometric space $X \subset \mathbb{F}^n$ defined by the vanishing of some equations $f_1, \dots f_k$ is a point $p \in X$, where the rank of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$ drops.

An example that plays an important role throughout the whole dissertation is the Whitney umbrella. This is a singular surface in the 3-space defined by the equation $xy^2 = z^2$. See its real picture in Figure 0.0.1. Its Jacobian rank is generically 1, but it drops to 0 on the x-axis. These points are all considered singular, however the origin appears to be the most special of them.

We study complex analytic singularities, which means that we work over $\mathbb{C}$ and our functions and mappings are local, that is, given by convergent power series. This theory establishes a bridge between local analytic geometry and topology. In his 1956 paper

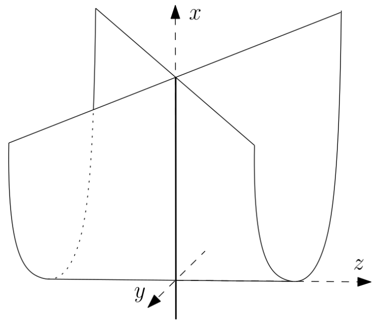


Figure 0.0.1: The Whitney umbrella.

[Mil56], Milnor provides the first examples of exotic spheres. These are topological spaces that are homeomorphic to the sphere – of dimension 7 in this case – but with a C^∞-structure that is not diffeomorphic to that of the sphere. It turns out that singularities are a rich source of such interesting topological spaces [Hir86]. Given an isolated singularity (X, p) – that is a singular point that only smooth points around it – we can surround the point p with a small real ball of the ambient space. The intersection between the space X and the boundary sphere of the ball is called the link of the singularity. This topological object carries a lot of information about the singularity and vice versa. For example, the singularity is topologically equivalent to the real cone over its link [BV72].

As singularities are unusual and rare phenomena, we can hope to smoothen them out by perturbing the defining equations. For isolated singularities defined by a single equation $f = 0$, called hypersurface singularities, Milnor created the theory of Milnor fibrations to handle the smooth spaces defined by the deformed equations $f = \varepsilon$ [Mil74]. Also, the boundary of Milnor fibre is diffeomorphic to the link. For more general singularities, deformations theory is needed to perturb the singularity in a meaningful way.

In this work, we restrict our attention to complex surface singularities. Isolated surface singularities are well studied. For example, the link of a normal (thus isolated) surface singularity is a real 3-dimensional manifold can be given a graph manifold structure [Mum61]. This structure can be related to the resolution of the singularity, which means that we 'untangle' the singularity replacing the set of singular points with something 'bigger' making the whole space smooth.

However, non-isolated surface singularities are much less known. For instance, their link is not a manifold, their deformation theory is more complicated, and even when they admit a Milnor fibration, its fibres are a mystery so far.

To tackle these challenges, we turn to two special classes of non-isolated surface singularities, which have some additional structure on them as support.

INTRODUCTION

First, we look at surfaces that can be parametrized by a mapping from the complex 2-space. Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined (also called $\mathscr{A}$-finite) holomorphic germ. In this case $\mathscr{A}$-finiteness means that Φ is a stable immersion off the origin [Wal81; MN20]. For these germs the number of the complex Whitney umbrella (cross cap) points $C(\Phi)$ and the triple values $T(\Phi)$ of a stable holomorphic deformation are well-defined analytic invariants [Mon85; Mon87]. Recently in [FPS22] J. Fernández de Bobadilla, G. Peñafort, and J. E. Sampaio proved that these invariants are topological, moreover they are determined by the embedded topological type of the image of Φ. One of the main ingredients of their proof is the formula

$$L(\Phi|_{\mathfrak{S}}) = C(\Phi) - 3T(\Phi) \tag{1}$$

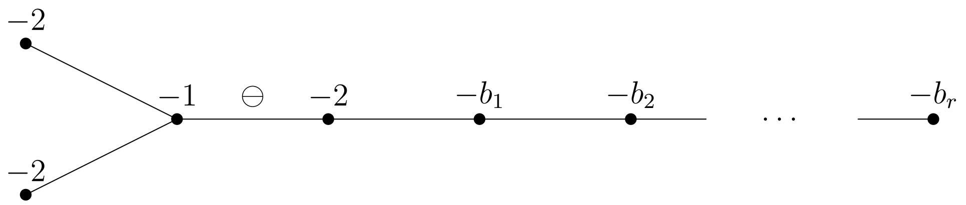
from [NP15], which expresses the naturally topological Ekholm–Szűcs invariant (also called triple point invariant or linking invariant) $L(\Phi|_{\mathfrak{S}})$ of the associated stable immersion $\Phi|_{\mathfrak{S}} : \mathfrak{S} \simeq S^3 \looparrowright S^5$ in terms of C and T. However, the formula (1) is proved in [NP15] in a rather complicated way, by using two Smale invariant formulas. Our purpose is to provide a new direct proof for this formula.

The Ekholm–Szűcs invariant $L(f)$ of a stable immersion $f : S^3 \looparrowright \mathbb{R}^5$ measures the linking of the image with a copy of the double values, shifted slightly along a suitable chosen normal vector field. In the literature, different versions of the definition of L can be found (see [Ekh01a; Ekh01b; ES03; SST02]), whose relations are not completely clarified. We verify their equivalence, i.e. $L_1(f) = -L_2(f)$, based on their opposite behavior through regular homotopies.

Although our proof of the main theorem (1) is self-contained, an independent secondary is to clarify the enigmatic relation between several variants of the linking invariant L and other related invariants, used in the study of generic C^∞ real maps and immersions.

Second, we study a family of non-normal toric surface singularities. Normal toric surface singularities are cyclic quotients with a huge literature. Altmann in [Alt95b] characterized the toric deformations – that is, deformations with a compatible toric total space – of normal affine toric variety. Also, in [ACF22], the total spaces over the negative degree components of the versal base of a normal toric singulity are constructed in a combinatorial manner. These give inspiration and a model for creating deformations for non-normal toric surfaces.

We define a family of non-normal toric surface singularities: one such singularity from each cyclic quotient surface $X_{p,p-q}$ introducing a 'mod 2' family of gaps to their defining semigroup $S \in M$. Then we build 1-parameter toric deformations for them. This results in a singular nearby fibration. We then describe the boundary of these nearby fibres as graph manifolds with plumbing graph



(2)

where the self intersections b_i are the coefficients in the negative continued fraction corresponding to the parameters $\frac{p}{q} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}} = [b_1, b_2, \dots, b_r]$.

Structure of the thesis

In the Preliminaries (Chapter 1), we give a short introduction to the basics of complex analytic singularities. We begin with the algebraic background in Section 1.1 then we turn to spaces and germs (Sections 1.2, 1.3). After an overview on resolutions in Section 1.4, we turn to the topology of singularites (Section 1.5) and we introduce the theory of Milnor fibration in Section 1.6 including a concise history. We end this chapter with a short section on plane curves.

The second chapter is about singularities of map germs. Its first half gives an introduction to the theory including unfoldings, the notion of $\mathscr{A}$-equivalence (2.1.1) and stability (2.1.2). We put special emphasis on the deformation theory in the two contexts of mappings and space germs, including infinitesimal deformations (2.1.3) and versality (2.1.5). Then we turn to the more specific aspects of the theory – to prepare for our findings – with finitely determined germs (2.1.4) and germs $\mathbb{C}^2 \to \mathbb{C}^3$ (2.1.6).

Section 2.2 is about an interplay between analytic and topological invariants corresponding to a finitely determined map germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. We recall the invariants C and T and outline their invariance for analytic, C^∞ and topological left-right equivalence. We introduce the associated immersion and we describe the double point structure of Φ.

In Section 2.2.3, we collect the different definitions of the Ekholm–Szűcs invariant L of stable immersions $S^3 \looparrowright S^5$ from the literature. We show that they agree up to sign and we clarify that sign. Then we define an invariant for finitely determined germs by applying L to the associated immersions, and we prove its topological left-right invariance.

In Section 2.2.4, we provide a new, direct proof for the correspondence $L = C - 3T$. We use local calculations near complex cross cap points and triple values.

We end the chapter with is a brief summary (2.2.5) of the applications of L and another similar linking invariant in the study of generic real maps and immersion theory. We collect the most relevant results and clear up the context of this article, including the main steps of the original proof of (1). Then we compare the new local calculation for the complex cross cap points with an older one from [NP15], and clarify its consequences for the Ekholm–Szűcs Smale invariant formula.

The third chapter is about non-isolated toric surface singularities. Section 3.1 gives a short introduction the deformation theory of space germs including their infinitesimal deformations (3.1.3). Then we turn to toric geometry in Section 3.2. We describe how the construction of affine non-normal toric varietes differ from the classical normal case (3.2.1). After a short subsection dedicated to cyclic quotient surface singularities (3.2.2), we give an overview of Altmann's work on toric deformations (3.2.3).

Section 3.3 is about a family of non-normal toric singularities. After setting up the context we state the result (3.3.2). Then we give a toric deformation (3.3.4), study the nearby fibres (3.3.5) and build the boundary of the singular Milnor fibre (3.3.6).

The original results of the author are included in Sections 2.2, 3.3.

The dissertation is not self-contained in the sense that we regularly assume some basics, omit proofs, but more importantly, sometimes foreshadow some results and concepts in order to motivate definitions or establish connections. We aim the thesis at mathematicians having some knowledge of commutative algebra, algebraic geometry and differential topology.

Chapter 1

Preliminaries

1.1 Algebraic foundations

...geometry provides intuition, while algebra provides rigour.

[GLS07]

Singularities are special points of geometric spaces and maps where they are not smooth. At these points a lot of interesting geometry is entangled, hidden. It has been the purpose of singularity theory to understand these enigmatic beings. In this section, we walk through the basics of the theory that we need. We create a language to be able to speak about singularities from different points of view. We follow [GLS07] as a guideline and main source in this section.

Throughout this thesis, we will concentrate on complex singularities. There are two possibilities for the class of functions we use: algebraic or analytic. We focus on the latter one and only mention the former. We start by introducing analytic $\mathbb{C}$-algebras that will serve as local rings of functions.

Notation 1.1.1. *We denote the* ***ring of convergent power series*** *in the variables* $\underline{x} = (x_1, ..., x_n)$ *by* $\mathbb{C}\{x_1, \ldots, x_n\}$.

Definition 1.1.2. *A* $\mathbb{C}$*-algebra is* **analytic** *if it is isomorphic to*

$$\mathbb{C}\{\underline{x}\} \,/_{I}$$

where I *is an ideal of* $\mathbb{C}\{\underline{x}\}$.

We think about such an analytic $\mathbb{C}$-algebra as the set of functions on the analytic space germ defined by I.

Definition 1.1.3. *The* **category of analytic $\mathbb{C}$-algebras** *is the set of analytic $\mathbb{C}$-algebras as objects with $\mathbb{C}$-algebra morphisms between them.*

Now we list some basic but crucial algebraic properties of analytic $\mathbb{C}$-algebras.

Proposition 1.1.4 (Properties of analytic $\mathbb{C}$-algebras).

(i) *$\mathbb{C}\{\underline{x}\}$ is a local ring with the unique maximal ideal consisting of the functions vanishing at the origin:*

$$\mathfrak{m}_{\mathbb{C}\{\underline{x}\}} = (x_1, \ldots, x_n) = \Big\{ f \in \mathbb{C}\{\underline{x}\} : f(\underline{0}) = 0 \Big\}.$$

(ii) *Any analytic $\mathbb{C}$-algebra is also local with the unique maximal ideal being the image of $\mathfrak{m}_{\mathbb{C}\{\underline{x}\}}$ under the factorization.*

(iii) *The units in $\mathbb{C}\{\underline{x}\}$ are the power series with nonzero constant term, and their images are the units in analytic $\mathbb{C}$-algebras.*

(iv) *The ring $\mathbb{C}\{\underline{x}\}$ is an integral domain, that is, there are no zero-divisors in – although there may be in an analytic algebra.*

(v) *The order – defined by the order of the smallest degree nonzero term – is additive:*

$$\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g).$$

Remark 1.1.5. *These facts hold over the real numbers too.*

From this point on, when we write analytic algebra, we mean analytic $\mathbb{C}$-algebra.

The following statement has great significance for dimension theory and for analytic geometry.

Theorem 1.1.6 (Noether property). *Every analytic algebra A is Noetherian, that is, each ideal in A is generated by finitely many elements.*

In particular, each analytic algebra can be written as

$$\mathbb{C}\{\underline{x}\} \Big/ (f_1, \ldots, f_s) .$$

We also need a few crucial notions describing analytic algebras. We present two notions to describe dimensions. Both of them are defined algebraically.

Definition 1.1.7. *Let A be an analytic algebra, and let $\mathfrak{m}_A$ be its maximal ideal.*

(i) *The (Krull)* ***dimension*** *of A is the supremum of the lengths of strictly increasing chains of prime ideals in A*

$$\dim A = \sup\{k : \ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k, \ \mathfrak{p}_i \lhd A \text{ prime } \}.$$

(ii) *The* ***cotangent space*** *of A is*

$$\mathfrak{m}_A \big/ \mathfrak{m}_A^2 .$$

(iii) *The* ***embedding dimension*** *of A is the dimension of the cotangent space*

$$\mathrm{edim} A = \dim_{\mathbb{C}} \left(\mathfrak{m}_A \big/ \mathfrak{m}_A^2 \right).$$

(iv) *The* ***Jacobian rank*** *of an ideal $I \lhd A$*

$$\mathrm{jrk}(I) = \dim_{\mathbb{C}} \left(I \big/ I \cap \mathfrak{m}_A^2 \right).$$

The geometric meaning of the Krull dimension is the following. Prime ideals correspond to closed analytic subspaces. So, essentially, the dimension counts how many times we can drop down to smaller and smaller analytic subsets from our space until we hit a single point. On the other hand, the embedded dimension is the dimension of the Zariski tangent space. In turn, it is the affine space of the lowest dimension to which our germ can be embedded. The difference between the two dimensions indicates a singularity.

If we consider $A = \mathbb{C}\{x_1, ..., x_n\}$ with the ideal $I = (f_1, ..., f_k)$, the Jacobian rank is the rank of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$. This rank also detects singularities as it drops at singular points. Moreover, it is related to the embedded dimension.

Proposition 1.1.8 (Jacobian rank lemma). *In the above setup,*

$$\mathrm{jrk}(I) = \mathrm{edim}(A) - \mathrm{edim}\left(A / I \right).$$

Let us show the proof, too, as it is quite simple.

Proof: Consider the short exact sequence of vector spaces.

$$\begin{tikzcd}
0 \arrow[r] & (I + \mathfrak{m}_A^2) \big/ \mathfrak{m}_A^2 \arrow[r] \arrow[d, "\cong"] & \mathfrak{m}_A \big/ \mathfrak{m}_A^2 \arrow[r] & \mathfrak{m}_A \big/ (I + \mathfrak{m}_A^2) \arrow[r] \arrow[d, "\cong"] & 0 \\
& I \big/ (I \cap \mathfrak{m}_A^2) & & \mathrm{m}_{A/I} \big/ \mathrm{m}_{A/I}^2 &
\end{tikzcd}$$

Then take $\mathbb{C}$-dimensions of the vector spaces obtaining the identity above. □

Let us see some simple examples involving convergent power series.

Example 1.1.9. *The Krull dimension of the power series ring equals the number of variables:* $\dim(\mathbb{C}\{x_1, ..., x_n\}) = n$. *The cotangent space is the complex n-vector space:* $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{C}^n \cong \mathbb{C}\langle x_1, ..., x_n\rangle$, *hence the embedding dimension is also n.*

A more interesting example where the dimensions do not match is the following.

Example 1.1.10. *Consider $A = \mathbb{C}\{x, y\}/xy$. Its Krull dimension is* $\dim A = 1$ *as a maximal prime chain is $(x) \subsetneq (x, y)$ – keeping in mind that the zero ideal is not prime. On the other hand, the cotangent space is $\mathfrak{m}_A/\mathfrak{m}_A^2 = (x, y)/(x, y)^2 = \mathbb{C}\langle x, y\rangle$, so* $\mathrm{edim} A = 2$.

It may also be interesting to mention that the Krull dimension gains another geometric interpretation by the following statement.

Remark 1.1.11 (Noether normalization lemma). *For an analytic algebra A, there exists a subalgebra of the form $A \supset B = \mathbb{C}\{y_1, ..., y_d\}$, such that A is a finitely generated module over B, where d is the Krull dimension of A.*

The inclusion $B \hookrightarrow A$ induces a surjective finite morphism $(X, 0) \to (\mathbb{C}^d, 0)$ that is a branched covering.

The finiteness of a map is defined later in Definition 1.3.21. Also, one should not confuse the Noether normalization with the normalization map, described in Definition 1.3.23.

Remark 1.1.12. *Given a map of analytic algebras $A \to B$, it induces a linear map of the respective cotangent spaces. This is called the* ***cotangent map****.*

1.2 Analytic spaces

The most elementary way to build complex spaces is to patch them together from pieces of $\mathbb{C}^n$. More precisely, we can define a reduced complex space as a Hausdorff topological space with an analytic atlas, meaning that the charts are locally closed analytic subsets of $\mathbb{C}^n$ and the transition maps are holomorphic maps.

However, we would like to define these spaces in a more modern way that is more flexible and carry more geometric meaning. We use the language of sheaves. For the minimal amount of sheaf theory needed here, consult Appendix A of [GLS07]. For a more comprehensive introduction to sheaves, with the focus on algebraic geometry, we recommend turning to [Vak23] and [EH00]. Here, we build on the basics. First, we give an analytic flavour to sheaves.

Definition 1.2.1.

*(i) A **ringed space** $(X, \mathcal{A}_X)$ is a topological space X together with a sheaf of rings $\mathcal{A}_X$ on X. The sheaf $\mathcal{A}_X$ is called the **structure sheaf** of X.*

*(ii) A ringed space $(X, \mathcal{A}_X)$ is called **locally ringed space** if each stalk $\mathcal{A}_p$ is a local ring. We denote the maximal ideal of $\mathcal{A}_{X,p}$ by $\mathfrak{m}_p$.*

*(iii) A **morphism of ringed spaces** is a pair of maps*

$$(\varphi, \varphi^{\#}) : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$$

where $\varphi : X \to Y$ is a continuous map of the underlying topological spaces and $\varphi^{\#} : \mathcal{A}_Y \to \varphi_* \mathcal{A}_X$ is a morphism of sheaves of local rings. (Or, equivalently, we could view it as a morphism $\varphi^{-1}\mathcal{A}_Y \to \mathcal{A}_X$.)

Being a morphism of sheaves means that $\varphi^{\#}$ is a collection of ring homomorphisms

$$\varphi_U^{\#} : \mathcal{A}_Y(U) \to \mathcal{A}_X(\varphi^{-1}(U))$$

for each $U \subset Y$ open that commutes with the restriction maps. In addition to that, being a morphism of local rings means for $\varphi^{\#}$ that for each $p \in X$ the local ring homomorphism $\varphi_p^{\#}$ takes the maximal ideal $\mathfrak{m}_{\varphi(p)}$ into the maximal ideal $\mathfrak{m}_p$.

Usually, instead of writing the pair $(\varphi, \varphi^{\#})$, we only write ϕ, but we always think the sheaf morphisms as part of the data.

Example 1.2.2. *For an open subset $D \subset \mathbb{C}^n$, the pair $(D, \mathcal{O}_D)$ is a locally ringed space where $\mathcal{O}_D$ denotes the sheaf of analytic (holomorphic) functions.*

Definition 1.2.3.

*(i) A $\mathbb{C}$**-analytic ringed space** is the ringed space $(X, \mathcal{A}_X)$, where the structure sheaf $\mathcal{A}_X$ is a sheaf of $\mathbb{C}$-algebras and all the stalks $\mathcal{A}_{X,p}$ are analytic C-algebras. (Note that the latter condition implies that $(X, \mathcal{A}_X)$ is a locally ringed space.)*

*(ii) A **morphism of** $\mathbb{C}$**-analytic ringed spaces** is a morphism of ringed spaces where the local morphisms are morphisms of $\mathbb{C}$-algebras.*

We are building complex spaces somewhat similarly to the mentioned manifold-like patchwork process. However, our pieces will be analytic subsets and the gluing is done via sheaves.

Definition 1.2.4. *Let $D \subset \mathbb{C}^n$ open. The ideal sheaf $\mathcal{I} \subset \mathcal{O}_D$ is said to be of **finite type** if it is locally generated by finitely many analytic functions. Precisely, if for any $p \in D$ there is a smaller open neighbourhood $p \in U \subset D$ where $\mathcal{I}|_U = f_1\mathcal{O}_U + .. + f_k\mathcal{O}_U$ for some $f_i \in \mathcal{O}(U)$.*

These functions locally define subsets of D that are significant from the analytic point of view.

Definition 1.2.5. *The **analytic set defined by** $\mathcal{I}$, or the vanishing set of $\mathcal{I}$, in D is*

$$V(\mathcal{I}) = \left\{p \in D : {\mathcal{O}_{D,p}} \big/ {\mathcal{I}_p} \neq 0\right\}.$$

This definition fits our intuition as away from its vanishing set, $\mathcal{I}$ contains units hence the quotient becomes 0.

Definition 1.2.6 (Complex spaces).

(i) *Let $\mathcal{I} \subset \mathcal{O}_D$ of finite type. Let $Y = V(\mathcal{I})$ be the corresponding analytic set and $\mathcal{O}_Y = (\mathcal{O}_D/\mathcal{I})|_Y$ be its structure sheaf. We call $(Y, \mathcal{O}_Y)$ the **complex model space** defined by $\mathcal{I}$.*

(ii) *A $\mathbb{C}$-analytic ringed space is a **complex space** if X is Hausdorff and the structure sheaf is locally like a model space. The latter means that for each $p \in X$ there exists an open neighbourhood $U \ni p$ such that $(U, \mathcal{O}_U)$ is isomorphic to a complex model space as a $\mathbb{C}$-analytic ringed space.*

(iii) *Holomorphic functions on U appear as sections of the above sheaf $f \in \Gamma(U, \mathcal{O}_X)$. Holomorphic or analytic maps are the morphisms of complex spaces. They are morphisms of $\mathbb{C}$-analytic ringed spaces $(\varphi, \varphi^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. An isomorphism of complex spaces is called a biholomorphism.*

Remark 1.2.7. *When it leads to no confusion, we only write $\varphi : X \to Y$ for a holomorphic map, omitting the sheaves whatsoever, but we always mean both parts.*

A stalk $\mathcal{O}_{X,p}$ of a complex space – defined as the direct limit $\lim_{U \ni p} \mathcal{O}_X(U)$ – is always an analytic algebra of finite type, meaning, it is isomorphic to $\mathbb{C}\{\underline{x}\}/(f_1, ..., f_k)$. In this case, $\{x_i\}$ is a set of local coordinates of $\mathbb{C}^n$, in turn, $\{f_i\}$ are the local equations that define the structure sheaf of X at p in those coordinates.

On the other hand, every analytic algebra of finite type $\mathbb{C}\{\underline{x}\}/(f_1, ..., f_k)$ appears as a stalk of a complex space. For this, we find an open neighbourhood U of $0 \in \mathbb{C}^n$ where all the f_i are holomorphic. Then we define $\mathcal{I} = f_1\mathcal{O}_U + ... + f_k\mathcal{O}_U$ and the complex space

$$(X, \mathcal{O}_X) = \Big(V(\mathcal{I}), (\mathcal{O}_U/\mathcal{I})|_{V(\mathcal{I})}\Big).$$

The stalk of the latter at 0 is isomorphic to the given analytic algebra $\mathbb{C}\{\underline{x}\}/(f_1, ..., f_k)$.

Definition 1.2.5 showed how ideals yield analytic sets. Now we see how we can go from sets to ideals.

Definition 1.2.8. *Given any subset $S \subset X$ of a complex space $(X, \mathcal{O}_X)$, the **vanishing ideal sheaf** of S is defined as*

$$\mathcal{I}(S)(U) = \Big\{f \in \mathcal{O}_X(U) : \ V(f) \supset S \cap U\Big\}.$$

This construction works for any subset of X not only the analytic ones. Of course, we only have nice properties in those cases.

Remark 1.2.9. *For an analytic set $S \subset D \subset \mathbb{C}^n$,*

$$S = V(\mathcal{I}(S)).$$

Note that each stalk $\mathcal{I}(S)_p$ is a radical ideal of $\mathcal{O}_{X,p}$ by construction. Moreover, the Hilbert-Rückert Nullstellensatz tells us that for a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_D$ the following holds

$$\mathcal{I}(V(\mathcal{J})) = \sqrt{\mathcal{J}}$$

where $\sqrt{\mathcal{J}}$ is the radical of $\mathcal{J}$.

Next, we define how complex subspaces are contained in a complex space.

Definition 1.2.10. *A **closed complex analytic subspace** of a complex space $(X, \mathcal{O}_X)$ is a $\mathbb{C}$-analytic ringed space (Y, O_Y) that is defined by an ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_X$ the following way. The underlying topological space is $Y = V(\mathcal{I})$ and the structure sheaf is $\mathcal{O}_Y = (\mathcal{O}_X/\mathcal{I}_Y)|_Y$.*
An open complex subspace $(U, \mathcal{O}_U)$ is just an open subspace $U \subset X$ with the restricted sheaf $\mathcal{O}_U = \mathcal{O}_X|_U$.

We can also say which subsets of the topological space X are analytic.

Definition 1.2.11. *A subset $S \subset X$ is a (closed)* ***analytic set*** *in the complex space $(X, \mathcal{O}_X)$ if it can be given a structure sheaf $\mathcal{O}_S$ (for example $\mathcal{O}_S = \mathcal{O}_X/\mathcal{I}(S)$) that makes it a closed complex subspace.*
An equivalent local description is the following. The subset $S \subset X$ is an analytic set in X if each point $p \in X$ has an open neighbourhood U such that

$$S \cap U = V(f_1, \dots, f_k)$$

for some $f_i \in \mathcal{O}_X(U)$.

It is important to note that a a closed analytic set can be given different structure sheaves that make it a complex subspace. The minimal one $\mathcal{O}_X/\mathcal{I}(S)$ makes it a so-called reduced space.

Definition 1.2.12. *A complex space $(X, \mathcal{O}_X)$ is* ***reduced*** *if each stalk $\mathcal{O}_{X,p}$ ($p \in X$) has no nilpotent elements – or in other words, is a reduced ring.*

We want to make sense of the natural geometric intuitions regarding dimensions. The notions at hand are the algebraic ones we have given in Definition 1.1.7. We apply them to complex spaces.

Definition 1.2.13. *The* ***(Krull) dimension*** *of the complex space X at $p \in X$ is the Krull dimension of the local ring*

$$\dim_p X = \dim \mathcal{O}_{X,p}.$$

On the other hand, the ***embedding dimension*** *at p is, again, the dimension of the cotangent space*

$$\mathrm{edim}_p X = \dim_{\mathbb{C}} \mathfrak{m}_p \big/ \mathfrak{m}_p^2.$$

If we look at a complex space X around a point p, it will be isomorphic to an analytic subset of some $\mathbb{C}^N$ – a complex model space. Precisely, there is an N, an open subset $D \subset \mathbb{C}^N$ and an ideal sheaf of the form

$$\mathcal{I} = f_1\mathcal{O}_D + \dots f_k\mathcal{O}_D \subset \mathcal{O}_D$$

such that

$$\left(U, \mathcal{O}_X\big|_U\right) \cong \left(V(\mathcal{I}), \mathcal{O}_D \big/ \mathcal{I}\right)$$

for some $p \in U \subset X$. We can see that the sheaf $\mathcal{O}_D/\mathcal{I}$ only depends on the choice of the neighbourhood U, but otherwise is encoded in $\mathcal{O}_X$. In particular, we do not know anything about N a priori. In fact, there is a smallest complex affine space where we can embed some neighbourhood of p. This minimal dimension is the embedding dimension.

These notions of dimension allow us to grasp the phenomenon of singularity.

Definition 1.2.14. *A complex space X is **regular** (or, equivalently **smooth** or **non-singular**) at p if the two concepts of dimension match:*

$$\dim_p X = \mathrm{edim}_p X.$$

*Otherwise, we say it is **singular** at p.*

Note that the embedded dimension is always greater or equal to the Krull dimension by Krull's principal ideal theorem ([Eis95, Section 8.2.2.]).

Remark 1.2.15. *We also want to mention that there is a natural way to define the **analytic fibre products** $X \underset{T}{\times} Y$ given three complex spaces X, Y, T with morphisms $X \to T, Y \to T$ between them. The situation can be summed up in the following commutative diagram.*

$$\begin{tikzcd}
 & X \underset{T}{\times} Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\psi"] \\
 & T &
\end{tikzcd}$$

Moreover, this satisfies the expected universal property that for any complex space Z with morphsisms $\zeta_X : Z \to X$ and $\zeta_Y : Z \to Y$ there exists a unique morphism $\zeta_{X\times Y} : Z \to X \underset{T}{\times} Y$ satisfying $\zeta_X = \pi_X \circ \zeta_{X\times Y}$ and $\zeta_Y = \pi_Y \circ \zeta_{X\times Y}$.

1.3 Germs

When we see a singular point in a complex space, we are interested in the geometry hidden in that point, we want to concentrate our attention to that point as closely as possible, forgetting about everything that happens further away from the singularity. The way to do this is by the notion of a germ.

Definition 1.3.1. *A **complex space germ** or a **singularity** (X, p), for a complex space X with a distinguished point $p \in X$, is the complex space $(U, \mathcal{O}_X|_U)$, where $U \subset X$ is an arbitrarily small neighbourhood of p.*

Similarly, a ***holomorphic map germ*** $\varphi : (X, p) \to (Y, q)$ *is a morphism of complex spaces respecting the distinguished point* $\varphi(p) = q$*, considering two morphisms equivalent if they agree in some neighbourhood of the distinguished point.*

They form a category.

Remark 1.3.2. *We can define* ***multi-germs*** *in similar fashion. We only need to replace* p *and* q *with some larger sets:* $f : (X, S) \to (Y, T)$ *is a multi-germ. In this case, for the equivalence, we need to look at the map in small neighbourhoods of* $S \subset X$*, and* $T \supset f(S)$ *is also needed. In the current thesis, we will see finite multi-germs, such as the triple point. However, one can also consider a germ 'along' a subspace* $S \subset X$*.*

When talking about germs, we usually take a representative (space or morphism) from the given equivalence class. The local ring $\mathcal{O}_{(X,p)}$ of a germ is the stalk $\mathcal{O}_{X,p}$ of a representative $(X, \mathcal{O}_X)$. We use these two notatitions for the stalk interchangeably, which is a slight abuse of notation as the latter requires the choice of a representative.

A holomorphic map germ $(X, p) \to (Y, q)$ induces a pair $(\varphi_p, \varphi_p^{\#})$, that consists of a continuous map germ φ_{top} and a morphism $\varphi_p^{\#} : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ of analytic $\mathbb{C}$-algebras.

On the relationship between the topological part (the underlying space) and the algebraic part (the structure sheaf) of the data carried by a germ, we can say the following.

Proposition 1.3.3.

(i) The underlying topological space does not determine its local algebra. An analytic set $X \subset \mathbb{C}^N$ *can be given different structure sheaves. Also, the continuous map germ* $\varphi : X \to Y$ *(with* $\varphi(p) = q$*) does not determine the morphism* $\varphi_p^{\#} : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ *between the stalks.*

(ii) The algebra determines the underlying topology. For pointed complex spaces (X, p) *and* (Y, q)*, a morphism* $\mu : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ *of analytic* $\mathbb{C}$*-algebras determines a unique holomorphic map germ* $(\varphi, \varphi^{\#}) : (X, p) \to (Y, q)$ *that satisfies* $\varphi_p^{\#} = \mu$*.*

The discrepancy in (i) motivates talking about reduced germs: in that case, the topological data above determines the algebra.

Definition 1.3.4. *The complex space germ* (X, p) *is* ***reduced*** *if the local ring* $\mathcal{O}_{X,p}$ *is reduced.*

Example 1.3.5. *A most simple but extremely important example of a nonreduced space germ is the* ***fat point of order two*** *or* ***double point****:*

$$\mathbb{D} = \Big(\{\mathrm{pt}\}, \mathbb{C}[\varepsilon]/\varepsilon^2\Big).$$

Topologically, this is only a point, but it has a richer, two dimensional local ring. Vaguely speaking, it sees the first derivatives of functions at this point besides their value.

A nonreduced space has a well-defined reduction.

Definition 1.3.6. *The* ***reduction*** *of a complex space* $(X, \mathcal{O}_X)$ *is*

$$X_{red} = (X, \mathcal{O}_X^{red}) = \Big(X, \mathcal{O}_X/\mathcal{I}(X)\Big),$$

where $\mathcal{I}(X)$ *is the vanishing ideal sheaf of* X*.*

That is, we obtain the reduced structure sheaf by factoring out by $\mathcal{I}(X)$ (see Definition 1.2.8) for $X \subset X$, removing the 'nilpotency' from its stalks.

The important geometric concepts that we introduced for complex spaces in the previous section are defined the following way.

Definition 1.3.7. *An* ***analytic subgerm*** *of* (X, p) *is defined by, first, choosing an ideal* $I \subset \mathcal{O}_{X,p}$ *in the local ring – that is always finitely generated* $I = (f_1, ..., f_k)$*. Then we take its vanishing set in a suitable neighbourhood* $U \ni p$ *as a representative*

$$\Big(V(I), p\Big) = \Big(V\Big(\sum f_i \mathcal{O}_U\Big), p\Big) \subset (X, p).$$

Definition 1.3.8.

(i) The dimension of a germ is defined as the dimension of a representative at the distinguished point $\dim(X, p) = \dim_p X$*. So is the embedded dimension:* $\mathrm{edim}(X, p) = \mathrm{edim}_p X$*.*

(ii) When $\dim(X, p) = 1$*, we call it a curve singularity. In case of* $\dim(X, p) = 2$*, it is a surface singularity. When a space germ can be embedded into a complex affine space one dimension greater, we call it a hypersurface singularity; in this case it can be described with one equation too.*

Note that a singularity or space germ in the sense of Definition 1.3.1 is not necessarily singular in the sense of Definition 1.2.14. In fact, regularity of germs can be described in different ways.

Proposition 1.3.9. *The following are equivalent.*

(i) (X, p) *is regular with* $\dim(X, p) = n$*.*

(ii) *A small open neighbourhood of* p *is a smooth complex manifold of dimension* n*.*

(iii) *Given that* $\mathcal{O}_{(X,p)} \cong \mathbb{C}\{x_1, ...x_N\}/(f_i, ..., f_k)$*, the rank of the Jacobian at every point* q *in a small neighbourhood is constant*

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial x_j}(q)\right)_{i,j} = N - n.$$

(iv) $\mathcal{O}_{(X,p)} \cong \mathbb{C}\{x_1, \ldots, x_n\}$*.*

We remark that the Jacobian rank of (iii) is lower semicontinuous. This means that for any $p \in X$ there is an open neighbourhood $U \subset X$ such that, for any $q \in U$, $\mathrm{rk}\left(\frac{\partial f_i}{\partial x_j}(q) \geq \left(\frac{\partial f_i}{\partial x_j}(p)\right.\right.$.

The complex space X is singular at p if the above equivalent conditions are not satisfied. We focus our attention to the set of singular points.

Definition 1.3.10. *The* ***singular locus*** *of a complex space* X *is*

$$\mathrm{Sing}(X) = \{p \in X : \ X \textit{ is singular at } p\}.$$

Let us look at a plane curve example to see the above notions in play.

Example 1.3.11. *Consider the crunode* $X = \{y^2 - x^2(x + 1) = 0\} \subset \mathbb{C}^2$*. As a complex subspace, it is irreducible. However, as a germ at* $(0, 0)$*, it is equivalent to two intersecting lines* $(X, 0) \cong (\{y^2 - x^2 = 0\}, 0) = (\{(y - x)(y + x) = 0\}, 0)$ *through a local analytic coordinate change.*

Definition 1.3.12 (Irreduciblity of spaces).

(i) *A complex space germ* (X, p) *is* ***irreducible*** *if the local ring* $\mathcal{O}_{X,p}$ *is an integral domain.*

(ii) *Accordingly, an analytic subset* $Y \subset X$ *is irreducible at* p *if the vanishing ideal* $\mathcal{I}(Y)_p \subset \mathcal{O}_{X,p}$ *is prime. Indeed, this makes the quotient* $\mathcal{O}_{X,p}/\mathcal{I}(Y)_p$ *an integral domain.*

When the vanishing ideal is not prime, it has a minimal prime decomposition as it is a radical ideal. This leads to the following in geometry.

Proposition 1.3.13. *Let $(Y,p) \subset (X,p)$ be an analytic subgerm. There is an* ***irreducible decomposition***

$$(Y,p) = (Y_1,p) \cup \ldots (Y_l,p)$$

where each germ (Y_i,p) is irreducible and they do not contain each other. They are called ***irreducible components****.*
The decomposition is unique up to permutation.

For global complex spaces, a similar characterization, that uses the structure sheaf as in Definition 1.3.12, is problematic. If we required the ring of global sections to be an integral domain, we would fail as there may be too few global sections – take projective spaces for instance. As a concrete example, consider two lines on the projective plane $X = \mathbb{P}^1 \cup \mathbb{P}^1 \subset \mathbb{P}^2$. The space X is clearly reducible but the global sections are only the constant functions $\mathcal{O}_X(X) \cong \mathbb{C}$ that is obviously an integral domain. On the other hand, requiring all rings $\mathcal{O}_X(U)$ to be integral domains is a too much. For the crudnode of Example 1.3.11, we can choose a small enough neighbourhood U around 0 for which $\mathcal{O}_X(U)$ has zero-divisors.

Instead, irreducibility of a complex space X is defined simply as not being decomposable to closed analytic subspaces $X = X_1 \cup X_2$ nontrivially. Although, the concept can be characterized using meromorphic functions instead. The complex space X is irreducible if and only if the global sections of the sheaf of meromorphic functions on X form a field.

Proposition 1.3.14. *An irreducible reduced complex space X is of* ***pure dimension****, that is the dimension is the same at each point of X.*

Proposition 1.3.15. *The singular locus* $\mathrm{Sing}(X)$ *of a complex space X is a closed analytic subset of X.*

Proof: It is enough to show it locally, so we can assume that X is a complex model space and let us treat irreducible components separately. Let the irreducible component X_i be defined by $\mathcal{I} = (f_1, ..., f_k) \cdot \mathcal{O}_D$ inside some open $D \subset \mathbb{C}^N$. According to Proposition 1.3.14, X_i is of pure dimension n. Then (iii) of Proposition 1.3.9 and the semicontinuity of the Jacobian rank tells us that the singular locus of X_i can be described using the Jacobian criterion:

$$\mathrm{Sing}(X_i) = \Big\{p \in X_i : \ \mathrm{rk}\Big(\tfrac{\partial f_i}{\partial x_j}(p)\Big) < N - n\Big\}.$$

Hence, with the assumptions above, the singular locus is a closed analytic subset. In case of a reducible space, we just need to take the union of the singular loci of the components and the intersection sets of the components $\{X_i\}$. This altogether is a union of closed analytic sets, thus closed analytic itself. $\square$

Note that if X is of pure dimension, then in the rank condition, we do not have to bother with the irreducible components separately. If it is of mixed dimension, however, then it is, indeed, necessary. Consider $X = V(xy, xz) \subset \mathbb{C}^3$. The space X is reducible as its vanishing ideal decomposes into $(xy, xz) = (x) \cdot (y, z)$ resulting the decomposition $X = V(x) \cup V(y, z) = X_1 \cup X_2$. Note that X_1 and X_2 are of dimensions 2 and 1, respectively. This leads to the Jacobian having different ranks on the components

$$\mathrm{rk}\Big(\frac{\partial f_i}{\partial x_j}(p)\Big) = \begin{cases} 0 & \text{if } p = \underline{0} \\ 1 & \text{if } p \in X_1 \setminus \underline{0} \\ 2 & \text{if } p \in X_2 \setminus \underline{0} \end{cases}$$

Remark 1.3.16. *We can collect all the data above – the local Jacobi matrix and the local equations – into a sheaf that defines the singular locus. This is denoted by* $\mathcal{I}_{\mathrm{Sing}(X)}$ *and we have* $V(\mathcal{I}_{\mathrm{Sing}(X)}) = \mathrm{Sing}(X)$.

Next, we discuss the regularity of maps.

Definition 1.3.17. *A map germ* $\varphi : (X, p) \to (Y, q)$ *is* ***regular*** *(or* ***smooth*** *or* ***nonsingular****) if it fits into the commutative diagram*

$$\begin{tikzcd} (X,p) \arrow[r, "\cong"] \arrow[dr, "\varphi"'] & (Y,q) \times (S,0) \arrow[d, "\pi_Y"] \\ & (Y,q). \end{tikzcd}$$

where $(S, 0)$ *is a regular space germ and* π_Y *is the projection. A holomorphic map is smooth if it is smooth at all points.*

Geometrically, regularity means that the map is a submersion. Regular maps have an equivalent algebraic description.

Proposition 1.3.18. *The map germ* $\varphi : (X, p) \to (Y, q)$ *is regular if and only if* $\mathcal{O}_{X,p}$ *is a free power series algebra over* $\mathcal{O}_{Y,q}$.

Regularity is an open condition, more precisely, those points $p \in X$ where φ is regular form an open subset of X.

Also, regularity of a space X is equivalent to the regularity of the map $X \to \mathrm{pt}$.

1.3.1 Normality

The next property of spaces that has an important effect on singularities is normality. This, too, is defined stalkwise.

Definition 1.3.19. *The complex space X is **normal** at $p \in X$, or the complex space germ (X, p) is normal if the local ring $\mathcal{O}_{X,p}$ is integrally closed in its total ring of fractions* $\mathrm{Frac}(\mathcal{O}_{X,p})$.

(We get the total ring of fractions if we localize our ring by the set S of non-zero-divisors: $\mathrm{Frac}(\mathcal{O}_{X,p}) = S^{-1}\mathcal{O}_{X,p}$. If X is irreducible at p, then $\mathcal{O}_{X,p}$ is an integral domain, hence $\mathrm{Frac}(\mathcal{O}_{X,p})$ is its field of fractions.)

Note that normality implies reducedness. Let us see an example.

Example 1.3.20. *The cusp $\mathcal{C} = \{x^3 = y^2\} \subset \mathbb{C}^2$ is not normal at the origin. Indeed, the local ring at $(0,0)$ is $\mathcal{O}_{\mathcal{C},(0,0)} = \mathbb{C}\{x, y\}/(x^3 - y^2)$ and the fraction $\frac{y}{x} \in \mathrm{Frac}(\mathcal{O}_{\mathcal{C},(0,0)})$ is not in the local ring but is in the integral closure of it as it solves the equation $Z^2 = x$ in Z.*

We can 'correct' non-normal spaces by normalizing them. But first, we need the notion of finiteness of maps in order to describe normalizations.

Definition 1.3.21. *A morphism $\varphi : X \to Y$ is **finite** at $p \in X$ if there are open neighbourhoods $U \ni p, V \ni \varphi(p)$ with $\varphi(U) \subset V$ such that $\varphi|_U$ has finite fibres.*

We could ask why we do not choose smaller neighbourhoods where the fibres consist of single points. However, the map $\mathbb{C} \to \mathbb{C}, x \mapsto x^k$ clearly satisfies the definition above and at any points around 0, it is an n-fold cover.

An important result concerning finite maps is that they take analytic subsets to analytic subsets.

Theorem 1.3.22. *Consider a finite morphism $\varphi : X \to Y$ of complex spaces. Then the image $\varphi(Z)$ of any closed complex subspace $Z \subset X$ is an analytic subset in Y.*

Back to normalization.

Definition 1.3.23. *Let X be a reduced complex space. The **normalization** is a morphism $n : X_{norm} \to X$, from a normal space X_{norm}, such that*

(i) the map n is finite and surjective,

(ii) the preimage of the non-normal locus is nowhere dense,

(iii) on the normal locus, n is a biholomorphism.

The normalization satisfies the universal property that any morphism from a normal space to X factors through $n : X_{norm} \to X$. In this sense, it is unique up to isomorphism of complex spaces.

The normalization of the above cusp singularity is the following. First, we normalize the local ring: $n^{\#} : \mathbb{C}[x,y]/x^3 - y^2 \to \mathbb{C}[t],\ x \mapsto t^2,\ y \mapsto t^3$. This implies the map of spaces $n : \mathbb{C} \to X,\ t \mapsto (t^2, t^3)$.

We mention two important consequence of normality on singularities.

Theorem 1.3.24. *For a normal complex space X, the dimension of the singular locus is at least two less than that of X*

$$\dim\Big(\mathrm{Sing}(X)\Big) \leq \dim(X) - 2.$$

The next theorem states that on a normal space, the values of a function outside of the singular locus determine its values on the singular locus.

Theorem 1.3.25. *Let X be a normal complex space. For every open subset $U \subset X$ the restriction*

$$\Gamma(U, \mathcal{O}_X) \to \Gamma\Big(U \setminus \mathrm{Sing}(X), \mathcal{O}_X\Big)$$

is bijective.

For reduced complex spaces this condition is an equivalent description of normality.

This is a good moment to clarify the relation between the properties of space germs introduced in this section.

Remark 1.3.26. *Regularity implies both normality and irreducibility. In turn, normality and irreducibility separately imply reducedness.*

In this thesis, we focus our attention to analytic spaces, germs and mappings. However the analogous notions in the algebraic category are similarly interesting. We say a few words about the difference between the two situations.

Remark 1.3.27. *Consider an algebraic variety $\mathcal{X}$ of finite type. We can naturally assign an associated complex space $\mathcal{X}_{an}$ in the following way. The variety $\mathcal{X}$ is covered by dense open charts that are isomorphic (in both the algebraic and the analytic sense) to subsets of some $\mathbb{C}^n$ defined by finitely many polynomial equations. These equations can be treated* *as analytic equations, so can be the gluing maps of $\mathcal{X}$ giving a collection of complex model spaces for $\mathcal{X}_{an}$.*

On the other hand, complex spaces have 'finer' structure on them. First, the topology is finer, there are more open subsets. Second, there are more analytic functions than algebraic ones on a space, and, in turn, more analytic sets. Hence, for a complex space X, there might not exist an algebraic variety whose associated complex space is X. Similarly, given two algebraic varieties $\mathcal{X}, \mathcal{X}'$ with (analytically) isomorphic associated complex spaces $\mathcal{X}_{an} \cong \mathcal{X}'_{an}$, the varieties might not be isomorphic in the algebraic sense.

However, if we restrict our attention to projective varieties, the two categories become equivalent. Chow proved in [Cho49] that for any closed analytic subset $X \subset \mathbb{P}^n$ of a projective space, there exists a projective variety $\mathcal{X} \subset \mathbb{P}^n$ such that $\mathcal{X}_{an} \cong X$ as complex spaces. Later, in [Ser56], Serre showed that for projective varieties the associated complex space functor is an equivalence of categories, including coherent sheaves, and implying an isomorphism on the respective sheaf cohomologies. Furthermore, in dimension 1, the statement is true for any compact complex space. That is, any compact Riemann surface is projective algebraic. However, in higher dimensions, it is no longer true that any compact complex space is algebraic and no condition is known that is necessary and sufficient.

On further details of this correspondence between the algebraic and analytic worlds, and on how this connection is used in algebraic geometry, see [Har77, Appendix B].

1.4 Resolution

One approach to 'smoothen' a singularity is resolution. This means that we replace the singular locus with something 'bigger', giving room to the enclosed geometry to 'set free', while we do not change the geometry of the smooth locus. We mostly follow [Ném22, Chapter 2.], however we also recommend the great book [Kol07] of Kollár on the topic that gives a much more extensive account for the topic showing how it has developed historically.

Definition 1.4.1 (Modification, resolution, good resolution). *Let $(X,0)$ be a space germ. A local* ***modification*** *(or partial resolution) of $(X,0)$ is a proper analytic map $\rho : \widehat{X} \to X$ from a normal space $\widehat{X}$ to a sufficiently small representative X, satisfying the conditions*

(i) the preimage $\widehat{X} \setminus \rho^{-1}(\mathrm{Sing}(X))$ of the smooth locus is dense in $\widehat{X}$;

(ii) the map ρ is an isomorphism over $X \setminus A$ for an analytic subset $X \supset A \supset \mathrm{Sing}(X)$ that does not contain any irreducible components of X.

*A modification is a local **resolution** if, in addition,*

(iii) $\widehat{X}$ is smooth.

*A resolution is **good** if the following conditions also hold*

(iv) the map ρ is an isomorphism over the whole smooth locus $X \setminus \mathrm{Sing}(X)$ (that is $A = \mathrm{Sing}(X)$ in the condition (ii)*);*

(v) the preimage $\rho^{-1}(\mathrm{Sing}(X))$ of the singular locus is a normal crossing divisor in $\widehat{X}$;

(vi) each irreducible component of $\rho^{-1}(\mathrm{Sing}(X))$ is smooth.

Notation 1.4.2. *The primage of the singular locus is denoted by $E = \rho^{-1}(\mathrm{Sing}(X)) \subset \widehat{X}$. Its irreducible components are usually numbered $E = \bigcup E_i$.*

Definition 1.4.3. *We say that a modification $\rho : \widehat{X} \to X$ **dominates** another modification $\rho' : X' \to X$, if the former factors through the latter, that is (maybe after taking smaller representatives) there exists an analytic map $\varphi : \widehat{X} \to X'$ such that*

$$\rho = \rho' \circ \varphi.$$

*A resolution is called **minimal** (or minimal good) if it does not dominate any other (good) resolution with a non-isomorphism.*

Remark 1.4.4. *The normalization map $n : X_{norm} \to X$ is a modification. For curves, the normalization is already a resolution. In the case of surfaces, the normalization X_{norm} can only have isolated singularities.*
Moreover, every resolution dominates the normalization.

The primary and motivating example of resolutions is the blowup.

Example 1.4.5 (Blowup of a point). *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an isolated singularity. We take the projective space of lines through the origin in $\mathbb{C}^n$ and consider the space of pairs of a point and a line where the line goes through the point:*

$$B = \{(p, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} : p \in \ell\}.$$

The blowup of $\mathbb{C}^n$ at the origin is the map

$$\pi : B \to \mathbb{C}^n, \quad (p, \ell) \mapsto p.$$

*This is a modification of $\mathbb{C}^n$ that is an isomorphism over $\mathbb{C}^n \setminus 0$. The preimage of 0 is $E = \pi^{-1}(0) = 0 \times \mathbb{P}^{n-1}$, the **exceptional divisor**.*

*The preimage $\pi^{-1}(X)$ of a small representative X of the singularity is the **total transform** of the singularity:*

$$\{(p, \ell) \in X \times \mathbb{P}^{n-1} : p \in \ell\}.$$

*Note that this contains the whole exceptional divisor as an irreducible component. We can remove this by looking at the preimage of X outside of 0 and then taking its closure in B, obtaining the so-called **strict transform** of X:*

$$\widehat{X} = \overline{\{(p, \ell) \in X \times \mathbb{P}^{n-1} : p \in \ell, p \neq 0\}}.$$

The map $\pi|_{\widehat{X}} : \widehat{X} \to X$ is the blowup of X at 0.

Example 1.4.6. *Instead of a (singular) point we can also blow up an analytic subspace. Consider the complex model space defined by some equations $Z = V(f_1, \ldots, f_k) \subset \mathbb{C}^n$, and let $\{y_1, \ldots, y_k\}$ be projective coordinates on $\mathbb{P}^{k-1}$. Then the blowup of $\mathbb{C}^n$ at Z is*

$$\widehat{\mathbb{C}^n} = \Big\{\Big(p, [y_1 : \cdots : y_k]\Big) \in \mathbb{C}^n \times \mathbb{P}^{k-1} : \quad y_i f_j(p) = y_j f_i(p) \ \forall 1 \geq i, j \leq k\Big\}.$$

The map $\widehat{\mathbb{C}^n} \to \mathbb{C}^n, (p, [y]) \to p$ is a modification that is an isomorphism over $\mathbb{C}^n \setminus Z$.

Let $(X, 0)$ be a normal surface singularity with a modification $\varphi : X' \to X$. Normality makes $(X, 0)$ an isolated singularity, furthermore, it implies that the modification is an isomorphism outside of the origin.

Theorem 1.4.7 (Zariski's main theorem). *In case of a modification of a normal surface singularity, the exceptional divisor $E = \varphi^{-1}(0)$ is a connected, compact curve.*

Remark 1.4.8. *If X is contractible to 0, then the space $\widehat{X}$ is retractable to E, hence they have the same homotopy type. Also, $\widehat{X}$ can be taken to be a closed complex space with boundary smoothly isomorphic to that of X: $\partial\widehat{X} \cong \partial X$.*

The way the components of the exceptional divisor intersect each other carries important information about the singularity.

Definition 1.4.9. *The **intersection matrix** of a resolution is the matrix $(E_i, E_j)_{i,j}$. For two different components, the intersection number is the number of intersection points counted with multiplicity. For the self-intersection number (E_i, E_i), we consider a slightly C^∞-shifted copy E'_i of E_i representing the same homology class $[E'_i] = [E_i] \in H_2(\widehat{X}, \mathbb{Z})$, and take the intersection number (E_i, E'_i). Note that the latter is independent of the choice of E'_i.*

Theorem 1.4.10. *A resolution $\rho : \widehat{X} \to X$ of a normal surface singularity $(X, 0)$ has negative definite intersection matrix.*

Proof: We want to prove that, for any divisor $Z = \sum_v \alpha_v E_v$ with $\alpha_v \in \mathbb{Q}$ and nonnegative self-intersection $Z^2 \geq 0$, we have $Z = 0$. For this, we use principal divisors.

Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a holomorphic function. We now that the corresponding principal divisor $\mathrm{div}(f \circ \rho) = \mathrm{div}_E(f \circ \rho) + \mathrm{st}(f)$ (where st is the strict transform) represents the class 0 in the relative homology group $H_2(\widehat{X}, \partial\widehat{X}, \mathbb{Z})$. Let us simplify notation by introducing $D = \mathrm{div}_E(f \circ \rho)$. Note that D is effective. The vanishing in homology means that $(D + \mathrm{st}(f), E_v) = 0$ for each v. Firstly, this implies

$$(D, Z') \leq 0 \tag{1.1}$$

for any effective divisor $|Z'| \subset E$, and, secondly, that

$$D^2 < 0 \tag{1.2}$$

as $(\mathrm{st}(f), D) > 0$ making $D^2 < D^2 + (\mathrm{st}(f), D) = (D + \mathrm{st}(f), D) = 0$. These two inequalities hold for any restriction $D|_{E'} = div_{E'}(f \circ \rho)$ of D.

Returning to the divisor Z, we can assume that it is effective, otherwise there is unique way to split it to $Z = Z_+ - Z_-$ where both Z_+ and Z_- and are effective and their supports are disjoint. At least one of the two parts have $Z_\pm^2 \geq 0$ as

$$0 \leq Z^2 = (Z_+ - Z_-)^2 = Z_+^2 + Z_-^2 - 2Z_+Z_- \leq Z_+^2 + Z_-^2.$$

(The last inequality is due to the disjoint supports.)

Let us assume that we have the statement for smaller support $|Z| \subsetneq E$ and now let $|Z| = E$. In this case, there exists a small positive $\lambda \in \mathbb{Q}$ such that $Z - \lambda D$ is still effective. This way, we did not decrease the self-intersection:

$$(Z - \lambda D)^2 = (Z, Z - \lambda D) - \lambda(D, Z - \lambda D) \geq (Z, Z - \lambda D) = Z^2 - \lambda(Z, D) \geq Z^2$$

using (1.1) twice.

Choosing the maximal λ with the above property, we decreased the support: $|Z - \lambda D| \subsetneq E$. The inductive statement implies that $Z - \lambda D = 0$, thus $Z^2 = \lambda^2 D^2$ holds. However, $Z^2 \geq 0$ according to the initial assumption and $D^2 < 0$ due to (1.2), hence $\lambda = 0$, which is a contradiction. □

Grauert [Gra62] proved the converse of this statement in the following sense: any smooth surface, with a given collection of curves inside it satisfying the above conditions, can be realized locally and analytically as the resolution of a surface singularity. More precisely:

Theorem 1.4.11. *Let X be a smooth complex space. Consider a collection $\{C_i\}_1^s$ of irreducible analytic curves such that $\bigcup_1^s C_i$ is connected and the intersection matrix (C_i, C_j) is negative definite. Then there exists a normal surface Y with a singularity at 0, and open neighbourhoods $0 \in V \subset Y$ and $\bigcup C_i \subset U \subset X$, such that there is an analytic map $\rho : U \to V$ that is the resolution of the singularity $(Y, 0)$ with the exceptional divisor $\rho^{-1}(0) = \bigcup C_i$.*

One beauty of normal surface singularities is that not only they have resolutions – that also holds for more general singularities – but they have minimal resolutions, too.

Theorem 1.4.12. *Every normal surface singularity has a good resolution. Moreover, there exist a unique minimal resolution and a unique minimal good resolution.*

One can check minimality with the following condition.

Proposition 1.4.13. *A resolution is minimal if and only if none of the rational smooth components E_v have -1 self-intersection. A resolution is minimal good if and only if each such rational (-1)-curve intersects at least 3 other components.*

Definition 1.4.14. *Let $\rho : \widehat{X} \to X$ be a good resolution of a normal surface singularity $(X, 0)$. We define the* ***resolution graph***

$$\Gamma_X = (\mathcal{V}, \mathcal{E})$$

of the singularity as the following undirected but not necessarily simple graph. For each irreducible component E_v of the exceptional divisor we assign a vertex $v \in \mathcal{V}$. If two components, E_v and E_w, intersect each other at k points, we connect the corresponding vertices v, w with k edges. We also introduce decorations on the vertices. For each vertex v, we mark the self-intersection number $e_v = E_v^2$ and the genus $[g_v]$ of the curve E_v. We usually omit the genus if it is 0.

The resolution graph Γ_X of a normal surface singularity is a connected graph as E itself is a connected topological space.

The resolution graph is associated to a particular resolution, however we can ask which graphs belong to the same singularity. For instance, blowing up a point of an exceptional component E_v gives another resolution graph, associated to the same singularity, with a new (-1)-vertex (of genus 0). In fact, two resolution graphs corresponding to a given normal surface singularity can always be connected with a sequence of such (-1)-blowups and (-1)-blow-downs.

Note that we can read off the intersection matrix from the resolution graph.

1.5 Topological structure and the link

If we concentrate to a small enough neighbourhood of the singularity, all topological data is encoded in the boundary, that is called the link.

In most cases from now on, when we talk about a singularity (X, p), we assume that p is the origin, and – abusing notation – we will denote it by $0 = \underline{0} \in X$.

In a general setup, we need a notion of ball around the singularity. For a complex space germ $(X, 0)$ let us consider a 'distance', a real analytic function $|.| : X \to \mathbb{R}_{\geq 0}$ such that it only takes the value 0 at the origin. We should imagine this 'distance' as one inherited from the ambient $\mathbb{C}^N$ of one of its complex model spaces, but keep in mind that we are free to choose any other 'distance' meeting the conditions. We denote balls and spheres accordingly:

$$B_\varepsilon = B_{X,\varepsilon}(0) = \{p \in X : |p| \leq \varepsilon\},$$

$$S_\varepsilon = S_{X,\varepsilon}(0) = \{p \in X : |p| = \varepsilon\}.$$

According to [BV72] we can take a small enough neighbourhood of 0, such that the singularity is topologically a real cone over its link. The precise statement is the following.

Proposition 1.5.1. *For a complex space germ $(X, 0)$, we can pick a small enough $\varepsilon > 0$ real number such that*

(i) for each $\varepsilon' < \varepsilon$, $S_{X,\varepsilon'}$ is smoothly isomorphic to $S_{X,\varepsilon}$, and

(ii) $B_{X,\varepsilon}$ is homeomorphic to the real cone over $S_{X,\varepsilon}$.

Moreover, having a complex subgerm $(Y, 0) \subset (X, 0)$ and a respectively small $\varepsilon > 0$, the following holds, too. (In case of non-isolated singularities, a stratified versions of the following statements hold, see [BV72] for the statements and [Mat12].)

(iii) For each $\varepsilon' \leq \varepsilon$, the intersection $S_{\varepsilon'} \cap Y$ is transverse.

(iv) The embedded topology is also cone-like: the pair $(B_{X,\varepsilon}, B_{X,\varepsilon} \cap Y)$ is homeomorphic to the real cone over $(S_{X,\varepsilon}, S_{X,\varepsilon} \cap Y)$. ([BV72, Lemma 3.2])

Finally, these homeomorphism types are independent of the choice of $|.|$.

Definition 1.5.2. *We call $S_{X,\varepsilon}$ (respectively $S_{X,\varepsilon} \cap Y$) the* ***link*** *of $(X, 0)$ (resp. $(Y, 0)$) for such an ε and we denote it by L_X (resp. L_Y).*

For such a pair of isolated singularities $(Y, 0) \subset (X, 0)$, we have an embedding of links: $L_Y \hookrightarrow L_X$. If $Y = V(f)$ defined by an equation, the above embedding is of real codimension 2 with a complex line bundle structure on the normal bundle of $L_Y \subset L_X$.

Proposition 1.5.3. *The link L_X of a complex hypersurface germ $(X, 0) \subset (\mathbb{C}^{n+1})$ of dimension n is $(n-2)$-connected.*

For hypersurface singularities of dimension 2, this means that their links are connected.

If $(X, 0)$ is smooth and of complex dimension n, then the link is a real sphere $L_X \cong S^{2n-1}$.

In the case of an isolated singularity, the link is a smooth manifold that yields nice topology. If Y is a curve in X of dimension 2, $L_Y \hookrightarrow L_X$ is a knot or more generally a link. If we consider a normal surface singularity, we can consider a resolution of it – that exists by Theorem 1.4.12, then we can recognise that the boundary of a tubular neighbourhood of the resolution is diffeomorphic to the link. Therefore Theorems 1.4.10 and 1.4.11 imply the following.

Corollary 1.5.4. *The link of a normal surface singularity is homeomorphic to a connected graph 3-manifold with negative definite plumbing graph. Moreover, all such graph manifolds appear as links of such a singularity.*

In the case of an isolated but non-normal singularity, the above statement still holds as the reslution of the singularity factors through the normalization.

If X is a non-isolated singulartiy, the link L_X is not a smooth manifold. However, we can still tell that the number of connected components of L_X equals the number of irreducible components of $(X, 0)$.

1.6 The Milnor fibration

The Milnor fibration, or the local Milnor package, is created to grasp the structure of one-parameter smoothings of singularities. It is a way to ‘smoothen’ the singularity and detect some additional information about it. In this section, we define the Milnor fibration in a more general setup that we need, mention some crucial properties and give an overview of the state of the art. We refer to [Ném22] for the details, especially for the isolated case. At the end of this section, we give an overview of the development of this area.

The Milnor fibration is primarily defined for isolated hypersurface singularities. We begin with this case.

Regarding the fibration, we have the following result by Hamm ([Ham71, Satz 1.6]) and Lê ([Lê77, Theorem 1.1]).

Theorem 1.6.1. *Let $(X, 0)$ be a reduced complex space germ. Also, let $f : (X, 0) \to (\mathbb{C}, 0)$ be a (nontrivial) holomorphic function germ such that $V(f) \supset \operatorname{Sing}(X)$, in other words $X \setminus V(f)$ is smooth. Then if $\varepsilon \in \mathbb{R}_{>0}$ is small enough and $0 < \delta \ll \varepsilon$, we have two diffeomorphic smooth fibrations:*

(i) $\frac{f}{|f|} : S_\varepsilon \setminus V(f) \to S^1$

(ii) $f : B_\varepsilon \cap \{f = \delta\} \to \delta \cdot S^1$

The former, (i), is called the **Milnor fibration**. The latter, (ii) is the **nearby fibration** or the Milnor-Lê fibration. Conceptually the two are very different: (i) provides a fibration of the small sphere around the singularity using the argument of the function, whereas (ii) gives a fibration closely around the zero locus of f. By **Milnor fibre**, we mean the closure of the fibres – that are diffeomorphic to each other – of the nearby fibration:

$$F \cong F_\theta = \{f = \theta\} \cap \overline{B}_\varepsilon.$$

In case of an isolated singularity $(X, 0)$, the **boundary of the Milnor fibre** is diffeomorphic the link:

$$\partial F \cong L.$$

Also, the fibration (i) yields an open book decomposition of the sphere, where the pages are the fibres and the binding is the link.

Remark 1.6.2. *If the defining function f does not vanish on the whole singular locus in Theorem 1.6.1, the map* (ii) *is, again, a locally trivial fibration. However, in this case, the nearby fibres are singular. We call this a* ***singular nearby fibration****.*

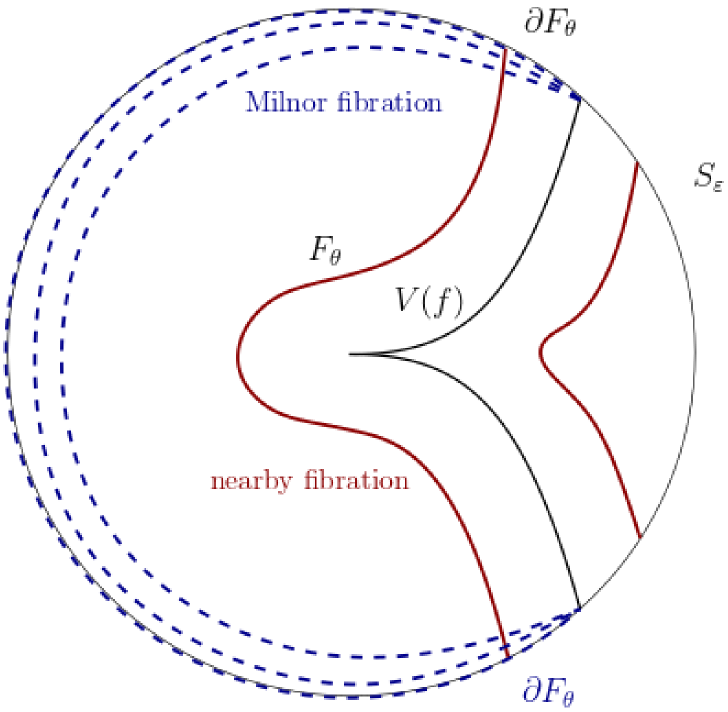


Figure 1.6.1: Milnor and nearby fibrations with their different kinds of fibres.

As we aim for a a more general application of this theory, we explore the possible generalizations. In case of higher codimension – that is when the singularity is defined by more equations – there is no given way to perturb the equations. Naturally, in case of a complete intersection singularity, we have independent perturbations for each equation, but we are interested in the case when this is not true, when the equations satisfy some nontrivial relations (or syzygies). There, we need a consistent way of perturbation. This is a highly nontrivial problem of deformation theory, and we will discuss it in Section 3.1. Here, we only show results informally to paint the picture.

To mimic the hypersurface case, we want a 1-parameter deformation with smooth fibres, or a **smoothing**. In other words, we want a space germ $(\widetilde{X}^{n+1}, 0)$ of one dimension higher than X with a map s making the following diagram commute.

$$\begin{tikzcd} (X,0) \arrow[r, hook] \arrow[d] & (\widetilde{X},0) \arrow[d, "\pi"] \\ 0 \arrow[r, hook] & (\mathbb{C},0) \end{tikzcd}$$

We also want the fibres $\pi^{-1}(t)$ to be smooth for $t \neq 0$ and a technical condition of flatness to be satisfied by π that actually makes it a deformation, that is a 'nice' and 'meaningful' perturbation.

All in all, provided such a smoothing, we ended up with an isolated hyperplane singularity $(X, 0) \subset (\widetilde{X}, 0)$.

On the other hand, in case of a non-isolated singularity, the boundary of a desired Milnor fibre would be smooth hence it cannot be isomorphic to the link that is singular. The two homeomorphic fibrations of Theorem 1.6.1 will have different boundaries. The open book decomposition of the sphere is problematic, too, with the binding being singular. For different versions of fibration theorem for non-isolated singularities and a nice description of the Whitney straitification needed, see [Lê16].

For an isolated surface singularity $(X, 0)$, resolutions help us describing the topology mentioned in this section.

Theorem 1.6.3. *Let $(X, 0)$ be an isolated surface singularity with a resolution graph Γ_X. Then the link L_X (and the boundary of the Milnor fibre ∂F when we have one) is homeomorphic to the graph (or plumbed) 3-manifold associated to Γ_X.*

In case of a non-isolated surface singuarity, we do not have such direct way to recover L_X. However, the following result by Curmi [Cur20] still holds.

Theorem 1.6.4. *Let (X , 0) be a 3-dimensional analytic space germ, and $f : (X, 0) \to (\mathbb{C}, 0)$ a reduced holomorphic germ, such that $V(f)$ contains the singular locus of $(X, 0)$. Then the boundary of the Milnor fibre ∂F of f is homeomorphic to an oriented plumbed 3-manifold.*

However, in contrast to the isolated case, we do not know how to construct the corresponding graph in general. This is why our result in Section 3.3 describing the graph associated to the Milnor fibre boundary of certain non-isolated surface singularities is relevant.

Going back to the case of isolated singularities, we can say a lot about the whole Milnor fibre.

Theorem 1.6.5. *Let $(X, 0)$ be an isolated singularity of dimension $n+1$ and $f : (X, 0) \to (\mathbb{C}, 0)$ a holomorphic germ on it. Let F be its Milnor fibre as defined in Theorem 1.6.1. Then the following hold true for F.*

(i) The Milnor fibre is a complex n-dimensional Stein manifold – that is a complex submanifold of a complex vector space.

(ii) It is homotopy equivalent to a real n-dimensional CW complex, see [AF59].

(iii) If, in addition, $(X, 0)$ is smooth, then F is $(n-1)$-connected, see [Mil74].

Hence, we can deduce the following.

Theorem 1.6.6. *Consider an isolated hypersurface germ $V(f) \subset \mathbb{C}^{n+1}$ corresponding to the holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Its Milnor fibre F is homotopy equivalent to a bouquet of spheres of real dimension n*

$$F \simeq \bigvee^{\mu} S^n.$$

The latter still holds for isolated complete intersection singularities (see [Ham71]).

The number of spheres appearing in the homotopy type is called the Milnor number, and it has an analytic description, too. This is a beautiful connection between the topological and analytic sides of singularities.

Definition 1.6.7. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an analytic germ. Its* ***Milnor number*** *is the codimension of its Jacobian ideal in the local ring at 0:*

$$\mu = \dim_{\mathbb{C}} \left(\mathcal{O}_{\mathbb{C}^{n+1},0} \Big/ \left(\tfrac{\partial f}{\partial x_1}, \ldots, \tfrac{\partial f}{\partial x_{n+1}} \right) \right).$$

In [Mil74], Milnor defined the 'multiplicity' μ for isolated singularities defined by $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ as the degree of the mapping

$$S^{2n+1}_{\varepsilon} \to S^{2n+1}_1, \quad \underline{x} \mapsto \frac{f(\underline{x})}{|f(\underline{x})|}.$$

Accordingly, the Milnor number also agrees with the maximal number of critical points in a small deformation of the holomorphic function.

Definition 1.6.8 (Signature). *According to the theorem above, all the homological data concentrates in H_n. As that is exactly half the real dimension of F, we have the intersection form*

$$\cap : H_2(F, \mathbb{Z}) \times H_2(F, \mathbb{Z}) \cong \mathbb{Z}^{\mu} \times \mathbb{Z}^{\mu} \to \mathbb{Z}$$

that is a $(-1)^n$-symmetric bilinear form.
For n even, we can define the ***signature***

$$\sigma_f = \mu_+ - \mu_-$$

where μ_+, μ_0, μ_- are Sylvester's indexes of inertia, that is the number of $+1$, 0 and -1 diagonal elements in the diagonal matrix corresponding to the form.

1.6.1 History

We summarize some of the most relevant results in the study of the Milnor fibre mostly following the great historical account in [Cur20].

The first spark to start this filed was Milnor's examples of exotic spheres [Mil56]. This result is relevant because links of singularities turned out to be an important source of exotic spheres. A couple of years later Mumford proved that, in case of a complex surface, being a topological manifold implies regularity [Mum61]. In fact, he proves that a surface germ is regular if and only if its link is simply connected. Later, Neumann showed that the link of a surface singularity can be recovered from its fundamental group except for the well-known families of cyclic quotients and cusps [Neu81]. However, shortly after that Brieskorn showed that this fails in higher dimensions, giving counterexamples [Bri66b]. Milnor, in 1968, published his book on isolated hypersurface singularities in $\mathbb{C}^n$, where he introduced the Milnor fibration [Mil74]. Later, Lê generalized this concept to any hypersurface singularities in complex spaces [Lê77]. Hamm proved in [Ham71] that for an equidimentional complex space germ $(X, 0)$ and a holomorphic function $f : (X, 0) \to (\mathbb{C}, 0)$ with $V(f) \supset \mathrm{Sing}(X)$, the Milnor fibration is a smoothing.

Assigning Milnor fibres to singularities is an extremely difficult task in general. Usually there is more than one possible smoothings, sometimes there is none of them. In case of an isolated surface singularity, there are important specific results. Brieskorn proved that for A, D, E singularities, the Milnor fibre is unique and diffeomorphic to the corresponding minimal resolution [Bri66a]. We have complete descriptions of the Milnor fibre in a few other cases: normal toric surface singularities by Lisca [Lis07] and Némethi and Popescu-Pampu [NP10]; sandwich singularities by de Jong and van Straten [JS90]. Finally, a non-isolated surface singularity whose Milnor fibre we know is the hypersurface singularity of the form $\{f(x, y) + z \cdot g(x, y) = 0\} \subset \mathbb{C}^3$ studied by Sigurðsson in [Sig16].

Studying the boundary ∂F of the Milnor fibre, however, appears to be a slightly less difficult question. Mumford [Mum61] and Grauert [Gra62] proved the two directions of Theorem 1.5.4 characterizing the links of normal surface singularities. However, the language of graph manifolds that we use was only introduced later by Waldhausen [Wal67b; Wal67a].

Although we aim for a similar classification in the case of non-isolated surface singularities, proving that the boundary of the Milnor fibre is a graph manifold is already challenging. Siersma [Sie91; Sie01] computed the homology of ∂F in certain cases and characterized when ∂F is a rational homology sphere. Michel, Pichon and Weber gave the plumbing graphs of ∂F for Hirzebruch surface singularities (that are of the form

$x^k y^l - z^m = 0$ with $\gcd(\mathrm{k},\mathrm{l},\mathrm{m}) = 1$) [MPW07] and for suspensions ($g(x,y) - z^m = 0$) [MPW09]. Michel and Pichon also showed that, in the case when the total space of the smoothing is smooth and the equation defining the singularity inside it is reduced, ∂F is a graph-manifold [MP03; MP16]. Fernández de Bobadilla and Menegon Neto extended this result to a wider context [FM14]. However, these proofs were not constructive. Némethi and Szilárd gave an algorithm for constructing the plumbing graph of ∂F in the case of reduced holomorphic functions $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ [NS12]. Curmi developed their strategy further for the case $f : (X, 0) \to (\mathbb{C}, 0)$ with $V(f) \supset \mathrm{Sing}(X)$ [Cur20].

1.7 Plane curve singularities

The simplest setup where we can find singularities, is the case of plane curves. Let us consider a reduced power convergent series $f \in \mathfrak{m} \subset \mathbb{C}\{x, y\}$. This defines an isolated hypersurface singularity on the plane $(\mathcal{C}, 0) = (V(f), 0) \subset (\mathbb{C}^2, 0)$. We can look at the irreducible decomposition of the defining equation $f = f_1 \cdot \dots \cdot f_k$ – where each factor is reduced, which leads to a decomposition of the curve: $(\mathcal{C}, 0) = (\mathcal{C}_1, 0) \cup \dots \cup (\mathcal{C}_k, 0)$. We call the components branches.

If we normalize an irreducible plane curve, we obtain a parametrization of the curve: $\varphi : (\mathbb{C}, 0) \to (\mathcal{C}, 0)$. The parametrization is usually described with two power series: $\varphi(t) = (x(t), y(t))$, $x, y \in \mathbb{C}\{t\}$ with $f(x(t), y(t)) = 0$.

As this is one of the most studied subfields of singularity theory, there are beautiful facts and puzzling open questions about plane curve germs. However, we restrict our attention to those concepts that we need later.

For a pair of plane curve singularities, we can assign a numerical invariant.

Definition 1.7.1. *Let $f, g \in \mathbb{C}\{x, y\}$ be irreducible plane curve singularities. Their* ***intersection multiplicity*** *is*

$$i(f, g) = \mathrm{ord}_t\, g(x(t), y(t))$$

where $(x(t), y(t))$ is a parametrization of f. If the germs are not irreducible, then the intersection multiplicity is that of the pairs of the respective branches summed up.

Proposition 1.7.2.

(i) The intersection multiplicity is independent of the parametrization.

(ii) The intersection multiplicity is symmetric: $i(f, g) = i(g, f)$.

(iii) $i(f,g) = \dim_{\mathbb{C}} \left(\mathbb{C}\{x,y\} \big/ (f,g) \right)$

When $i(f,g) = 1$, we say that the two germs intersect each other **transversely**. This fits the notion of transversality in differential topology.

Remark 1.7.3. *If we take small enough generic perturbations of two plane curves f, g with no common factor, they will have $i(f,g)$-many transverse intersection points – in a previously chosen – small neighbourhood of* 0.

The parametrization $(\mathbb{C}, 0) \to (\mathcal{C}, 0)$ of an irreducible curve germ $(\mathcal{C}, 0) = V(f)$ yields an embedding morphism of the local rings

$$n^* : \mathbb{C}\{x,y\}/(f) \cong \mathcal{O}_{\mathcal{C},0} \hookrightarrow \mathcal{O}_{\mathbb{C},0} \cong \mathbb{C}\{t\}.$$

When f is reducible, the embedding has the target $\mathcal{O}_{\bigcup(\mathbb{C},0)_i} \cong \bigoplus_1^k \mathbb{C}\{t_i\}$.

More generally, for any singularity $(X, 0)$, normalization $n : (\widehat{X}, 0) \to (X, 0)$ yields an embedding $n^* : \mathcal{O}_{X,0} \hookrightarrow \mathcal{O}_{\widehat{X},0}$ of local algebras.

Definition 1.7.4. *The* δ***-invariant*** *is the codimension of the embedding* n^**:*

$$\delta(f) = \dim_{\mathbb{C}} \left(\mathcal{O}_{X,0} \Big/ \mathcal{O}_{\widehat{X},0} \right),$$

if it is finite.

The δ-invariant behaves nicely if we take the union of two curve germs.

Proposition 1.7.5. *If $f, g \in \mathbb{C}\{x,y\}$ are reduced power series with no common factor, then*

$$\delta(f,g) = \delta(f) + \delta(g) + i(f,g).$$

It is also related to the Milnor number.

Theorem 1.7.6. *Let $f \in \mathfrak{m} \subset \mathbb{C}\{x,y\}$ be a reduced plane curve singularity. Then*

$$\mu(f) = 2\delta(f) - k + 1,$$

where $\mu(f)$ is the Milnor number of the singularity and k denotes the number of irreducible components of f.

Chapter 2

Maps and invariants

Holomorphic map germs $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ can have different kinds of isolated singularities that are stable under perturbation, as opposed to the case of hypersurface singularities ($\mathbb{C}^n \to \mathbb{C}$), where there is only one kind: the Morse singularity, $x_1^2 + ... + x_n^2$. However, the number of points of each type of isolated stable singularitie in a 'stabilization' of a given singularity does not depend on the stabilization. We will study the respective invariants in the case of $\mathbb{C}^2 \to \mathbb{C}^3$ maps after establishing the notions needed.

In her paper [Rua22], Ruas tells the origins of this field – called Thom–Mather theory – in the following way:

"In 1944, Whitney [Whi44] studied the first pair of dimensions not covered by his immersion theorem. For mappings f from $\mathbb{R}^n$ to $\mathbb{R}^{2n-1}$ Whitney proved that singularities cannot be avoided (with perturbations) in general. He introduced the semi regular mappings as proper mappings $f : \mathbb{R}^n \to \mathbb{R}^{2n-1}$ whose only singularities are the generalized cross-caps (Whitney umbrellas) points. Away from singular points, f is an immersion with transverse double points, and when $n = 2$ a finite number of triple points may also appear in the image of f. These are the only stable singularities in these dimensions. However, only later, Whitney introduced the notion of stable mappings."

Though we study complex mappings, the leading questions and results of this chapter resonates very well with Whitney's endeavours mentioned above.

2.1 Map germs

In this section the great book [MN20] of Mond and Nuño-Ballesteros is closely followed and we include several examples from there.

2.1.1 Mappings up to $\mathscr{A}$-equivalence

Singularity theory begins where f is neither a submersion nor an immersion though it is also concerned with the way that multiple immersions can interact.

[MN20]

Consider an analytic map $\Phi : \mathbb{C}^n \to \mathbb{C}^p$. We want to define the 'regularity' of this map. However, it turns out that depending on the dimensions, there are two 'regular' behaviours.

(i) If $n \geq p$, a regular mapping is a submersion. That is, when the differential is surjective. This matches the definition 1.3.17 we gave earlier. This is also how we define singularities of a subspace defined by an equation $f : X \to \mathbb{C}$.

(ii) If $n \leq p$, a regular mapping is an immersion. That is, when the differential is injective. This is the dual of submersion in a sense.

We can summarize this phenomenon saying that a mapping is regular at a point $x \in \mathbb{C}^n$ if the rank of the Jacobian at x is the possible maximum (that is, $\min(n, p)$). Otherwise Φ has a singularity at x. Note that there is the following discrepancy between this notion of singularity and the singularity of spaces described in Definition 1.2.14.

Remark 2.1.1. *A prominent example of singularities of $\mathbb{C}^2 \to \mathbb{C}^3$ maps is the Whitney umbrella (or cross-cap):*

$$\Phi : \mathbb{C}^2 \to \mathbb{C}^3,\ (s, t) \mapsto (s^2, st, t).$$

This map is not an immersion at $(0, 0)$, hence we say it has a singularity there, and at all the other points it is regular in the above sense. On the other hand, the image of Φ is an analytic subspace $X \subset \mathbb{C}^3$. As a space, it, indeed, has a singularity at $\Phi(0, 0) = (0, 0, 0)$, but the singular locus $\mathrm{Sing(X)}$ is the whole line $\Phi(s, 0) = (s^2, 0, 0)$.

In this section, we would like to handle both complex and real mappings. For this, we introduce the following notations.

Notation 2.1.2.

(i) $\mathbb{F}$ *denotes* $\mathbb{C}$ *and* $\mathbb{R}$.

(ii) When we talk about a ***smooth*** *map germ $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$, we mean either complex analytic map germs or real C^∞ mappings.*

(iii) In this context, a diffeomorphism means biholomorphism for $\mathbb{F} = \mathbb{C}$. In turn, a 'smooth space' means a regular analytic space in the complex case and a manifold in the real case.

Definition 2.1.3. *Let $F : \mathbb{F}^n \to \mathbb{F}^p$ be a smooth map.*

*(i) The map F is a **submersion** at $x \in \mathbb{F}^n$ if its differential*

$$\mathrm{d}F_x : T_x\mathbb{F}^n \to T_{F(x)}\mathbb{F}^p$$

is a surjective linear map.

*(ii) It is a **immersion** at x if the differential $\mathrm{d}F_x$ is injective.*

*(iii) A **critical point** is a point in the source where the map is not a submersion. The set of critical points of F is denoted by $\mathrm{Crit}(F)$.*

*(iv) A **critical value** of F is the image of a critical point. The set of critical values is called the **discriminant** of F and is denoted by $\Delta(F) = F(\mathrm{Crit}(F))$.*

Definition 2.1.4. *Two smooth map germs $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ and $\Psi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ between smooth space germs are **left-right equivalent** (or $\mathscr{A}$**-equivalent**) if there exist germs of diffeomorphisms $F : (\mathbb{F}^n, 0) \to (\mathbb{F}^n, 0)$, $G : (\mathbb{F}^p, 0) \to (\mathbb{F}^p, 0)$ such that*

$$\Psi = G \circ \Phi \circ F^{-1}.$$

This condition can be expressed with the commutative diagram

$$\begin{tikzcd}
(\mathbb{F}^n, 0) \arrow[r, "\Phi"] \arrow[d, "F"] & (\mathbb{F}^p, 0) \arrow[d, "G"] \\
(\mathbb{F}^n, 0) \arrow[r, "\Psi"] & (\mathbb{F}^p, 0).
\end{tikzcd}$$

Note that the notions of left ($\mathscr{L}$-)equivalence and right ($\mathscr{R}$-)equivalence can be defined in similar manner. However, we want to concentrate on left-right equivalence in this thesis.

The name $\mathscr{A}$-equivalence refers to the following. Let

$$\mathscr{A} = \mathscr{A}_{n,p} = \mathrm{Diff}(\mathbb{F}^n, 0) \times \mathrm{Diff}(\mathbb{F}^p, 0).$$

the set of pairs of diffeomorphisms. This is, in fact, a group with respect to composition. This acts on the set of map germs $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ accordingly:

$$(F, G)(\Phi) = G \circ \Phi \circ F^{-1}.$$

Two map germs are $\mathscr{A}$-equivalent if they are in the same orbit of this action.

An important project in Thom–Mather theory is the classification of them up to different equivalences – left-right equivalence in our case. An important notion in the topic is finite determinacy, for which we need the definition of jets.

Definition 2.1.5. *The k-**jet** of a map germ $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ is the degree k Taylor polynomial of f.*

For complex analytic germs, we can obtain their jets by cutting their power series at the given degree. However, it depends on the local coordinates we choose.

One can also define the k-jet bundle $J^k(X, Y)$ for a pair of smooth spaces X, Y. Its fibre $J^k(X, Y)_{(x,y)}$ is the set of k-jets of map germs $(X, x) \to (Y, y)$. This is a locally trivial fibre bundle over $X \times Y$.

Definition 2.1.6. *The smooth germ $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ is k-**determined for $\mathscr{A}$-equivalence** if given any Ψ with the same k-jet as Φ, the germs Ψ and Φ are $\mathscr{A}$-equivalent. A smooth germ is **finitely determined** if it is finitely l-determined for some k.*

2.1.2 Stable perturbations

Definition 2.1.7 (Unfolding).

*(i) A d-**parameter unfolding** of a holomorphic map germ $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ is a smooth map germ*

$$\widetilde{\Phi} : (\mathbb{F}^n \times \mathbb{F}^d, 0) \to (\mathbb{F}^p \times \mathbb{F}^d, 0)$$

of the form

$$\widetilde{\Phi}(x, t) = (\Phi_t(x), t)$$

such that $\Phi_0(x) = \Phi(x)$.

Equivalently, we require $\widetilde{\Phi}$ to fit into the following commutative diagram.

$$\begin{tikzcd}
(\mathbb{F}^n, 0) \arrow[rr, "\Phi"] \arrow[dr] \arrow[dd, hook] & & (\mathbb{F}^p, 0) \arrow[dl] \arrow[dd, hook'] \\
& 0 \arrow[dd, hook] & \\
(\mathbb{F}^n \times \mathbb{F}^d, 0) \arrow[rr, "\widetilde{\Phi}"] \arrow[dr, "\pi"] & & (\mathbb{F}^p \times \mathbb{F}^d, 0) \arrow[dl, "\pi"'] \\
& (\mathbb{F}^d, 0) &
\end{tikzcd}$$

(ii) Two unfoldings $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ of Φ are ***equivalent as unfoldings*** *if there are germs of diffeomorphisms*

$$F : (\mathbb{F}^n \times \mathbb{F}^d, 0) \to (\mathbb{F}^n \times \mathbb{F}^d, 0) \quad \text{and} \quad G : (\mathbb{F}^p \times \mathbb{F}^d, 0) \to (\mathbb{F}^p \times \mathbb{F}^d, 0)$$

that are d-parameter unfoldings of the respective identity mappings:

$$F(x,t) = (F_t(x), t),\ G(y,t) = (G_t(y), t) \text{ with } F_0(x) = x, G_0(y) = y;$$

and satisfy

$$\widetilde{\Phi}_2 = G \circ \widetilde{\Phi}_1 \circ F^{-1}.$$

The latter relation can be phrased in terms of the following commutative diagram, too.

$$\begin{tikzcd}
(\mathbb{F}^n \times \mathbb{F}^d, 0) \arrow[r, "\widetilde{\Phi}_1"] \arrow[d, "F"] & (\mathbb{F}^p \times \mathbb{F}^d, 0) \arrow[d, "G"] \\
(\mathbb{F}^n \times \mathbb{F}^d, 0) \arrow[r, "\widetilde{\Phi}_2"] & (\mathbb{F}^p \times \mathbb{F}^d, 0)
\end{tikzcd}$$

(iii) We call an unfolding ***trivial*** *if it is equivalent to the constant unfolding*

$$\Phi \times \mathrm{id} : (x,t) \mapsto (\Phi(x), t).$$

Definition 2.1.8. *A map germ $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ is* ***stable*** *if all of its unfoldings are trivial.*

Stability also means that the germ cannot be changed up to $\mathscr{A}$-equivalence with perturbations.

Let us see some examples.

Example 2.1.9. *Consider the germ $f : (\mathbb{F},0) \to (\mathbb{F},0)$, $f(x) = x^2$ and its unfolding $\widetilde{f} : (\mathbb{F} \times \mathbb{F}) \to (\mathbb{F} \times \mathbb{F})$, $\widetilde{f}(x,t) = (x^2 + tx, t)$. This unfolding is, in fact, trivial. For example, the diffeomorphism germs $g(x,t) = (x + \frac{1}{2}t, t)$ and $h(x,s) = (x - \frac{1}{4}s^2, s)$ trivialize it.*

$$\begin{array}{ccc} (\mathbb{F} \times \mathbb{F}, 0) & \xrightarrow{\widetilde{f}} & (\mathbb{F} \times \mathbb{F}, 0) \\ \Big\downarrow g & & \Big\downarrow h \\ (\mathbb{F} \times \mathbb{F}, 0) & \xrightarrow{\textit{triv.}} & (\mathbb{F} \times \mathbb{F}, 0) \end{array}$$

Indeed, the composition is the trivial unfolding:

$$\begin{aligned} h \circ \widetilde{f} \circ g^{-1}(x,u) &= h\Big(\widetilde{f}(x - \tfrac{1}{2}u, u)\Big) \\ &= h\Big((x - \tfrac{1}{2}u)^2 + ux, u\Big) \\ &= h(x^2 - ux + \tfrac{1}{4}u^2 + ux, u) \\ &= h(x^2 + \tfrac{1}{4}u^2, u) \\ &= (x^2, u). \end{aligned}$$

Example 2.1.10. *On the other hand, the unfolding $\widetilde{g}(x^3 + tx, t)$ of the germ $g(x) = x^3$ is nontrivial. We can prove this by noticing that g_t has two critical points for $t \neq 0$ whereas g has only one and showing that this makes it impossible to trivialize $\widetilde{g}$.*

We can compare unfoldings of the same map in the following way. Let $\widetilde{\Phi} : (\mathbb{F}^n \times \mathbb{F}^d, 0) \to (\mathbb{F}^p \times \mathbb{F}^d, 0)$ be an unfolding of $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$. Furthermore, let $g : (\mathbb{F}^k, 0) \to (\mathbb{F}^d, 0)$ be a smooth map germ – we call this a base change. Then the pull-back of $\widetilde{\Phi}$ by g, or the unfolding induced from $\widetilde{\Phi}$ by g is

$$g^*\widetilde{\Phi} : (\mathbb{F}^n \times \mathbb{F}^k, 0) \to (\mathbb{F}^p \times \mathbb{F}^k, 0), \quad g^*\widetilde{\Phi}(x,s) = \Big(\Phi_{g(s)}(x), s\Big).$$

Definition 2.1.11. *An unfolding $\widetilde{\Phi}$ of Φ (as above) is **versal** if, for any unfolding $\widetilde{\Phi}' : (\mathbb{F}^n \times \mathbb{F}^k, 0) \to (\mathbb{F}^p \times \mathbb{F}^k, 0)$, there exists a base change $g : (\mathbb{F}^k, 0) \to (\mathbb{F}^d, 0)$ such that the pull-back $g^*\widetilde{\Phi}$ is equivalent as unfolding to $\widetilde{\Phi}'$.*

Unfortunately, the base change map and the equivalence of unfoldings in the definition are usually not unique. This is why we call this notion 'versal' instead of 'universal'. However, we can define a notion of minimality.

Definition 2.1.12. *An unfolding $\widetilde{\Phi}$ is **miniversal** if it is versal and its paremeter space is of minimal dimension among the versal unfoldings.*

2.1.3 Infinitesimal deformations and $\mathscr{A}_e$-codimension

The 'space' of all unfoldings is unmanageably large as we can perturb our map germ in any way – while paying attention to keeping it analytic. This makes it extremely hard to study unfoldings, for example showing that an unfolding is nontrivial (without tricks such as in Example 2.1.10), checking stability, or proving the existence of versal deformations. For this reason, we introduce infinitesimal deformations. Essentially, we are only interested in very small perturbations of maps.

Let $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ be a smooth germ.

Definition 2.1.13. *The* ***space of infinitesimal deformations*** *is*

$$\mathrm{ID}(\Phi) = \left\{ \left.\frac{\mathrm{d}\Phi_t}{\mathrm{d}t}\right|_{t=0} : \quad \widetilde{\Phi}(x,t) = (\Phi_t(x), t) \text{ is a 1-parameter unfolding of } \Phi \right\}.$$

This is an $\mathbb{F}$-vector space with respect to the following operations. The identity element is given by the trivial deformation $\Phi_0 \times 1$. The addition $\frac{\mathrm{d}\Phi_t}{\mathrm{d}t}|_{t=0} + \frac{\mathrm{d}\Psi_t}{\mathrm{d}t}|_{t=0} = \frac{\mathrm{d}(\Phi_t+\Psi_t-\Phi_0)}{\mathrm{d}t}|_{t=0}$ uses the unfolding $\Phi_t + \Psi_t - \Phi_0$. The scalar multiple by $\alpha \in \mathbb{C}$ is defined as $\alpha \cdot \frac{\mathrm{d}\Phi_t}{\mathrm{d}t}|_{t=0} = \frac{\mathrm{d}\Phi_{\alpha t}}{\mathrm{d}t}|_{t=0}$. These operations do not depend on the representative we take for the elements, and they satisfy the relations required.

However, at this point we do not even know whether this vector space is of finite dimension, let alone any finer structure on it.

Note that an unfolding does not need to respect the base point, meaning $\Phi_t(0) = 0$ might not hold for $t \neq 0$. It makes sense to require this, too, which leads to analogue notions, see Remark 2.1.21.

We can ask which infinitesimal unfoldings are trivial in the left-right sense. Trivializations $G \circ \widetilde{\Phi} \circ F^{-1} = \Phi \times \mathrm{id}$ are used in the following definition.

Definition 2.1.14. *The* ***extended tangent space of*** Φ *is*

$$T\mathscr{A}_e\Phi = \left\{ \left.\frac{\mathrm{d}}{\mathrm{d}t}(G_t^{-1} \circ \Phi \circ F_t)\right|_{t=0} : \quad F_0 = \mathrm{id}, G_0 = \mathrm{id} \right\}.$$

This is a subspace of $\mathrm{ID}(\Phi)$. The word 'extended' stands for not respecting the base point.

Taking the quotient of the two spaces above, we get the classes of nontrivial unfoldings.

Definition 2.1.15.

$$T^1_{\mathscr{A}_e}\Phi = \mathrm{ID}(\Phi) \big/ T\mathscr{A}_e\Phi$$

Indeed, this is isomorphic to the tangent space at 0 to the base space of a miniversal deformation. As the notion of flatness is not relevant in case of our unfoldings, there will be no obstruction space T^2 (see Section 3.1). The dimension of this tangent space will be extremely important.

There is another way to look at infinitesimal deformations. The elements of the above spaces can be interpreted as germs of vector fields. First, consider an infitesimal deformation $\left.\frac{\mathrm{d}\Phi_t}{\mathrm{d}t}\right|_{t=0}$ and fix a point $x \in \mathbb{F}^n$ in the source. The image $\Phi_t(x)$ is a curve through $\Phi_0(x) = \Phi(x)$. Hence the infinitesimal deformation at x results in a tangent vector in $T_{\Phi(x)}\mathbb{F}^p$. Therefore, in this sense, $\left.\frac{\mathrm{d}\Phi_t}{\mathrm{d}t}\right|_{t=0}$ is a vector field making the diagram

$$\begin{tikzcd} & T\mathbb{F}^p \arrow[d, "\pi"] \\ \mathbb{F}^n \arrow[ur, "\left.\frac{\mathrm{d}\Phi_t}{\mathrm{d}t}\right|_{t=0}"] \arrow[r, "\Phi"] & \mathbb{F}^p \end{tikzcd} \tag{2.1}$$

commute, where π is the natural projection of the tangent bundle. On the other hand, any vector field ζ fitting into the diagram (2.1) defines an infinitesimal deformation through the unfolding $\widetilde{\Phi}(x,t) = (\Phi(x) + \hat{\zeta}_x(t), t)$ where $\hat{\zeta}_x$ is a flow of ζ: a smooth x-family of curves with $\hat{\zeta}_x(0) = x$.

We introduce notations for vector fields.

Definition 2.1.16.

(i) *We denote the space of vector fields on* $\mathbb{F}^k$ *by* θ_k.

(ii) *Let* $\Phi : \mathbb{F}^n \to \mathbb{F}^p$ *be a smooth map. The space* $\theta(\Phi)$ *of* ***vector fields along*** Φ *consists of smooth maps* $\zeta : \mathbb{F}^n \to T\mathbb{F}^p$ *such that* $\pi \circ \zeta = \Phi$. *The latter condition is equivalent to fitting into the commutative diagram* (2.1).

We can create vector fields along Φ *from vector fields on the source and the target.*

(iii) *A vector field* $\xi \in \theta_n$ *can be pushed forward by* $\mathrm{d}\Phi$ *yielding a vector field along* Φ. *We denote this by* $t\Phi(\xi) = \mathrm{d}\Phi \circ \xi$.

(iv) *Similarly, let* $\eta \in \theta_p$ *be a vector field over the target* $\mathbb{F}^p$. *This, we can pull back to get* $\omega\Phi(\eta) := \eta \circ \Phi$ *that is, again, a vector field along* Φ.

The latter two constructions can be summed up in the following diagram.

$$\begin{tikzcd} T\mathbb{F}^n \arrow[r, "\mathrm{d}\Phi"] \arrow[d] & T\mathbb{F}^p \arrow[d] \\ \mathbb{F}^n \arrow[u, dashed, bend left, "\xi"] \arrow[ur, dashed, "t\Phi(\xi)"] \arrow[ur, squiggly, "\omega\Phi(\xi)"'] \arrow[r, "\Phi"'] & \mathbb{F}^p \arrow[u, squiggly, bend right, "\eta"'] \end{tikzcd}$$

With these notations, we can say that $\mathrm{ID}(\Phi) = \theta(\Phi)$.

Moreover, the elements of the extended tangent space of Φ can also be interpreted as vector fields along Φ – as $T\mathscr{A}_e\Phi \subset \mathrm{ID}(\Phi)$. Using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}(G_t^{-1} \circ \Phi \circ F_t)\bigg|_{t=0} = \mathrm{d}\Phi \circ \left(\frac{\mathrm{d}F_t}{\mathrm{d}t}\bigg|_{t=0}\right) + \left(\frac{\mathrm{d}G_t^{-1}}{\mathrm{d}t}\bigg|_{t=0}\right) \circ \Phi.$$

In turn, the two summands can be interpreted as a vector field $\frac{\mathrm{d}F_t}{\mathrm{d}t}\big|_{t=0} = \xi$ on the source pushed forward and one $\frac{\mathrm{d}G_t^{-1}}{\mathrm{d}t}\big|_{t=0} = \eta$ on the target pulled back. This turns elements of the extended tangent space into the form $t\Phi(\xi) + \omega\Phi(\eta)$ for vector fields $\xi \in \theta_n, \eta \in \theta_p$. This argument can also be reversed: any pair of vector fields define an element in the extended tangent space. To sum up, we can say the following.

Proposition 2.1.17. *For a smooth map germ $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$,*

$$T^1_{\mathscr{A}_e}\Phi = \frac{\mathrm{ID}(\Phi)}{T\mathscr{A}_e\Phi} = \frac{\theta(\Phi)}{t\Phi(\theta_n) + \omega\Phi(\theta_p)}.$$

The dimension of this T^1-space is an important invariant of the germ Φ.

Definition 2.1.18. *The $\mathscr{A}_e$-codimension of Φ is*

$$\mathrm{codim}_{\mathscr{A}_e}(\Phi) = \dim_{\mathbb{F}} T^1_{\mathscr{A}_e}\Phi.$$

This codimension measures how unstable a germ is. In particular, it characterizes stability. Roughly speaking, it gives us the minimal number of parameters of a family where the particular map occurs 'irremovably' up to $\mathscr{A}$-equivalence.

Theorem 2.1.19. *[Mat69] A smooth germ Φ is stable if and only if*

$$\mathrm{codim}_{\mathscr{A}_e}(\Phi) = 0.$$

The latter condition is called infinitesimal stability. The theorem, thus, states that infinitesimal stability is equivalent to stability for germs. The direction that stability implies infinitesimal stability easily follows from the definition: we can see that any infinitesimal deformation can be trivialized. The other direction is much more difficult [Mat69].

Definition 2.1.20. *The germ Φ is called **$\mathscr{A}$-finite** if its $\mathscr{A}_e$-codimension is finite:*

$$\mathrm{codim}_{\mathscr{A}_e}(\Phi) < \infty.$$

Remark 2.1.21 (On deformations respecting the base point). *We can define base-point-preserving versions of all the notions in this subsection, leading to a slightly different codimension. The infinitesimal deformations with this property form a subspace* $\mathrm{ID}_0(\Phi) \subset \mathrm{ID}(\Phi)$*, as does the tangent space of* Φ*:*

$$T\mathscr{A}\Phi = \left\{ \frac{\mathrm{d}}{\mathrm{d}t}(G_t^{-1} \circ \Phi \circ F_t)\bigg|_{t=0} : F_0 = \mathrm{id}, G_0 = \mathrm{id}, F_t(0) = 0, G_t(0) = 0 \ \forall t \right\} \subset T\mathscr{A}_e\Phi.$$

The function $(G_t^{-1} \circ \Phi \circ F_t)$ *of* t *can be regarded as a curve in the* $\mathscr{A}$*-orbit of* Φ*. Informally,* $T\mathscr{A}\Phi$ *is the tangent space to the orbit.*

The elements of these spaces can also be expressed as vector fields involving the maximal ideal $\mathfrak{m}$ *that is responsible for fixing the base point:* $\mathfrak{m}_n\theta_n$ *and* $\mathfrak{m}_p\theta_p$ *are these vector spaces on the source and the target, and* $\mathfrak{m}_n\theta(\Phi)$ *is the set of such vector fields along* Φ*. This makes the* $\mathscr{A}$*-codimension*

$$\mathrm{codim}_{\mathscr{A}}(\Phi) = \dim_{\mathbb{F}} T^1_{\mathscr{A}}\Phi = \dim_{\mathbb{F}} \frac{\mathfrak{m}_n\theta(\Phi)}{t\Phi(\mathfrak{m}_n\theta_n) + \omega\Phi(\mathfrak{m}_p\theta_p)}.$$

The $\mathscr{A}$-codimension is closely related to the $\mathscr{A}_e$-codimension.

Proposition 2.1.22. *[MN20, p. 65] Let* $\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ *be a smooth germ. Then the following hold true.*

(i) $\mathrm{codim}_{\mathscr{A}}(\Phi) < \infty \iff \mathrm{codim}_{\mathscr{A}_e}(\Phi) < \infty$

(ii) If $0 < \mathrm{codim}_{\mathscr{A}_e}(\Phi) < \infty$*, then* $\mathrm{codim}_{\mathscr{A}}(\Phi) = \mathrm{codim}_{\mathscr{A}_e}(\Phi) - n + 2p$*.*

2.1.4 Finitely determined germs

The $\mathscr{A}$-finiteness condition is equivalent to being finitely $\mathscr{A}$-determined.

Theorem 2.1.23 (Mather's finite determinacy). *The following are equivalent for a smooth map germ* $\Phi : (\mathbb{F}^p, 0) \to (\mathbb{F}^p, 0)$*:*

(i) Φ *is* $\mathscr{A}$*-finite,*

(ii) Φ *is finitely* $\mathscr{A}$*-determined,*

(iii) $\mathfrak{m}_n^k\theta(\Phi) \subset T\mathscr{A}_e(\Phi)$ *for some* $k \in \mathbb{N}$*.*

For Mather's original proof, see [Mat68b]. For another, more efficient proof, see [Wal81] or [MN20, Section 6.1.]. From this point on, being finitely $\mathscr{A}$-determined and $\mathscr{A}$-finiteness are synonyms and we will usually use the latter.

We want to characterize finitely $\mathscr{A}$-determined – or equivalently $\mathscr{A}$-finite – germs geometrically.

Definition 2.1.24. *A smooth map $\Phi : \mathbb{F}^n \to \mathbb{F}^p$ is* ***locally stable*** *if*

(i) the restriction $\Phi|_{\mathrm{Crit}(\Phi)}$ to the critical set is finite – a closed map with finite fibres, and

(ii) for any critical value $y \in \Delta(\Phi)$, the multi-germ $\Phi : (\mathbb{F}^n, \Phi^{-1}(y)) \to (\mathbb{F}^p, y)$ is stable.

In condition (ii), we only care about the critical locus because regular germs are inherently stable. On the other hand, we should note that for $n < p$, all points are critical, that makes local stability simpler.

Remark 2.1.25. *For a smooth map $\Phi : \mathbb{F}^n \to \mathbb{F}^p$, with $p \geq n$, local stability is equivalent to being finite and stable as a multi-germ at each value.*

When we say that a map $\Phi : \mathbb{F}^n \to \mathbb{F}^p$ is locally stable 'over' a set $Y \subset \mathbb{F}^p$, we mean that for any $y \in Y$, the multi-germ

$$\Phi : (\mathbb{F}^n, \mathrm{Crit}(\Phi) \cap \Phi^{-1}(y)) \to (\mathbb{F}^p, y)$$

is locally stable.

Theorem 2.1.26 (Mather–Gaffney criterion). *A holomorphic germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is $\mathscr{A}$-finite if and only if it has a small representative $\Phi : U \to V$ such that*

(i) the only critical point in $\Phi|_U^{-1}(0)$ is 0, and

(ii) the restriction $\Phi : U \setminus \Phi^{-1}(0) \to V$ is locally stable.

For the proof and for the real version, see [Wal71].

Again, if the source has smaller or equal dimension than the target, the criterion simplifies significantly.

Corollary 2.1.27. *A holomorphic germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $p \geq n$ is $\mathscr{A}$-finite if and only if there is a small enough neighbourhood $0 \in U \subset \mathbb{C}^n$ such that only 0 gets mapped to 0 ($\Phi|_U^{-1}(0) = 0$), and $\Phi|_U$ is finite and stable outside the origin.*

If $\Phi|_U^{-1}(0) = 0$, then finiteness in a small neighbourhood is provided in case of an analytic map. In fact, the proof of Theorem 2.1.26 uses the following lemma, that will come in handy for us later.

Lemma 2.1.1. *An $\mathscr{A}$-finite holomorphic germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ has a small enough representative for which 0 is the only possible critical point getting mapped to 0.*

2.1.5 Stabilizations and versal deformations

In principle, $\mathscr{A}_e$-codimension measures how unstable the given germ is. Now, we want to see, how we can stabilize $\mathscr{A}$-finite germs with a small perturbation. We are looking for unfoldings that are generically 'stable' except for the original singularity. We will express this by avoiding 'bifurcations'.

Consider an $\mathscr{A}$-finite smooth germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with an unfolding $\widetilde{\Phi} : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$. Take a small $\varepsilon > 0$ such that Φ satisfies the Mather–Gaffney criterion over $B_\varepsilon \subset \mathbb{F}^p$, in particular Φ is locally stable outside the origin over this ε-ball. Then we look at the perturbations Φ_t in the unfolding $\widetilde{\Phi}$. As the sphere $S_\varepsilon = \partial B_\varepsilon$ is a compact set in $\mathbb{C}^p$ and disjoint from the origin, we can pick a small enough radius $\eta \ll \varepsilon$ such that for all $t \in \mathbb{C}$ with $|t| < \eta$ the mapping Φ_t is locally stable over S_ε. However, the perturbed mapping Φ_t might not be locally stable over the whole ε-ball.

Definition 2.1.28. *The **bifurcation set** of the unfolding $\widetilde{\Phi}$ in the above setup is the set*

$$\mathscr{B}(\widetilde{\Phi}) = \left\{ t \in B_\eta \subset \mathbb{C}^d : \ \Phi_t \text{ is not locally stable over } B_\varepsilon \subset \mathbb{C}^p \right\}.$$

Proposition 2.1.29. *For an $\mathscr{A}$-finite mapping Φ, the bifurcation set germ $(\mathscr{B}(\widetilde{\Phi}), 0)$ of any unfolding $\widetilde{\Phi}$ is an analytic space germ.*

However, in the real case, the bifurcation set is not real analytic, only semianalytic.

Example 2.1.30. *Consider the map germ $f(x) = (x^2, x^5)$. Its versal unfolding is $\tilde{f}(x, s, t) = (x^2, x^5 + sx^3 + tx)$. In the complex case, after a straightforward calculation, we get that the bifurcation set is*

$$\mathscr{B}_{\mathbb{C}}(\tilde{f}) = \left\{ (s,t) \in \mathbb{C}^2 : \ t(s^2 - 4t) = 0 \right\},$$

that is an analytic subspace consisting of two components.

Furthermore, describing the bifurcation set explicitly is a difficult task in general. For example, if we consider the germ $f(x) = (x^3, x^4)$ instead – that looks just as simple as the previous example, we find that the versal deformation has 3 parameters, and the bifurcation set has three complicated components that are hard to compute.

Definition 2.1.31. *A **stabilization** of a smooth mapping Φ is a one-parameter unfolding $\widetilde{\Phi}$ with bifurcation germ*

$$(\mathscr{B}(\widetilde{\Phi}), 0) = (0, 0).$$

In other words, Φ_t is stable over a small ball for a small enough $t \neq 0$.

The bifurcation set is key for the existence of a stabilization.

Proposition 2.1.32. *For an $\mathscr{A}$-finite holomorphic germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ the following are equivalent.*

(i) The germ Φ has a stabilization.

(ii) There exists an unfolding $\widetilde{\Phi}$ of Φ such that the bifurcation germ $(\mathscr{B}(\widetilde{\Phi}, 0))$ is a proper subgerm in the parameter space.

(iii) For any versal unfolding, the bifurcation germ is a proper subgerm of the parameter space.

2.1.6 Holomorphic germs $\mathbb{C}^2 \to \mathbb{C}^3$

In this subsection, let Φ be a holomorphic mapping $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$.

Corollary 2.1.33 (Mather-Gaffney criterion for $\mathbb{C}^2 \to \mathbb{C}^3$). *An $\mathscr{A}$-finite $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ holomorphic germ has a small representative that only has regular points and ordinary double points away from the origin.*

This follows from Theorem 2.1.26.

For the converse statement, we need the additional condition that it is one-to-one over the origin. Precisely, if a holomorphic germ $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ has a small representative $f : U \to V$ with $f^{-1}(0) = \{0\}$ that has only regular points and ordinary double points away from of 0, then the germ is $\mathscr{A}$-finite.

About stable germs in this pair of dimensions, we know the following, see also in [Mon87].

Proposition 2.1.34. *The stable holomorphic $(\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ multi-germs are the following:*

(i) Cross-cap or Whitney umbrella.

(ii) Triple point with regular intersection.

(iii) Transverse crossing of two branches.

(iv) Regular embedding.

The first two are isolated (multi-)germs

In any given pair of dimensions, there is a list of possible isolated singularities that can occur. The numbers of these points in the stabilization of a given $\mathscr{A}$-finite map germ is independent of the stabilization for the following reason. Consider a versal unfolding. The bifurcation set is a proper subgerm – this is why $\mathscr{A}$-finiteness is important (Proposition 2.1.29) – hence, we can connect any two deformations with a real curve avoiding $\mathscr{B}$. Along this curve, the above numbers remain unchanged.

The invariant counting cross-caps is the following.

Definition 2.1.35. *For a holomorphic map germ* Φ*, let*

$$C(\Phi) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} \big/ \mathcal{R}(\Phi)$$

*be the codimension of the **ramification ideal** $\mathcal{R}$, where the latter is the ideal in $\mathcal{O}_{\mathbb{C}^2,0}$ generated by the three 2×2 minors of the Jacobian of Φ at 0.*

Theorem 2.1.36. *[Mon87] The number of cross-cap points of the stabilization of the map germ Φ in the above sense is $C(\Phi)$.*

In case of a corank 1 mapping of the form $\Phi(x, y) = (x, f(x, y), g(x, y))$, the invariant can be computed slightly easier as

$$C(\Phi) = \dim \left(\mathcal{O}_{\mathbb{C}^2,0} \big/ (\partial f/\partial y, \partial g/\partial y) \right).$$

The other invariant concerning triple points can be defined with fitting ideals, defined usually for mappings $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. Take an $\mathscr{A}$-finite mapping Φ between these dimensions. Due to the Mather–Gaffney criterion (Corollary 2.1.27), such a mapping is finite, thus $\Phi_* \mathcal{O}_{\mathbb{C}^n,0}$ is a finitely generated $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module [Mat68a]. We have the corresponding exact sequence

$$\mathcal{O}^s_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^q_{\mathbb{C}^{n+1},0} \xrightarrow{g} \Phi_* \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow 0,$$

called the presentation of $\Phi_* \mathcal{O}_{\mathbb{C}^n,0}$.

The **presentation matrix** λ contains the relations between the generators g_i of $\Phi_* \mathcal{O}_{\mathbb{C}^n,0}$ as columns. According to Teissier – see [Tei77] – the minimal presentation has a square matrix, whose determinant is an equation for the image of Φ. Let us denote the size of this minimal matrix by q.

Definition 2.1.37. *The k**-th fitting ideal***

$$\mathcal{F}_k(\Phi) = \mathrm{Fitt}_k^{\mathcal{O}_{\mathbb{C}^{n+1},0}} (\Phi_* \mathcal{O}_{\mathbb{C}^n,0})$$

corresponding to $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ *is the ideal generated by the* $(q-k) \times (q-k)$ *minors of the minimal presentation matrix* λ*.*

In general, the k-th fitting ideals describe the multiple-point loci in the target. That is, $\mathcal{F}_k(\Phi)$ vanishes exactly at those points that have at least $k+1$ preimages – counted with multiplicity. These points form complex space germs. We focus our attention to triple points.

Definition 2.1.38. *For a holomorphic map germ* Φ*, let*

$$T(\Phi) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3,0} \big/ \mathcal{F}_2(\Phi).$$

Theorem 2.1.39. *[MP89] The number of triple points of a stabilization of the* $\mathscr{A}$*-finite holomorphic germ* $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ *is* $T(\Phi)$*.*

Example 2.1.40. *Consider the* A_2 *surface singularity* $(X, 0) = (\mathbb{C}^2, 0)/\mathbb{Z}_2$ *as the image of the mapping*

$$\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0),\ (s, t) \mapsto (s^2, t^2, st).$$

Its presentation matrix is

$$\lambda = \begin{pmatrix} -z & 0 & 0 & xy \\ 0 & -z & y & 0 \\ 0 & x & -z & 0 \\ 1 & 0 & 0 & -z \end{pmatrix}.$$

The determinant of λ *is* $\det(\lambda) = (z^2 - xy)^2$*. This defines the image of* Φ*, however it is not reduced – due to the fact that* Φ *is a double cover over the regular values of its image. The first fitting ideal* $\mathcal{F}_1(\Phi)$ *is the ideal generated by* $z^2 - xy$*, that is the vanishing ideal of the double values of* Φ*. The second fitting ideal is*

$$\mathcal{F}_2(\Phi) = (x, y, z) = \mathfrak{m}_{(\mathbb{C}^3,0)},$$

*that is a codimension-*1 *ideal in* $\mathcal{O}_{\mathbb{C}^3,0}$*, which means that a stable deformation of* Φ *has a single triple point.*

2.2 Invariants

This section is an expanded version of our paper [PS23] with Gergő Pintér.

Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined (also called $\mathcal{A}$-finite) holomorphic germ. In this case $\mathcal{A}$-finiteness means that Φ is a stable immersion off the origin [Wal81;

MN20]. For these germs the number of complex Whitney umbrella (cross cap) points $C(\Phi)$ and the triple values $T(\Phi)$ of a stable holomorphic deformation are well-defined analytic invariants. Recently in [FPS22] J. Fernández de Bobadilla, G. Peñafort, and J. E. Sampaio proved that these invariants are topological, moreover they are determined by the embedded topological type of the image of Φ. One of the main ingredients of their proof is the formula

$$L(\Phi|_{\mathfrak{S}}) = C(\Phi) - 3T(\Phi) \tag{2.2}$$

from [NP15], which expresses the naturally topological Ekholm–Szűcs invariant (also called triple point invariant or linking invariant) $L(\Phi|_{\mathfrak{S}})$ of the associated stable immersion $\Phi|_{\mathfrak{S}} : \mathfrak{S} \simeq S^3 \looparrowright S^5$ in terms of C and T. However, the formula (2.2) is proved in [NP15] in a rather complicated way, by using two Smale invariant formulas. The main purpose of this section is to provide a new direct proof for this formula.

The Ekholm–Szűcs invariant $L(f)$ of a stable immersion $f : S^3 \looparrowright \mathbb{R}^5$ measures the linking of the image with a copy of the double values, shifted slightly along a suitable chosen normal vector field. In the literature, different versions of the definition of L can be found (see [Ekh01a; Ekh01b; ES03; SST02]), whose relation is not completely clarified. We verify their equivalence, i.e. $L_1(f) = -L_2(f)$, based on their opposite behavior through regular homotopies.

Although our proof of the main theorem (2.2) is self-contained, an independent secondary goal of this section is to clarify the enigmatic relation between several versions of the linking invariant L and other related invariants, used in the study of generic $\mathcal{C}^\infty$ real maps and immersions.

2.2.1 Invariants of a stabilization

By Mather–Gaffney criterion (Corollary 2.1.33), a finitely determined germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is finitely determined if and only if its restriction $\Phi|_{\mathbb{C}^2 \setminus \{0\}}$ has a sufficiently small representative that has only (1) regular simple points and (2) double values with transverse intersection of the regular branches.

As said in Proposition 2.1.34, the only possible multi-germs of a stabilization (stable deformation) of a holomorphic germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ are regular simple points, double values with transverse intersection of the regular branches, triple values with regular intersection of the regular branches and simple *Whitney umbrella (cross cap)* points. The Whitney umbrellas and the triple values are isolated points, up to analytic $\mathscr{A}$-equivalence they have local normal forms

$$\text{Whitney umbrella (cross cap):} \quad (s,t) \mapsto (s^2, st, t) \tag{2.3}$$

$$\text{Triple value:} \begin{cases} (s_1,t_1) & \mapsto & (0,s_1,t_1) \\ (s_2,t_2) & \mapsto & (t_2,0,s_2) \\ (s_3,t_3) & \mapsto & (s_3,t_3,0) \end{cases} \tag{2.4}$$

The numbers $C(\Phi)$ of the cross caps and $T(\Phi)$ of the triple values are independent of the stabilization, they are analytic invariants of the finitely determined germs Φ. Both invariants were introduced by Mond [Mon85; Mon87], they can be defined in algebraic way as well, without referring to a stabilization, as follows.

Let $C_{alg}(\Phi)$ be the codimension of the *ramification ideal*, which is the ideal in the local ring $\mathcal{O}_{(\mathbb{C}^2,0)}$ generated by the determinants of the 2×2 minors of the Jacobian matrix of $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. $T_{alg}(\Phi)$ is the codimension of the second Fitting ideal associated with Φ in $\mathcal{O}_{(\mathbb{C}^3,0)}$ [MP89]. If Φ is finitely determined, then both $C_{alg}(\Phi)$ and $T_{alg}(\Phi)$ are finite, and any stabilization of Φ has $C(\Phi) = C_{alg}(\Phi)$ number of cross caps and $T(\Phi) = T_{alg}(\Phi)$ number of triple values. The invariants T and C appear in several different contexts, see for example [Mon91; MM89; MNP12; MN14; Pin19].

The analytic invariance of C and T means the following. Let Φ_1 and Φ_2 be finitely determined germs, analytic $\mathscr{A}$-equivalent to each other. That is, there exist germs of biholomorphisms $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and $\psi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that

$$\Phi_2 = \psi \circ \Phi_1 \circ \phi \tag{2.5}$$

holds, i.e. the diagram below commutes.

$$\begin{array}{ccc} (\mathbb{C}^2,0) & \xrightarrow{\Phi_1} & (\mathbb{C}^3,0) \\ \phi \uparrow & & \downarrow \psi \\ (\mathbb{C}^2,0) & \xrightarrow{\Phi_2} & (\mathbb{C}^3,0) \end{array} \tag{2.6}$$

Then

$$C(\Phi_1) = C(\Phi_2) \text{ and } T(\Phi_1) = T(\Phi_2). \tag{2.7}$$

In [NP15] it is proved that C and T are C^∞-invariants as well. That is, (2.7) holds also for C^∞ left-right equivalent germs, i.e. for two holomorphic finitely determined germs for which (2.6) holds with some germs of C^∞-diffeomorphisms $\phi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ and $\psi : (\mathbb{R}^6, 0) \to (\mathbb{R}^6, 0)$. (Here, $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ are naturally identified.)

The topological invariance of C and T would mean that (2.7) holds also for topologically left-right equivalent germs, that is when we only require ϕ and ψ to be germs of homeomorphisms. This invariance was an open question for a long time. In [NP15] Némethi and the Pintér proved that the linear combination $C-3T$ is a topological invariant. This follows from $L = C - 3T$ (formula (2.2)) which expresses a topological invariant (the Ekholm–Szűcs invariant) of the associated immersion, see the next sections. In this section, we present a new direct proof of formula $L = C - 3T$. (We also prove the topological invariance of the Ekholm–Szűcs invariant, see Proposition 2.2.5. This fact is very natural and has been implicitly used previously, but according to the our knowledge, it has not been published yet.)

In [FPS22] J. Fernández de Bobadilla, G. Peñafort, and J. E. Sampaio proved that C and T are topological invariants, moreover they are determined by the embedded topological type of the image of Φ. A key ingredient of their proof is the topological invariance of $C-3T$, which follows from the formula $L = C - 3T$.

2.2.2 The associated immersion and the double points

Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined germ. Such a germ, on the level of links of the spacegerms $(\mathbb{C}^2, 0)$ and $(\mathbb{C}^3, 0)$, provides a stable immersion $\Phi|_{S^3} : S^3 \looparrowright S^5$ as follows. The preimage $\mathfrak{S} := \Phi^{-1}(S^5_\varepsilon)$ of the 5-sphere $S^5_\varepsilon \subset \mathbb{C}^3$ around the origin, with a sufficiently small radius ε, is diffeomorphic to S^3. The restriction $\Phi|_{\mathfrak{S}} : \mathfrak{S} \looparrowright S^5_\varepsilon$ is the immersion associated with Φ. The regular homotopy class of $\Phi|_{\mathfrak{S}}$ is independent of all the choices. The immersions obtained by different choices are regular homotopic to each other through stable immersions. See [NP15, p. 2.1.] or [Pin19, Subsection 1.1.2.].

Write $(X, 0)$ for $(\mathrm{im}(\Phi), 0)$ and let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be the reduced equation of $(X, 0)$. Note that $(X, 0)$ is a *non-isolated* hypersurface singularity, except when Φ is a regular map (see [NP15]). We denote by $(\Sigma, 0) = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)^{-1}(0) \subset (\mathbb{C}^3, 0)$ the *reduced* singular locus of $(X, 0)$ – that is the closure of the set of double values of Φ. Also, we denote by $(D, 0)$ the *reduced* double point curve $\Phi^{-1}(\Sigma) \subset (\mathbb{C}^2, 0)$. The reduced equation of D is $d : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. (In fact, the finite determinacy of the germ Φ is equivalent with the fact that the double point curve D is reduced; see e.g. [MNP12].)

Let $\Upsilon \subset S^5_\varepsilon$ be the link of Σ. It is exactly the set of double values of $\Phi|_{\mathfrak{S}}$. Let $\gamma = \Phi^{-1}(\Upsilon) \subset \mathfrak{S}^3$ denote the set of double points of $\Phi|_{\mathfrak{S}}$, that is, $\gamma \subset \mathfrak{S}$ is the link of D. All link components are considered with their natural orientations.

$$\begin{array}{ccc} (\mathbb{C}^2, 0) & \to & (\mathbb{C}^3, 0) \\ \cup & & \cup \\ (D, 0) & \to & (\Sigma, 0) \\ \cup & & \cup \\ \gamma = D \cap \mathfrak{S}^3 & \to & \Upsilon = \Sigma \cap S^5_\varepsilon \end{array} \tag{2.8}$$

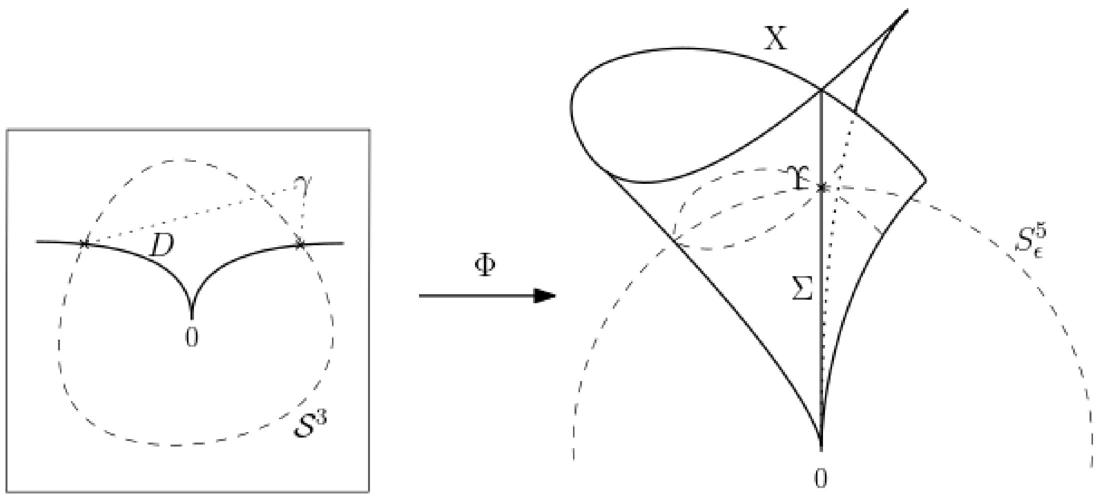


Figure 2.2.1: Notations of the various parts of the space germs.

2.2.3 The Ekholm–Szűcs linking invariant

The invariant $L(f)$ of a stable immersion $f : S^3 \looparrowright \mathbb{R}^5$ measures the linking of a shifted copy of the double values with the whole image of f. Different versions of the definition can be found in the literature, for references see below. In this paragraph, we review these definitions and prove their equivalence via their behavior along regular homotopies. We present the whole argument in the simplest case, for immersions $S^3 \looparrowright \mathbb{R}^5$, although originally they were introduced for different levels of generality (for other manifolds, higher dimensions) in [Ekh01a; Ekh01b; ES03; SST02]. This discussion is an extended version of the summery in [Pin19, p. 2.2.2.].

A stable immersion $f : S^3 \looparrowright \mathbb{R}^5$ has only simple values and double values with transverse intersection of the two branches. Let $\gamma \subset S^3$ be the double point locus of f, that is $\gamma = \{p \in S^3 \mid \exists p' \in S^3 : \ p \neq p' \text{ and } f(p) = f(p')\}$. The locus γ is a closed 1-manifold, i.e. a link in S^3 with possibly more components. The map $f|_\gamma : \gamma \to f(\gamma)$ is a 2-fold covering. γ is endowed with an involution $\iota : \gamma \to \gamma$ such that $\iota(p) \neq p$ and $f(p) = f(\iota(p))$ hold for all $p \in \gamma$.

The first definition of $L(f)$ is from [Ekh01a, p. 6.2.]. Let v be a vector field along γ tangent to S^3 and nowhere tangent to γ, i.e. v represents a section of the normal bundle $TS^3|_\gamma/T\gamma$ of $\gamma \subset S^3$. We also require that $[\tilde{\gamma}]$ is 0 in $H_1(S^3 \setminus \gamma, \mathbb{Z})$, where $\tilde{\gamma} \subset S^3$ is the result of pushing γ slightly along v. Such a vector field v is unique up to homotopy, and for instance each of the two vectors of a Seifert framing provides such a vector field. If v is such a vector field, then the linking number $\mathrm{lk}_{S^3}(\gamma, \tilde{\gamma})$ equals to 0, but the reverse is not true, since $\mathrm{lk}_{S^3}(\gamma, \tilde{\gamma})$ is the sum of the components of $[\tilde{\gamma}] \in H_1(S^3 \setminus \gamma, \mathbb{Z})$. (All the linking numbers appearing are considered with respect to the natural orientation of the curves and submanifolds involved.) Let $q = f(p) = f(\iota(p))$ be a double value of f. Then $w(q) = df_p(v(p)) + df_{\iota(p)}(v(\iota(p))$ defines a vector field w along $f(\gamma)$ that is nowhere tangent to the branches of f. In this sense w is a normal vector field of f along $f(\gamma)$. Let $\widetilde{f(\gamma)} \subset \mathbb{R}^5$ be the result of pushing $f(\gamma)$ slightly along w, then $\widetilde{f(\gamma)}$ and $f(S^3)$ are disjoint. The first invariant is the linking number

$$L_1(f) := \mathrm{lk}_{\mathbb{R}^5}(\widetilde{f(\gamma)}, f(S^3)) \tag{2.9}$$

(or equivalently, $L_1(f) = [\widetilde{f(\gamma)}] \in H_1(\mathbb{R}^5 \setminus f(S^3), \mathbb{Z}) \cong \mathbb{Z}$). Note that Ekholm used an other notation: in [Ekh01a, pp. 2.2., 6.2.] our $L_1(f)$ is denoted by $\mathrm{lk}(f)$, and $L(f)$ is defined as $\lfloor \mathrm{lk}(f)/3 \rfloor$.

The second definition is [ES03, Definition 11.], [SST02, Definition 2.2.]. It works only with further assumptions, see Remark 2.2.1 below. The normal bundle $\nu(f)$ of f is trivial, since the oriented rank–2 vector bundles over S^3 are classified by $\pi_2(SO(2)) = 0$. Any two trivializations are homotopic, since their difference represents an element in $\pi_3(SO(2)) = 0$. Let (v_1, v_2) be the homotopically unique normal framing of f, and at a double value $q = f(p) = f(\iota(p))$ define $u(q) = v_1(p) + v_1(\iota(p))$. u is a normal vector field along $f(\gamma)$, and let $\overline{f(\gamma)} \subset \mathbb{R}^5$ be the result of pushing $f(\gamma)$ slightly along u. Then $\overline{f(\gamma)}$ and $f(S^3)$ are disjoint. The invariant is the linking number (or equivalently, the homology class)

$$L_2(f) := \mathrm{lk}_{\mathbb{R}^5}(\overline{f(\gamma)}, f(S^3)) = [\overline{f(\gamma)}] \in H_1(\mathbb{R}^5 \setminus f(S^3), \mathbb{Z}) \cong \mathbb{Z}. \tag{2.10}$$

Note that the framing (v_1, v_2) can be replaced by an arbitrary nonzero normal vector field v of f, since it can be extended to a framing whose first component is v.

Remark 2.2.1. *Without further assumptions it is possible that $u(q)$ is tangent to one of the branches of f, hence it can happen that $\overline{f(\gamma)} \cap f(S^3) \neq \emptyset$. To avoid this problem one has to choose a unit normal vector field v or has to assume that the intersection of* *the branches is orthogonal, which can be reached by a regular homotopy through stable immersions. In this paper all the calculations uses L_1 and not L_2.*

The third definition is in [ES03, Definition 4.], see also [Ekh01b, pp. 4.5., 4.6.]. Let v be a nonzero normal vector field of f along γ, that is, a nowhere zero section of $\nu(f)|_\gamma$. Let $[v]$ be the homology class represented by v in $H_1(E_0(\nu(f)), \mathbb{Z}) \cong \mathbb{Z}$, where $E_0(\nu(f))$ denotes the total space of the bundle of nonzero normal vectors of f. Let $u_v(q) = v(p) + v(\iota(p))$ be the value of the vector field u_v along $f(\gamma)$ at the point $q = f(p) = f(\iota(p))$. Let $\overline{f(\gamma)}^{(v)}$ be the result of pushing $f(\gamma)$ slightly along u_v, then $\overline{f(\gamma)}^{(v)}$ and $f(S^3)$ are disjoint. The invariant is

$$L_v(f) := \mathrm{lk}_{\mathbb{R}^5}(\overline{f(\gamma)}^{(v)}, f(S^3)) - [v] = [\overline{f(\gamma)}^{(v)}] - [v], \tag{2.11}$$

where $[\overline{f(\gamma)}^{(v)}] \in H_1(\mathbb{R}^5 \setminus f(S^3), \mathbb{Z}) \cong \mathbb{Z}$.

By [Ekh01b, Lemma 4.15.] $L_v(f)$ is well-defined, that is, $L_v(f)$ does not depend on the choice of the normal field v. Moreover, if v is the restriction of a (global) normal vector field of f to γ, then $[v] = 0$. Indeed, the restriction of the normal field of f to a Seifert surface H of γ results a surface $\overline{H} \subset E_0(\nu(f))$, whose boundary is the image of $v : \gamma \to E_0(\nu(f))$. Hence $L_v(f) = L_2(f)$.

The invariants L_1, L_2 are equal to each other with opposite sign. This follows from the fact that they behave in an inverse way along regular homotopies, i.e. they change with the same number with opposite sign when a stable regular homotopy steps through first order instabilities: immersions with (1) one triple value ('triple point moves') or (2) a self-tangency ('self-tangency moves'). For definitions we refer to [Ekh01a; Ekh01b]. The proof of Proposition 2.2.2 is a result of a discussion with Tamás Terpai and András Szűcs.

Proposition 2.2.2. *(a) $L_1(f)$ and $L_2(f)$ are invariants of stable immersions. They change by ± 3 under triple point moves and do not change under self tangency moves. In other words: if f and g are regular homotopic stable immersions, $h : S^3 \times [0,1] \to \mathbb{R}^5$ is a stable regular homotopy between them, then $\pm(L_i(f) - L_i(g))$ is equal to three times the algebraic number of the triple values of the map $H : S^3 \times [0,1] \to \mathbb{R}^5 \times [0,1]$, $H(x,t) = (h(x,t), t)$.*

(b) In the above setup $L_1(f) - L_1(g) = -(L_2(f) - L_2(g))$.

(c) The three definitions are equivalent:

$$L_1(f) = -L_2(f) = -L_v(f).$$

Proof: Part (a) is proved for L_1 in [Ekh01a, Lemma 6.2.1.] and for $L_2 = L_v$ in [Ekh01b, Theorem 1.].

For part (b), we compare the change of L_1 and L_2 through a triple point move. In the proof of [Ekh01a, Lemma 6.2.1.] Ekholm defines a local model of the triple point move where L_1 increases by 3. On the other hand, in the discussion preceding [Ekh01b, Definition 6.3] he provides a convention to measure the change of L_2. If we check this convention on the previous local model, we obtain that L_2 decreases by 3 through that triple point move. Hence L_1 and L_2 changes in opposite ways at each triple point move.

Using part (a) and part (b), we prove part (c) as follows. Since L_1 and L_2 changes in opposite way along a regular homotopy, L_1+L_2 is a regular homotopy invariant. Moreover L_1 and L_2 are additive under connected sum, see [Ekh01b, Lemma 5.2., Proposition 5.4.], [Ekh01a, p. 6.5.]. It follows that $L_1 + L_2$ defines a homomorphism from $\mathrm{Imm}(S^3, \mathbb{R}^5)$ to $\mathbb{Z}$. If $f : S^3 \hookrightarrow \mathbb{R}^5$ is an embedding, then $L_1(f) = L_2(f) = 0$, hence $L_1 + L_2$ is 0 on the 24-index subgroup $\mathrm{Emb}(S^3, \mathbb{R}^5)$ of $\mathrm{Imm}(S^3, \mathbb{R}^5) \cong \mathbb{Z}$. It follows that L_1+L_2 is 0 for every stable immersion, hence $L_1 = -L_2$. □

We fix the following convention.

Notation 2.2.3. *$L(f) := L_1(f)$.*

The definition of $L_1(f)$ and $L_2(f)$ of immersions $f : S^3 \looparrowright \mathbb{R}^5$ cannot be applied directly for $\Phi|_{\mathfrak{S}} : \mathfrak{S} \looparrowright S^5$. In fact, the shifted copy of $v \in S^5$ by a normal vector field is a curve in $\mathbb{C}^3 = \mathbb{R}^6$, but not exatly in S^5. To solve this technical difficulty we recall one of the definitions of the linking number.

Definition 2.2.4. *Let $N^n, M^m \subset S^k = \partial B^{k+1}$ be two closed oriented submanifolds with dimensions $n + m + 1 = k$. Choose any oriented homological membranes $\widetilde{M}, \widetilde{N} \subset B^{k+1}$ for them, that is, $\widetilde{M}$ and $\widetilde{N}$ are singular chains in B^{k+1} of dimensions $n + 1$, respectively $m+1$, with coefficients in $\mathbb{Z}$, whose boundaries are $\partial\widetilde{N} = N$, $\partial\widetilde{M} = M$. Then the linking number $\mathrm{lk}_{S^k}(N, M)$ of N and M in S^k is defined as the intersection number $int_{B^{k+1}}(\widetilde{M}, \widetilde{N})$ of $\widetilde{M}$ and $\widetilde{N}$ in B^{k+1}.*

For the definition of $L_1(\Phi|_{\mathfrak{S}})$ consider $\overline{\mathrm{grad}(d)}$, the conjugate of the gradient vector field of d defined on D. Its restriction to $\gamma \subset \mathfrak{S}$ is a representative of the homotopically unique Seifert framing of γ. Then the sum of the two copies of $d\Phi(\overline{\mathrm{grad}(d)})$ is a nonzero normal vector field along $\Sigma \setminus \{0\}$, which extends to the origin with 0. Let $\tilde{\Sigma}$ be a copy of Σ shifted along this vector field. Define $\tilde{\Upsilon} := \tilde{\Sigma} \cap S^5$ and $L_1(\Phi|_{\mathfrak{S}}) = \mathrm{lk}_{S^5}(\tilde{\Upsilon}, \Phi(\mathfrak{S}))$. The invariant $L_1(\Phi|_{\mathfrak{S}})$ is equal to the intersection number of any pair of membranes in B^6 with boundaries $\tilde{\Upsilon}$ and $\Phi|_{\mathfrak{S}}$. Especially, $L_1(\Phi|_{\mathfrak{S}})$ is the intersection number of $\tilde{\Sigma}$ and X. Unfortunately, however, they intersect each other only at the origin, which is a singular point of possibly both membranes, hence the intersection number cannot be calculated directly. Instead, we will repeat the whole procedure with the *analytic stabilization* of Φ, and that will lead to the formula $L_1(\Phi|_{\mathfrak{S}}) = C(\Phi) - 3T(\Phi)$.

$L_2(\Phi|_{\mathfrak{S}})$ can be defined in a similar way, by using $\overline{\partial_s\Phi \times \partial_t\Phi}$ as a representative of the homotopically unique global normal field of $\Phi|_{\mathfrak{S}}$. We can define the shifted copy $\widetilde{\Sigma}^{(2)}$ of Σ, and $\widetilde{\Upsilon}^{(2)} := \widetilde{\Sigma}^{(2)} \cap S^5$. However, by Remark 2.2.1, we cannot guarantee that $\widetilde{\Upsilon}^{(2)}$ and $\Phi(\mathfrak{S})$ are disjoint. Although the formula $L_2(\Phi|_{\mathfrak{S}}) = 3T(\Phi) - C(\Phi)$ can be supported by local calculation, the precise proof in this way is technically complicated. On the other hand, L_2 can be computed directly for the Whitney umbrella to support that $L_1 = -L_2$ holds, see Subsection 2.2.5.

The topological invariance of $L(\Phi|_{\mathfrak{S}}) = L_1(\Phi|_{\mathfrak{S}})$ is almost trivial, since the linking number is a topological (homological) invariant. However, its proof has been nowhere explained in detail.

Proposition 2.2.5. *$L(\Phi|_{\mathfrak{S}})$ is a topological invariant of Φ. That is if Φ_1 and Φ_2 are finitely determined germs topologically $\mathscr{A}$-equivalent to each other (see Subsection 2.2.1), then*

$$L(\Phi_1|_{\mathfrak{S}_1}) = L(\Phi_2|_{\mathfrak{S}_2}). \tag{2.12}$$

Proof: The topological equivalence of the germs means that there exist germs of homeomorphisms $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and $\psi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that $\Phi_2 = \psi \circ \Phi_1 \circ \phi$ holds. The double point curves $(D_1, 0) = (d_1^{-1}(0), 0)$ of Φ_1 and $(D_2, 0) = (d_2^{-1}(0), 0)$ of Φ_2 are topologically equivalent germs of curves, in fact, $D_1 = \phi(D_2)$. Their links γ_1, γ_2 are of the same type as links in $\mathfrak{S}_1 \cong \mathfrak{S}_2 \cong S^3$.

Although the normal vector field $\mathrm{grad}(d_2)$ along γ_2 cannot be pushed forward by ϕ since it is not necessarily differentiable, the slightly pushed out copy $\widetilde{\gamma_2}$ can be. The image $\phi(\widetilde{\gamma_2})$ determines a normal vector field denoted by $\phi_*(\mathrm{grad}(d_2))$ along γ_1, which is homotopic to $\mathrm{grad}(d_1)$ since both vector field represent the Seifert framing. Hence the sum of the two copies of $d\Phi_1(\mathrm{grad}(d_1))$ and $d\Phi_1(\phi_*(\mathrm{grad}(d_2)))$ are homotopic normal fields along Υ_1, thus the pushed out copies $\widetilde{\Upsilon}_1$ and $\widetilde{\Upsilon}_1^{(2)}$ of Υ_1 along these vector fields are homotopic in $S^5 \setminus \Phi_1(\mathfrak{S}_1) = S^5 \setminus \Phi_1(\phi(\mathfrak{S}_2))$. Therefore, $\mathrm{lk}_{S^5}(\Phi_1(\mathfrak{S}_1), \widetilde{\Upsilon}_1) = \mathrm{lk}_{S^5}(\Phi_1(\mathfrak{S}_1), \widetilde{\Upsilon}_1^{(2)})$. Finally, applying ψ to the whole configuration does not change the linking numbers, and $\psi(\Phi_1(\mathfrak{S}_1)) = \Phi_2(\mathfrak{S}_2)$, $\psi(\Upsilon_1) = \Upsilon_2$, $\psi(\widetilde{\Upsilon}_1^{(2)}) = \widetilde{\Upsilon}_2$. $\square$

Remark 2.2.6. *$L(f)$ can be defined for stable immersions $f : M^3 \looparrowright \mathbb{R}^5$ of closed oriented 3-manifolds M^3, with trivial normal bundle, see [SST02, Definition 2.5.]. Especially*

M^3 can be a disjoint union of some copies of S^3. In this way for multi-germs $\Phi = (\Phi_i) : \sqcup(\mathbb{C}^2,0)_i \to (\mathbb{C}^3,0)$ the invariant $L(\Phi|_{M^3})$ is defined, where $M^3 = \sqcup\mathfrak{S}_i$ with $\mathfrak{S}_i = \Phi_i^{-1}(S^5_\varepsilon)$. We will use this extension of L for ordinary triple values.

Remark 2.2.7. *Recall Remark 2.2.7 from [Pin19]. L can be defined also for nonstable immersions which do not have triple values, by the following argument. Any immersion f admits a small perturbation by regular homotopy to a stable immersion $\widetilde{f}$, and if f does not have triple values, then any two stable perturbations can be joined with a regular homotopy without stepping through a triple point. Thus $L(f)$ can be defined as $L(\widetilde{f})$ of any small stable perturbation $\widetilde{f}$ of f.*

Consequently $L(\Phi|_\mathfrak{S})$ can be defined not only for finitely determined germs but for germs with finite C and T, since for these germs $\Phi|_\mathfrak{S}$ is not a stable immersion, but it does not have triple points. Moreover the equation (2.2) *holds for these germs too, since the proof uses an analytic stabilization of Φ, not Φ itself. See also Corollary 3.6.3., Remark 3.6.4. in [Pin19] or [NP15]. An interesting example is $\Phi(s,t) = (s^2, t^2, st)$, which is the double cover of the A_1 singularity. See Subsection 3.7.2. of [Pin19].*

However it is not clear for these germs, how can $L(\Phi|_\mathfrak{S})$ be computed directly from the topology of Φ, without stabilizing it.

2.2.4 Agreeing invariants

Theorem 2.2.8. *For a finitely determined holomorphic germ $\Phi : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$*

$$L(\Phi|_\mathfrak{S}) = C(\Phi) - 3T(\Phi).$$

Proof: Let $\Phi_\lambda : \mathfrak{B}_\lambda \to B^6_\varepsilon$ be an analytic stabilization of $\Phi_0 = \Phi$. Here, $\mathfrak{B}_\lambda = \Phi_\lambda^{-1}(B^6_\varepsilon)$ with boundary $\partial\mathfrak{B}_\lambda = \mathfrak{S}_\lambda = \Phi_\lambda^{-1}(S^5_\varepsilon)$.

Decreasing λ to 0 induces a diffeomorphism $\mathfrak{B}_\lambda \simeq \mathfrak{B}$, respectively $\mathfrak{S}_\lambda \simeq \mathfrak{S}$, and a regular homotopy through stable immersions between $\Phi_\lambda|_{\mathfrak{S}_\lambda}$ and $\Phi|_\mathfrak{S}$. It implies – as recognised in [NP15, Section 9.] – that

$$L(\Phi|_\mathfrak{S}) = L(\Phi_\lambda|_{\mathfrak{S}_\lambda}). \tag{2.13}$$

We denote the corresponding double point sets respectively by D_λ, Σ_λ, γ_λ, Υ_λ, as defined in Subsection 2.2.3 . The reduced equation of D_λ is $d_\lambda : \mathfrak{B}_\lambda \to \mathbb{C}$.

We are going to count $L(\Phi_\lambda) = L_1(\Phi_\lambda)$. According to definitions 2.9 and 2.2.4, we want to construct membranes bounding $\Phi_\lambda(\mathfrak{S}_\lambda)$ and $\widetilde{\Upsilon}_\lambda$, and count their intersection number. The first membrane is simply the whole image $\Phi_\lambda(\mathfrak{B}_\lambda)$.

For the second membrane, consider the normal vector field $w = \overline{\mathrm{grad}(d_\lambda)}$ on $D_\lambda \subset \mathfrak{B}_\lambda$, its restriction represents the Seifert framing on $\gamma_\lambda \subset \mathfrak{S}_\lambda$. The pushforward $\mathrm{d}\Phi_\lambda(w)$ gives a double valued vector field at each point of Σ_λ. We add up the two vectors pointwise and pushout Σ_λ slightly along the obtained vector field v to get $\widetilde{\Sigma}_\lambda$. (Notice that at triple points the vector field v has three values, but they are all zeroes.)

By the construction in Subsection 2.2.3, the boundary is $\partial\widetilde{\Sigma}_\lambda = \widetilde{\Upsilon}_\lambda$ and

$$L(\Phi_\lambda|_{\mathfrak{S}_\lambda}) = \mathrm{int}(\Phi_\lambda(\mathfrak{B}_\lambda), \widetilde{\Sigma}_\lambda). \tag{2.14}$$

As the two components of v are tangent to the two branches of the image at a double point, the pushout $\widetilde{\Sigma}_\lambda$ has no intersection point with the whole image near an ordinary double point.

Besides double points, the only two types of singular points that may occur in the stabilized map Φ_λ are Whitney umbrella points and triple points. With the above remark, it means that we only have to count the intersection points near these points.

Umbrella points and triple points are left-right equivalent to the standard copies of them, see (2.3) and (2.4). In the following two lemmas, we calculate the intersection numbers for these normal forms – which are, in fact, the Ekholm–Szűcs invariants of these (multi)-germs. After stating the lemmas we will deduce the global invariant by gluing these pieces together to complete the proof.

Lemma 2.2.1. *The Ekholm–Szűcs invariant of the standard Whitney umbrella*

$$\Phi(s,t) = (s^2, st, t)$$

is

$$L(\Phi|_{\mathfrak{S}}) = 1$$

where $\mathfrak{S} = \Phi^{-1}(S^5_\varepsilon)$.

Lemma 2.2.2. *The standard triple value is the regular intersection of three branches. We parametrize it the following way*

$$\Phi \begin{cases} \Phi_1 : (s_1, t_1) & \mapsto & (0, s_1, t_1) \\ \Phi_2 : (s_2, t_2) & \mapsto & (t_2, 0, s_2) \\ \Phi_3 : (s_3, t_3) & \mapsto & (s_3, t_3, 0) \end{cases}$$

(The pairs (s_i, t_i) are local coordinates around the three preimages of the triple point.) The Ekholm–Szűcs invariant of this multi-germ is

$$L(\Phi|_{\mathfrak{S}}) = -3.$$

The above results suggest that each umbrella point and triple point contributes 1 and respectively -3 to the global Ekholm–Szűcs invariant. This is, in fact, the case and the brief argument is the following. The (multi)germs at the umbrella and triple points of Φ_λ are left-right equivalent of their standard form, hence, by the left-right invariance of L (see 2.2.5), the membrane of Φ_λ shall be replaced locally by the one coming from the standard forms.

More precisely, let us take an umbrella point or a triple value p_i in $\mathbb{C}^3$ and take a small balls $U_i \subset \mathbb{C}^3$ around p_i and $V_i \subset \mathbb{C}^2$ around $\Phi_\lambda^{-1}(p_i)$ and biholomorphisms $\phi_i : (U_i, p_i) \to (\mathbb{C}^3, 0)$ and $\psi_i : (\mathbb{C}^2_r, \underline{0}_r) \to (V_i, \Phi_\lambda^{-1}(p_i))$ so that

$$\phi_i \circ \Phi_\lambda \circ \psi_i : (\mathbb{C}^2_r, \underline{0}_r) \to (\mathbb{C}^3, 0)$$

is a standard umbrella (respectively triple point) at $\underline{0}_r$. (Here we use the notation for multi germs: $(\mathbb{C}^2_r, \underline{0}_r) = \bigsqcup_{i=1}^r (\mathbb{C}^2, 0)$ with $r = 1$ for a Whitney umbrella point and $r = 3$ for a triple value p_i.)

We pull back the two membranes of the standard Whitney umbrella (resp. triple value) via ϕ_i to define new membranes inside U_i. On one hand we obtain another pushout of Σ_λ instead of $\tilde{\Sigma}_\lambda$, let us denote it by $M_i \subset U_i$. On the other hand, we get back a piece of the other original membrane, $\Phi_\lambda(\mathfrak{B}_\lambda) \cap U_i$.

Taking a look at the boundary of U_i, we find that both $\tilde{\Sigma}_\lambda \cap \partial U_i$ and $M_i \cap \partial U_i$ have the same linking number with $\Phi_\lambda(\mathfrak{B}_\lambda) \cap \partial U_i$: that is the L_1 invariant of the umbrella point or the triple point. Therefore we can construct a collar N_i that connects $\tilde{\Sigma}_\lambda \cap \partial U_i$ and $M_i \cap \partial U_i$ in ∂U_i, in a way that N_i has an intersection number 0 with $\Phi_\lambda(\mathfrak{B}_\lambda) \cap \partial U_i$.

Gluing all these pieces together, we obtain a membrane replacing $\tilde{\Sigma}_\lambda$:

$$\mathcal{M} = (\tilde{\Sigma}_\lambda \setminus \bigcup_i U_i) \cup \bigcup_i (N_i \cup M_i). \tag{2.15}$$

The intersection number $\operatorname{int}(\Phi_\lambda(\mathfrak{B}_\lambda), M_i)$ equals 1 for an umbrella point and -3 for a triple value, $\operatorname{int}(\Phi_\lambda(\mathfrak{B}_\lambda), N_i) = 0$ and $\operatorname{int}(\Phi_\lambda(\mathfrak{B}_\lambda), (\tilde{\Sigma}_\lambda \setminus \bigcup_i U_i)) = 0$, hence

$$L(\Phi|_{\mathfrak{S}}) = \operatorname{int}(\Phi_\lambda(\mathfrak{B}_\lambda), \mathcal{M}) = C(\Phi) - 3T(\Phi). \tag{2.16}$$

$\square$

Proof: [Proof of Lemma 2.2.1] Consider the standard Whitney umbrella $\Phi(s,t) = (s^2, st, t)$. The closure of the set of double values of Φ is

$$\Sigma = \{y = z = 0\} = \{(x, 0, 0) : x \in \mathbb{C}\}.$$

This is the image of the double point curve $D = \{t = 0\} = \{(s, 0) : s \in \mathbb{C}\}$. The link of D is γ and $\Phi(\gamma) = \Upsilon$. We compute the linking number $\mathrm{lk}_{S^5}(\widetilde{\Upsilon}, \Phi(\mathfrak{S}))$ by defining membranes bounded by $\widetilde{\Upsilon}$ and $\Phi(\mathfrak{S})$ and taking their intersection multiplicity.

Let the membrane of $\widetilde{\Upsilon}$ be the shifted copy of the curve of double values $\widetilde{\Sigma}$. More precisely, we push Σ out from $X = \mathrm{im}(\Phi)$ along the pushforward $\mathrm{d}\Phi(v)$ of the vector field $v(s,0) = \overline{\mathrm{grad}(t)}(s,0) = (0,1)$ that is normal to D. The differential of our germ is

$$\mathrm{d}\Phi(s,t) = \begin{pmatrix} 2s & 0 \\ t & s \\ 0 & 1 \end{pmatrix}$$

making the pushforward of the normal vector field

$$\mathrm{d}\Phi(v(s,0)) = \begin{pmatrix} 2s & 0 \\ 0 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ 1 \end{pmatrix}.$$

At any double point $(x, 0, 0) \in \Sigma$, we have two preimages

$$\Phi^{-1}\{(x,0,0)\} = \{(\sqrt{x}, 0), (-\sqrt{x}, 0)\}.$$

The pushforward of the normal vectors at these points are $\mathrm{d}\Phi(v(\pm\sqrt{x}, 0)) = (0, \pm\sqrt{x}, 1)$, hence the sum of the two vectors provides the vector field

$$w(x,0,0) = \begin{pmatrix} 0 \\ \sqrt{x} \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -\sqrt{x} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

along $\Sigma \setminus \{0\}$. The vector field w can be extended continuously to the origin as it is constant. Therefore, when we push the double point out by w, we obtain $(x,0,0) + \delta w(x,0,0) = (x,0,2\delta)$ for a some $\delta \ll \varepsilon$.

Thus the resulting membrane is

$$\widetilde{\Sigma} = \{(x, 0, 2\delta) : x \in \mathbb{C}\} \cap B_\varepsilon.$$

On the other hand, let the membrane of $\Phi(\mathfrak{S})$ be simply the image of the ball $\Phi(\mathfrak{B}) = X \cap B_\varepsilon$. That is

$$\{(x,y,z) : xz^2 = y^2\} \cap B_\varepsilon.$$

The two membranes $\widetilde{\Sigma}$ and $X \cap \mathfrak{B}$ intersect transversely at $(0, 0, 2\delta)$. The sign of the intersection is positive as the two membranes have the complex orientations. □

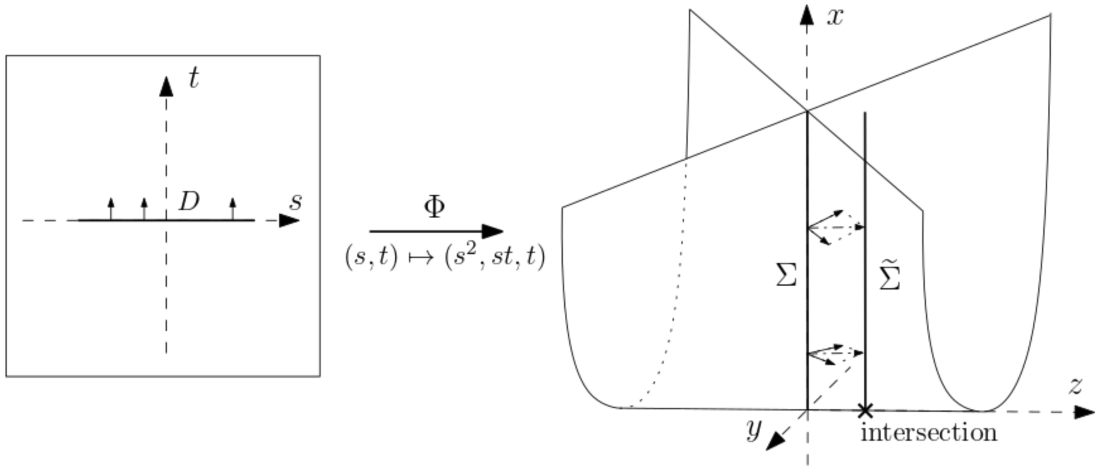


Figure 2.2.2: Pushing out Σ with the sum of the two pushforward vector fields.

Proof: [Proof of Lemma 2.2.2] Consider the standard triple value

$$\Phi \begin{cases} \Phi_1 : (s_1, t_1) & \mapsto & (0, s_1, t_1) \\ \Phi_2 : (s_2, t_2) & \mapsto & (t_2, 0, s_2) \\ \Phi_3 : (s_3, t_3) & \mapsto & (s_3, t_3, 0) \end{cases}$$

In this case, the set of double values is

$$\Sigma = \{(x,0,0)\} \cup \{(0,y,0)\} \cup \{(0,0,z)\}$$

with $x, y, z \in \mathbb{C}$. The curve Σ has three components meeting at the origin. Also, Σ has preimages in each two-dimensional chart:

$$D_i = \{(s_i, 0)\} \cup \{(0, t_i)\} = \{s_i t_i = 0\}$$

for $i \in \{1,2,3\}$.

The membrane we pull over $X \cap S^5$ is again the whole of the image $X \cap B^6$. Note that $X \cap S^5$ is diffeomorphic to the disjoint union of three copies of S^3. Thus the membrane consists of three components $X_x = \{x = 0\} \cap B$, $X_y = \{y = 0\} \cap B$, and $X_z = \{z = 0\} \cap B$, meeting at the origin.

Now, we describe the membrane for $\widetilde{\Upsilon}$. Let us see what happens if we push out the double values using the sum of the normal vector fields in the preimage – as before.

The normal vector fields corresponding to $D_i = \{s_i t_i = 0\}$ are $v_i(s_i, t_i) = \overline{\mathrm{grad}(s_i t_i)} = (\overline{t_i}, \overline{s_i})$. The differentials of the three map germs are

$$\mathrm{d}\Phi_1(s_1,t_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{d}\Phi_2(s_2,t_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{d}\Phi_3(s_3,t_3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let us show how our construction works on one component of Σ. We denote $X_y \cap X_z = \{(x,0,0) : x \in \mathbb{C}\}$ by Σ_x. A point of Σ_x has, again, two preimages: $\Phi_2^{-1}(x,0,0) = (0,x) \in \mathbb{C}_{s_2,t_2}$ and $\Phi_3^{-1}(x,0,0) = (x,0) \in \mathbb{C}_{s_3,t_3}$. The corresponding normal vectors are $v_2(0,x) = (\overline{x},0)$ and $v_3(x,0) = (0,\overline{x})$. When we push these vectors forward with the respective differentials, we obtain

$$\mathrm{d}\Phi_2(v_2(0,x)) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \overline{x} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \overline{x} \end{pmatrix}$$

and similarly $\mathrm{d}\Phi_3(v_3(0,x)) = (0,\overline{x},0)^T$. Hence, by pushing the initial point $(x,0,0)$ out with the sum of these, we reach $(x,0,0) + \delta(0,0,\overline{x}) + \delta(0,\overline{x},0) = (x,\delta\overline{x},\delta\overline{x}) \in \widetilde{\Sigma}_x$.

Because of the cyclic symmetry of the presentation, the other two components behave similarly, resulting in the membrane

$$\widetilde{\Sigma} = \{(x,\delta\overline{x},\delta\overline{x})\} \cup \{(\delta\overline{y},y,\delta\overline{y})\} \cup \{(\delta\overline{z},\delta\overline{z},z)\} =: \widetilde{\Sigma}_x \cup \widetilde{\Sigma}_y \cup \widetilde{\Sigma}_z$$

for some $x,y,z \in \mathbb{C}$ with $\widetilde{\Sigma}$ being in B. One problem with this membrane is that each vector field vanishes at the origin hence in the end we have not moved the point of Σ at the origin. Thus $\widetilde{\Sigma}$ meets X only at the origin but with some multiplicity that is somewhat difficult to count. Fortunately, each pair of components $(X_\alpha, \widetilde{\Sigma}_\beta)$ intersect transversely. We only need to compute the sign of each such intersection and sum them up.

Take $\widetilde{\Sigma}_x = \{(x,\delta\overline{x},\delta\overline{x})\}$ first. It intersects $X_x = \{x=0\}$ with positive sign, and the other two with negative – as the corresponding coordinate functions are antiholomorphic. The membranes $\widetilde{\Sigma}_y$ and $\widetilde{\Sigma}_z$ behave similarly. We can summerize this in the formula

$$\mathrm{int}_0\left(\widetilde{\Sigma}_\alpha, X_\beta\right) = \begin{cases} +1 & \text{if } \alpha = \beta \\ -1 & \text{if } \alpha \neq \beta. \end{cases}$$

Therefore the total intersection number is

$$\mathrm{int}_0\left(\widetilde{\Sigma}, X\right) = \sum_{\alpha,\beta \in \{x,y,z\}} \mathrm{int}_0\left(\widetilde{\Sigma}_\alpha, X_\beta\right) = 3 \cdot 1 + 6 \cdot (-1) = -3.$$

□

Remark 2.2.9. *Note that we could also move the components of $\widetilde{\Sigma}$ away from the origin in order to see the nine points of intersection apart. A perturbation of the form*

$$\Sigma' = \{(x - \varepsilon_1, \delta(\overline{x - \varepsilon_2}), \delta(\overline{x - \varepsilon_3}))\} \cup \ldots$$

with $|\varepsilon_i| \ll \varepsilon$ would do so. In turn, these modifications would not change the topology of the membrane on the boundary of B.

2.2.5 A similar linking invariant

The aim of this section is to clarify the role of L and another related linking invariant in the study of generic C^∞ real maps and immersions, and clear up the context of our result. We also clear up some sensitive sign ambiguities related to the Ekholm–Szűcs Smale invariant formula.

The other linking invariant l is defined for real generic maps. While L measures the linking of the double values of an immersion with the image of it, l measures the linking of the set of singular points in the target of a generic map with the image of the map.

The Ekholm–Szűcs formula for the Smale invariant of an immersion uses both linking invariant, L of the immersion and l of a singular Seifert surface of the immersion. The original proof of our main formula (2.2) is based on the Ekholm–Szűcs Smale invariant formula and the 'holomorphic Smale invariant formula' of Némethi and Pintér.

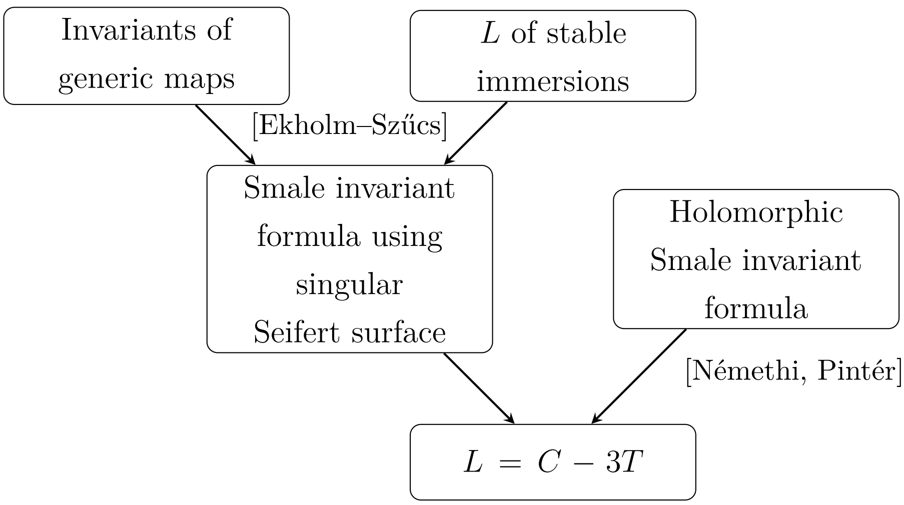


Figure 2.2.3: Mindmap for the original proof of (2.2).

The $\mathbb{Z}_2$ or integer valued invariant $l(f)$ is defined for real generic maps $f : M^{2k} \to \mathbb{R}^{3k}$ of closed smooth manifolds M^{2k}. It measures the linking of a pushout copy of the singular values with the image of the map as follows. (See, for reference, [ES02; ES03; SST02].)

Such a map f has (1) regular simple points, (2) double values with transverse intersection of the regular branches, (3) triple values with regular intersection of the regular branches and (4) singular values. The dimension of the set of double values is k, and the triple values are isolated. The set of singular values is a $k-1$ dimensional family of generalized real Whitney umbrella points, whose local form is

$$f_{\mathrm{wh}} : (\mathbb{R} \times \mathbb{R}^k, 0) \to (\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k, 0), \tag{2.17}$$

$$f_{\mathrm{wh}}(s,\underline{t}) = (s^2, s\underline{t}, \underline{t}). \tag{2.18}$$

The closure $\Delta(f)$ of the set of double values of f is an immersed manifold with boundary. $\Delta(f)$ has triple self intersection at the triple values of f and the boundary of $\Delta(f)$ is the set of the Whitney umbrella points (singular values) $\Sigma(f) = \partial\Delta(f)$.

The invariant $l(f)$ is defined as the linking number

$$l(f) = \mathrm{lk}_{\mathbb{R}^{3k}}(\Sigma'(f), f(M^{3k})) \tag{2.19}$$

of the copy $\Sigma'(f)$ of $\Sigma(f)$ shifted slightly along the outward normal field of $\Sigma(f) \subset \Delta(f)$ and the image $f(M^{3k})$ of f.

In general $l(f)$ and the number of triple values is defined only modulo 2 – because the lack of orientation on $\Delta(f)$ – and these $\mathbb{Z}_2$ versions are denoted by $l_2(f)$ and $t_2(f)$ respectively. If k is even and M^{2k} is oriented, then $l(f)$ is a well defined as an integer, [ES02]. In these cases each triple value can be given a sign, and the sum of these signs is the integer $t(f)$.

Ekholm and Szűcs expressed some characteristic numbers of M^{2k} in terms of l and t. Namely, in [ES02] they proved the equality

$$l_2(f) + t_2(f) = \overline{w}_k^2[M] + \overline{w}_{k-1}\overline{w}_{k+1}[M] \tag{2.20}$$

in $\mathbb{Z}_2$, where the terms on the right hand side are products of the normal Stiefel-Whitney classes of M^{2k} evaluated on the fundamental class $[M]$ of M^{2k}.

For $k = 2n$ and M^{2k} oriented, the equation of integers

$$3t(f) - 3l(f) = \overline{p}_n[M] \tag{2.21}$$

is proved in [ES03], where $\overline{p}_n[M]$ is the n-th normal Pontryagin number of M^{4n}. By using Hirzebruch signature theorem, for $k = 2$ one can rewrite the formula (2.21) as

$$l(f) - t(f) = \sigma(M^4), \tag{2.22}$$

where $\sigma(M^4)$ is the signature of M^4, see [ES03].

The proofs of these formulas use methods similar to that of our proof. Namely, each of them considers a set of certain type singularities of a map, and deals with the pushout copy of it along a suitably defined normal vector field, then counts the intersection point, see for example [ES03, Lemma 3]. When proving the formula (2.20) in [ES02, Theorem 1], the set of double values of the map $f : M^{2k} \to \mathbb{R}^{3k}$ is shifted slightly along a vector field, which is defined as the sum of the two vectors coming from a suitable normal vector field of the double point set in the source. At this rate it is even more similar to the method we use to prove equation (2.2).

Despite the similarity in methods, none of the equations (2.20), (2.21) and (2.22) can be directly applied for the setup of this result, that is for holomorphic stabilizations Φ_λ of holomorphic germs $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, for the following reasons. First, the domain of Φ_λ is a 4-ball, which is not a closed manifold, moreover it is topologically trivial. Second, the stabilization Φ_λ is stable as a holomorphic map, but it is not stable (not generic) in the C^∞-sense, considered as a map from $\mathbb{R}^4$ to $\mathbb{R}^6$. Indeed, each isolated complex cross cap point can be further deformed to obtain a stable real C^∞ map with a circle of generalized real cross cap points, see [NP15]. Also, in contrast to the real case, the complex cross cap points are not boundary points of the set of double values. Third, one could try to relate the above results to the immersion on the boundary in our case. Then, however, the dimensions do not match and these immersions do not have triple points or singular points whatsoever. What is more, the Smale invariant formula (2.25) hints that $t(f)$ and $l(f)$ should really be considered for the membranes and not the boundary.

Smale invariant formulas

If M^4 is an oriented 4-manifold with boundary, the 'defect' of the equation (2.22) provides information about the restriction of the map to the boundary. In the simplest case, the manifold M^4, with boundary ∂M^4 diffeomorphic with S^3, is mapped to the upper half space $\mathbb{R}^6_+ = \{(x_1, \dots, x_6) \in \mathbb{R}^6 \mid x_6 > 0\}$ with a generic map

$$\hat{f} : M^4 \to \mathbb{R}^6_+, \tag{2.23}$$

whose restriction is assumed to be a stable immersion

$$f := \hat{f}|_{\partial M^4} : S^3 \looparrowright \mathbb{R}^5. \tag{2.24}$$

In this case $\hat{f}$ is referred as a singular Seifert surface of the immersion f.

Recall that the immersions of S^3 to $\mathbb{R}^5$ are classified up to regular homotopy by an integer valued invariant called Smale invariant and denoted by Ω. That is, two immersions $f_1, f_2 : S^3 \looparrowright \mathbb{R}^5$ are regular homotopic if and only if $\Omega(f_1) = \Omega(f_2)$, and for every integer $n \in \mathbb{Z}$ there is an immersion $g : S^3 \looparrowright \mathbb{R}^5$ with Smale invariant $\Omega(g) = n$. The Smale invariant can be constructed in many different ways, see for example [Sma59; HM85; NP15; Pin19]. Eventually the Smale invariant $\Omega(f)$ is constructed as an element of the homotopy group $\pi_3(SO(5))$, which is isomorphic with the infinite cyclic group $(\mathbb{Z}, +)$.

Then by [ES03] the Smale invariant $\Omega(f)$ of a stable immersion $f : S^3 \looparrowright \mathbb{R}^5$ can be expressed with the invariants of a singular Seifert surface $\hat{f} : M^4 \to \mathbb{R}^6_+$ and L as

$$\Omega(f) = \frac{1}{2}(3\sigma(M^4) + 3t(\hat{f}) - 3l(\hat{f}) + L(f)). \tag{2.25}$$

Several variants and generalizations of the Ekholm–Szűcs formula (2.25) appeared in the literature, see [HM85; SST02; Juh05; ES06] or the brief summary of these results in [Pin19, Ch.2].

For immersions $\Phi|_{\mathfrak{S}} : \mathfrak{S} \cong S^3 \looparrowright S^5$ associated with finitely determined holomorphic germs $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ Némethi and Pintér [NP15] proved the 'holomorphic Smale invariant formula'

$$\Omega(\Phi|_{\mathfrak{S}}) = -C(\Phi). \tag{2.26}$$

The proof of this formula is self-contained in the sense that it is independent of the above results.

A singular Seifert surface for $\Phi|_{\mathfrak{S}}$ can be constructed from a holomorphic stabilization Φ_λ of Φ by a canonical C^∞ stabilization of the complex Whitney umbrella points. In this way the Ekholm–Szűcs formula (2.25) can be applied. By comparing it with the equation (2.26) and using calculations on concrete examples, [NP15] proves the main theorem (2.2) of this section, namely

$$L(\Phi|_{\mathfrak{S}}) = C(\Phi) - 3T(\Phi). \tag{2.27}$$

The evolution of these results is summed up by Figure 2.2.3.

However, the proof of each Smale invariant formula is rather complicated, and the identification of the signs of the terms are widely nontrivial (see the next paragraph). Furthermore the correspondence (2.2) becomes important in the proof of the topological invariance of C and T. This was the motivation to publish a new direct proof for (2.2), which does not use any of the above results – although the techniques are similar to those ones used in their proofs. An additional benefit of our proof is the simple identification of the sign: (2.2) is sign correct with the L_1 version of L. This fact has further consequences for the singular Seifert surface formula (2.25) as explained in the next paragraph.

Remarks on sign and orientation

This paragraph is a brief summary of the issues related to the signs in the Smale invariant formulas. We unravel an imprecision in [ES03] and [SST02]: although the linking invariant L is defined in these articles using the construction denoted by L_2 in Subsection 2.2.3, the Smale invariant formula (2.25) is satisfied by using $L = L_1$.

The Smale invariant does not have a canonical sign, since by default it is an element of the group $\pi_3(SO(5)) \cong (\mathbb{Z}, +)$. To identify this group with $\mathbb{Z}$, one has to fix a generator in $\pi_3(SO(5))$ and declare it to be $+1$. That was done in [NP15], and the formula (2.26) is sign-correct with that fixed generator. In other words it is proved that the Smale invariant of the immersion associated with the complex Whitney umbrella is -1 times the fixed generator.

The formula (2.25) is proved in [ES03] without considering the sign of the Smale invariant. More precisely, they proved that the right hand side of the formula is a complete regular homotopy invariant, therefore it must agree with the Smale invariant up to sign. Nevertheless it is shown in [NP15] that the foruma (2.25) is correct with the fixed generator of $\pi_3(SO(5))$.

However the sign of L is not specified directly in [NP15; Pin19]. It is chosen to satisfy the Ekholm–Szűcs formula (2.25) with this choice. For example, the invariant $L(\Phi|_{\mathfrak{S}})$ of the complex Whitney umbrella $\Phi(s,t) = (s^2, st, t)$ is computed in [NP15, p. 10.1.] up to sign by using the 'L_2' construction (see also [Pin19, p. 3.7.1]), resulting $L(\Phi|_{\mathfrak{S}}) = \pm 1$. Using the sign convention adapted to the formula (2.25), $L(\Phi|_{\mathfrak{S}})$ of the complex Whitney umbrella is declared to be $+1$, and $L = C - 3T$ is concluded with this sign convention.

Now, from the proof of Lemma 2.2.1 it is clear that $L_1(\Phi|_{\mathfrak{S}}) = +1$ for the Whitney umbrella, hence $L_2(\Phi|_{\mathfrak{S}}) = -1$ by part (c) of Proposition 2.2.2. Therefore to make the Ekholm–Szűcs Smale invariant formula (2.25) correct, L has to defined to be L_1, in contrast to the definitions given in [ES03] and [SST02]. Note that by changing the sign of L in the formula not only the sign of the right hand side changes, but the absolute value changes as well.

On the other hand, in the calculation of $L_2(\Phi|_{\mathfrak{S}})$ of the complex Whitney umbrella in [NP15, p. 10.1.] the sign of the intersection point can be determined directly. Both membranes ($\Phi(\mathfrak{B})$ and H in [NP15] and [Pin19]) has complex (but not holomorphic) parametrization. These parametrizations induce the correct orientations in the sense that the induced orientation on the boundary agrees with the original orientation of the boundary. A direct calculation of the determinant shows that the intersection point has negative sign. Hence $L_2(\Phi|_{\mathfrak{S}}) = -1$ can be discovered directly, which is equal to $-L_1(\Phi|_{\mathfrak{S}})$ according to Proposition 2.2.2.

Remark 2.2.10. *By default, the orientation induced on the boundary of an oriented manifold depends on a choice of a convention, called 'boundary convention', for example 'outward normal first'. Although, at first sight, the boundary convention seems to play a key role in the identification of the signs of the Smale invariant formulas and L, this is* *not the case.*

The correct sign of the formulas (2.25) *and* (2.26) *are independent of the choice of the boundary convention. Briefly speaking its reason is that in the construction of the Smale invariant* S^3 *is considered as the boundary of the 4-ball in* $\mathbb{R}^4$*. By changing the boundary convention, the orientation of the boundary of the singular Seifert surface changes, as well as the orientation of* $S^3 = \partial B^4$ *in the construction of the Smale invariant, but the value of the Smale invariant and the right hand side of the formulas remain the same. See [NP15] or [Pin19, Ch.3] for details.*

The invariant L *of finitely determined holomorphic germs* Φ *is also independent of the choice of the boundary convention. Recall that* $L(\Phi|_{\mathfrak{S}})$ *is defined as the intersection number of two oriented 'membranes' in* B^6 *whose boundaries are* $\Phi(\mathfrak{S})$ *and* $\widetilde{\Upsilon} = \partial\widetilde{\Sigma}$ *respectively. Although* $\Phi(\mathfrak{S})$ *and* $\widetilde{\Upsilon} = \partial\widetilde{\Sigma}$ *are originally oriented as the boundaries of* $\Phi(\mathfrak{B})$ *and* $\widetilde{\Sigma}$ *after choosing a boundary convention, all in all, the correct orientations of the membranes do not depend on the choice of the boundary convention. Indeed, the correct orientation means that the membrane induces the same orientation on the boundary as the original membrane, whichever boundary convention is used. Cf. [NP15; Pin19].*

Chapter 3

Non-isolated toric singularities

3.1 Deformation of singularities

"Knowing the existence of versal deformations of singularities we can go on and ask questions about the structure of the base space (is it reduced, what is the number of components?), or ask if for some t_0 the fibre $(F_1(x,t_0),\dots,F_k(x,t_0))$ is smooth. In general these question cannot be answered, because the equations are just one enormous mess."

[Ste03, Introduction]

This quote from Stevens hints the depth and difficulty of deformation theory and the discrepancy between the beautiful results of abstract deformation theory and the concrete computations.

In this section, we give a general introduction to the deformation theory of complex space germs highlighting those concepts that we will need in the subsequent sections. As a reference, we recommend [GLS07, Section II.1.] and the book [Ste03] of Stevens.

The purpose of deformation of singularities is to perturb our given germ to make it simpler – less singular or completely smooth – without changing it too drastically. The latter ensures that the perturbation can still tell us something meaningful about the original singularity. Drastic change would mean, for example, a jump in the dimension of the germ. Let us see an example of this.

Example 3.1.1. *Consider the subspace X of $\mathbb{C}^3 \cong \mathbb{C}\langle x,y,z\rangle$ consisting of the three coordinate axes, defined by $X = \{xy = yz = zx = 0\}$. It is a natural idea to perturb the defining equations letting $X_t = \{xy = yz = zx = t\}$. However, for $t \neq 0$ the space X_t only consists of two points: $(\pm\sqrt{t},\pm\sqrt{t},\pm\sqrt{t})$. We do not want to call this family a deformation of X.*

From a different perspective, it also makes sense to take a look at the whole family as a single space defined by the perturbed equations. Let

$$\widetilde{X} = \{(x, y, z, t) \in \mathbb{C}^4 : \ xy = yz = zx = t\} \subset \mathbb{C}\langle x, y, z, t\rangle.$$

This is a one-dimensional subspace consisting of three lines and a curve, with the lines being contained in the $t = 0$ fibre. This is the situation we would like to avoid when constucting a deformation: having irreducible components of the 'total space' contained in the original space.

However naive perturbations work nicely when the space X is defined by one equation, or is a complete intersection, as we will see in Example 3.1.11.

Definition 3.1.2. *A **deformation** of a complex singularity $(X, 0)$ is a flat morphism of complex space germs*

$$\pi : (\widetilde{X}, 0) \to (S, 0)$$

together with an isomorphism

$$i : (X, 0) \xrightarrow{\cong} (X_0, 0) = (\pi^{-1}(0), 0).$$

*The germ $(\widetilde{X}, 0)$ is the **total space**, $(S, 0)$ is the **base space** of the deformation, and $(X_0, 0)$ is the **special fibre**.*

The situation can also be summarized in the commutative diagram

$$\begin{tikzcd} (X,0) \arrow[r, hook, "i"] \arrow[d] & (\widetilde{X},0) \arrow[d, "\pi"] \\ (0,0) \arrow[r, hook] & (S,0) \end{tikzcd}$$

In short, we refer to deformations by the pair of maps (i, π), or only by π when i is clearly specified.

Definition 3.1.3.

*(i) A **morphism of deformations** between $(X, 0) \overset{i}{\hookrightarrow} (\widetilde{X}, 0) \xrightarrow{\pi} (S, 0)$ and $(X, 0) \overset{i'}{\hookrightarrow} (\widetilde{X}', 0) \xrightarrow{\pi'} (S', 0)$ of the same singularity $(X, 0)$ is a pair of map germs:*

$$\varphi : (S, 0) \to (S', 0), \ \widetilde{\varphi} : (\widetilde{X}, 0) \to (\widetilde{X}', 0)$$

making the following diagram commute:

$$\begin{tikzcd}
 & (X,0) \arrow[dl, hook, "i"'] \arrow[dr, hook, "i'"] & \\
(\widetilde{X},0) \arrow[rr, "\widetilde{\varphi}"] \arrow[d, "\pi"] & & (\widetilde{X}',0) \arrow[d, "\pi'"] \\
(S,0) \arrow[rr, "\varphi"] & & (S',0)
\end{tikzcd}$$

(ii) *Two deformations over the same base space* $(S,0)$ *are* ***isomorphic*** *if there is a morphism of deformations* $(\varphi,\widetilde{\varphi})$ *between them where* φ *is the identity of* $(S,0)$ *and* $\widetilde{\varphi}$ *is an isomorphism of complex spaces between the total spaces.*

The technical condition of flatness is the key for the notion of deformation.

3.1.1 Flatness

"The concept of flatness is a riddle that comes out of algebra, but which is technically the answer to many prayers."

[Mum67]

We begin with an analytic definition by Stevens – originally by Grothendieck. Consider a map of analytic germs $\pi : (\widetilde{X},0) \to (S,0)$. The special fibre $(X_0,0) = (\pi^{-1}(0),0)$ can be embedded into some $(\mathbb{C}^N,0)$ with a map f. According to Fischer ([Fis76, 0.35. Proposition]), the embedding $(X_0,0) \to (\mathbb{C}^N,0)$ can be extended to a small neighbourhood of $0 \in \widetilde{X}$. That is, we have an analytic map germ $(\widetilde{X},0) \to (S \times \mathbb{C}^N,0)$ that is an immersion at 0 and the diagram

$$\begin{tikzcd}
 & (X_0,0) \arrow[r, hook] \arrow[dl, hook] & (\widetilde{X},0) \arrow[dl, hook] \arrow[d, "\pi"] \\
(0\times\mathbb{C}^N,0) \arrow[r, hook] & (S\times\mathbb{C}^N,0) \arrow[r, "\mathrm{proj}_S"] & (S,0)
\end{tikzcd}$$

commutes. Let $(F_1,\dots,F_k)$ be the vanishing ideal of the image of $(\widetilde{X},0)$ in $\mathcal{O}_{S,0} \otimes \mathbb{C}\{z_1,\dots,z_N\}$. Then the equations $f_i(z) = F_i(0,z) \in \mathbb{C}\{z_1,\dots,z_N\}$ define the image of $(X_0,0)$ in $(\mathbb{C}^N,0)$.

Definition 3.1.4. *The above map germ* $\pi : (\widetilde{X},0) \to (S,0)$ *is* ***flat*** *if every relation of the form* $\sum_1^k r_i f_i = 0$ *with* $r_i \in \mathbb{C}\{z_1,\dots,z_N\}$ *lifts to a relation* $\sum_1^k R_i F_i = 0$ *with* $R_i \in \mathcal{O}_{S,0} \otimes \mathbb{C}\{z_1,\dots,z_N\}$*,* $R_i(0,z) = r_i(z)$*.*

This definition, however, heavily uses the representation $(\widetilde{X},0) \to (S \times \mathbb{C}^N, 0)$ which makes it a little clumsy to use. We want a more versatile definition, and for that, we need to turn to algebra. The notion of flatness, there, involves the exactness of tensor product. We recall some definitions and facts we need.

Let R be a commutative ring with identity. The tensor product $\otimes_R$ over R will be denoted simply by $\otimes$ if there is no need to specify the ring.

Proposition 3.1.5. *The tensor product of R-modules is right exact. That is, for every R-module M and exact sequence $N' \to N \to N" \to 0$ of R-modules, the sequence $M \otimes N' \to M \otimes N \to M \otimes N" \to 0$ is also exact.*

Definition 3.1.6. *We say that M is a* ***flat R-module*** *if the functor $M\otimes$ is exact. This means that for every short exact sequence of R-modules $0 \to N' \to N \to N" \to 0$, the induced sequence*

$$0 \to M \otimes N' \to M \otimes N \to M \otimes N" \to 0$$

is also exact.

According to the above proposition, the exactness of M is equivalent to the condition that for every injective morphism $N' \to N$, the induced morphism $M \otimes N' \to M \otimes N$ is also injective.

Remark 3.1.7. *Another equivalent condition of the R-flatness of M is that for each R-module N the first* Tor *group of the pair is trivial: $\mathrm{Tor}_1^R(M,N) = 0$. In fact, it is enough to have $\mathrm{Tor}_1^R(M,R/I) = 0$ for each finitely generated ideal $I \subset R$ for M to be flat. Also, the flatness of M implies the vanishing of all the Tor_i^R groups $(i \geq 1)$. In principle, the functors $\mathrm{Tor}_i^R(M,\cdot)$ measure how far is M from being flat.*

Let us quickly link the notions of flatness.

Proposition 3.1.8. *A complex map germ $\pi : (\widetilde{X},0) \to (S,0)$ is flat if and only if $\mathcal{O}_{\widetilde{X},0}$ is a flat $\mathcal{O}_{S,0}$-module via the induced morphism $\pi_0^{\#}$ of stalks.*

Proof: (Sketch) For a local $\mathbb{C}$-algebra, such as $\mathcal{O}_{\widetilde{X},0}$, flatness is equivalent to $\mathrm{Tor}_1^{\mathcal{O}_{S,0}}(\mathcal{O}_{\widetilde{X},0},\mathbb{C}) = 0$. We can build a free $\mathcal{O}_{S,0}$-resolution of $\mathcal{O}_{\widetilde{X},0}$ starting with the equations $\{F_1,\ldots,F_k\}$ and the relations between them. When we tensor the free resolution with $\mathbb{C}$, we obtain the analogous resolution of $\mathcal{O}_{X_0,0}$. The vanishing of the Tor^1 group – that is the exactness of the latter sequence – and in turn the flatness of the local ring of $(\widetilde{X},0)$, is equivalent to the lifting of the relations f_i to F_i. (See [Ste03, p. 8.])
□

Definition 3.1.9. *A map $\pi : \widetilde{X} \to S$ of complex spaces is* ***flat*** *if its germ at each point $x \in \widetilde{X}$ is flat.*

Revealing the geometric meaning of flatness is complicated. For instance, it is already nontrivial to prove that, for two analytic spaces, X and S, the projection $X \times S \to S$ is flat (see [Dou68, Theorem 2.] or [GLS07, Corollary 1.88.]).

Let us see some basic geometric implications of flatness.

Theorem 3.1.10. *Let $\varphi : Y \to S$ be a morphism of complex spaces.*

(i) The flat locus of φ is analytically open. That is, the set of those points $y \in Y$ where φ is flat is the complement of a closed analytic subset of Y. (See [Fri67].)

(ii) If φ is flat, then it is an open map, meaning, it maps open subsets of Y to open subsets of S. (See [Dou68].)

(iii) If φ is flat at $y \in Y$, then it is locally surjective onto some neighbourhood of $\varphi(y) \in S$. (Corollary of (i) *and* (ii)*.)*

(iv) For any points $y \in Y$ and $s = \varphi(y) \in S$

$$\dim(Y, y) \leq \dim(Y_s, y) + \dim(S, s)$$

with equality if φ is flat at y. (Follows from the algebraic property.)

Condition (iii) implies that a closed embedding of a proper subset is never flat. Let us see some examples that are flat – besides the projections we mentioned above.

Example 3.1.11. *[GLS07, p. 225]*

(i) Let $X \subset \mathbb{C}^n$ be defined by a nonconstant holomorphic map $f : \mathbb{C}^n \to \mathbb{C}$. Then the map f is flat making

$$(X, 0) \hookrightarrow (\mathbb{C}^n, 0) \xrightarrow{f} (\mathbb{C}, 0)$$

a deformation of $(X, 0)$.

(ii) Similarly, let $X \subset \mathbb{C}^n$ be a complete intersection defined by $f_1, \dots, f_k$. This, again, gives a flat morphism $(f_1, \dots, f_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ and, in turn, a deformation of $(X, 0)$ over $(\mathbb{C}^k, 0)$.

(iii) If, however, $(X, 0) = (V(f_1, \dots, f_k), 0) \subset (\mathbb{C}^n, 0)$ is not a complete intersection, then the corresponding morphism $(\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ is not flat.

Proposition 3.1.12. *A one-parameter family $\varphi : (\widetilde{X}, 0) \to (\mathbb{C}, 0)$ is flat if and only if no irreducible component of $(\widetilde{X}, 0)$ is contained in $(X_0, 0) = (\varphi^{-1}(0), 0)$. Equivalently, a $\mathbb{C}\{s\}$-module M is flat if and only if s is not a zero-divisor in M. (See [GLS07, Proposition 2.7.])*

This statement means that, if we have a one-parameter family of germs that is non-flat, there is always a component of the total space germ sitting inside the special fibre.

We have to be careful, though, because this is not true anymore over a base space of higher dimensions. See the following example of Douady.

Example 3.1.13. *Consider the space*

$$Y = \{xz - y\} \subset \mathbb{C}^3$$

and the map

$$\varphi : Y \to \mathbb{C}^2,\ (x, y, z) \mapsto (x, y).$$

The fibre over $(0, 0)$ is the whole z-axis in $\mathbb{C}^3$, but all the other fibres are finite. Precisely, for $x = 0, y \neq 0$, the fibres are empty, otherwise they consist of a single point each. Therefore, φ is clearly not flat at $(0, 0)$. However, there is no component inside the special fibre that we could remove, no zero divisors in $\mathcal{O}_{Y,0} = \mathbb{C}\{x, y, z\}/(xz - y)$.

(The right way to generalize the condition in Proposition 3.1.12 to more parameters is through regular sequences.)

Finally, let us see how we can deform the singularity of Example 3.1.1.

Example 3.1.14. *Consider the morphism of complex space germs*

$$\pi : (\widetilde{X}, 0) \cong (V(xy - xt, yz, zx), 0) \to (\mathbb{C}, 0), \quad (x, y, z, t) \mapsto t.$$

We claim that this is a (one-parameter) deformation of $(X_0, 0) \cong (V(xy, yz, zx), 0) \subset (\mathbb{C}^3, 0)$. The generic fibre $\pi^{-1}(t)$ for $t \neq 0$ consists of two coordinate axes and a third line moving away from the origin with t:

$$X_t = \{x = y = 0\} \cup \{x = z = 0\} \cup \{z = 0, y = t\}.$$

To show the flatness of π, we use Definition 3.1.4. To avoid confusion, let us introduce the notations

$$f_1 = xy, f_2 = yz, f_3 = zx, \quad \textit{and} \quad F_1 = xy - xt, F_2 = yz, F_3 = zx$$

We want to prove that all relations $\sum r_i f_i = 0$ lift to relations $\sum R_i F_i = 0$ with $r_i \in \mathbb{C}\{x,y,z\}, R_i \in \mathbb{C}\{x,y,z,t\}$. We pick two relations:

$$zf_1 - yf_3 = 0, \quad xf_2 - yf_3 = 0$$

that generate the whole $\mathbb{C}\{x,y,z\}$-module of possible relations. As we can lift these to

$$zF_1 - (y-t)F_3 = z(xy - xt) - (y-t)zx = 0$$

and

$$xF_2 - yF_3 = xyz - yzx = 0,$$

we have proved the flatness of π.

As another strategy, we could show that t is not a zero-divisor in

$$\mathbb{C}\{x,y,z,t\}/(xy - xt, yz, zx)$$

and use Proposition 3.1.12.

3.1.2 Versality

A versal deformation of a singularity is a deformation that essentially contains all possible flat ways to perturb our space germ.

Definition 3.1.15. *Let $(i,\pi) : (X,0) \hookrightarrow (\widetilde{X},0) \to (S,0)$ be a deformation of the complex space germ $(X,0)$. Consider a map germ $\varphi : (T,0) \to (S,0)$. The* ***deformation induced by φ from π****, or the* ***pullback deformation****, is the fibre product $(\widetilde{X},0) \times_{(S,0)} (T,0)$ that fits into the commutative diagram*

$$\begin{tikzcd}
& (X,0) \arrow[dl, "\varphi^* i"'] \arrow[dr, hook, "i"] & \\
(\widetilde{X},0) \times_{(S,0)} (T,0) \arrow[rr, "\widetilde{\varphi}"] \arrow[d, "\varphi^*\pi"] & & (\widetilde{X},0) \arrow[d, "\pi"] \\
(T,0) \arrow[rr, "\varphi"] & & (S,0)
\end{tikzcd}$$

together with the induced map germs $(\varphi^* i, \varphi^* \pi)$. The mappings $\widetilde{\varphi}$ and $\varphi^*\pi$ are the projections of the fibre product (see Remark 1.2.15) and $\varphi^* i$ is induced by the universal property of the construction. The total space of the induced deformation is denoted by $\varphi^*(\widetilde{X},0)$.

Note that the induced map germ $\varphi^*\pi$ is flat, hence this is, indeed, a deformation of $(X, 0)$. Such a mapping $\varphi : (T, 0) \to (S, 0)$ is called a **base change**.

Definition 3.1.16. *A deformation $(X, 0) \overset{i}{\hookrightarrow} (\widetilde{X}, 0) \overset{\pi}{\to} (S, 0)$ is **versal** if it satisfies the following conditions.*

(i) Any other deformation can be pulled back from it. That is, given a deformation $(X, 0) \overset{j}{\hookrightarrow} (\widetilde{Y}, 0) \overset{\tau}{\to} (T, 0)$, there exists a morphism $\varphi : (T, 0) \to (S, 0)$ of complex spaces such that the induced deformation $(\varphi^* i, \varphi^*\pi)$ – with $\varphi^*\pi : \varphi^*(\widetilde{X}, 0) \to (T, 0)$ – is isomorphic to (j, τ).

(ii) Furthermore, we can prescribe the map φ on a closed analytic subgerm of $(T, 0)$. Precisely, given (j, τ) as above and a map germ $\varphi' : (T', 0) \to (S, 0)$ from a closed analytic subgerm $(T', 0) \subset (T, 0)$ such that the pullback $(\varphi'^* i, \varphi^*\pi)$ is isomorphic to the restriction $(j, \tau|_{T'})$, we can extend φ' to a mapping $\varphi : T \to S$ as above. The following commutative diagram sums up the second condition.

$$\begin{tikzcd}
 & (X,0) \arrow[dl, hook, "j"'] \arrow[dr, hook, "i"] & \\
(\widetilde{Y},0) \arrow[rr, "\widetilde{\varphi}"] \arrow[d, "\tau"] & & (\widetilde{X},0) \arrow[d, "\pi"] \\
(T,0) \arrow[rr, "\varphi"] & & (S,0) \\
 & (T',0) \arrow[ul, hook, "\supsetneq"] \arrow[ur, "\varphi'"'] &
\end{tikzcd}$$

*Moreover, the deformation (i, π) is **semiuniversal** or **miniversal** if, in addition to the two conditions above, the following also holds.*

(iii) In the above situation, the Zariski tangent map $T(\varphi) : T_{T,0} \to T_{S,0}$ is uniquely determined by the deformations (i, π) and (j, τ).

It follows from the definition that, if a complex space germ $(X, 0)$ admits a semiuniversal deformation π, then π is uniquely determined up to isomorphism. Although, this isomorphism is not unique in general. In particular, the dimension of the semiuniversal base space is well-defined if it exists.

It is not true that all singularities have versal deformations. However, Tjurina in [Tju69], and Grauert in [Gra71], showed that isolated singularities are special in this sense.

Theorem 3.1.17. *Every isolated singularity* $(X, 0)$ *has a semiuniversal deformation.*

Unfortunately, constructing a semiuniversal deformation for a given isolated singularity is a difficult task. Even the dimension of its base space is not known in general.

We continue with the notion parallel to stability of mappings.

Definition 3.1.18.

(i) A deformation $\pi : (\widetilde{X}, 0) \to (S, 0)$ *is* ***trivial*** *if it is isomorphic to the product deformation* $\mathrm{proj}_{(S,0)} : (X, 0) \times (S, 0) \to (S, 0)$.

(ii) A singularity $(X, 0)$ *is* ***rigid*** *if it only admits trivial deformations.*

Smooth germs are rigid. On the other hand, non-smooth hypersurface singularities and complete intersections are never rigid – the deformations shown in Example 3.1.11 are nontrivial. It is conjectured that there are no rigid reduced curve singularities and rigid normal surface singularities.

3.1.3 Infinitesimal deformations: T^1, T^2

Definition 3.1.19.

(i) The ***category of deformations*** $\mathcal{D}ef_{(X,0)}$ *of a complex space germ* $(X, 0)$ *consists of deformations* (i, π) *of* $(X, 0)$ *as objects and morphism of deformations. Its (non-full) subcategory* $\mathcal{D}ef_{(X,0)}(S, 0)$ *is the category of deformations over a fixed base* $(S, 0)$ *with those morphisms that are the identity on the base space.*

(ii) The set of ***isomorphism classes of deformations of*** $(X, 0)$ ***over*** $(S, 0)$ *is denoted by* $\underline{\mathcal{D}ef}_{(X,0)}(S, 0)$. *Its elements are denoted as* $[(i, \pi)]$.

A map germ $\varphi : (T, 0) \to (S, 0)$ induces the pullback map

$$[\varphi^*] : \underline{\mathcal{D}ef}_{(X,0)}(S, 0) \to \underline{\mathcal{D}ef}_{(X,0)}(T, 0).$$

Definition 3.1.20. *The (first order)* ***infinitesimal deformations*** *of a singularity* $(X, 0)$ *are deformations over the fat point of order two, that is, elements of* $\mathcal{D}ef_{(X,0)}(\mathbb{D})$. *The isomorphism classes of infinitesimal deformations are denoted by*

$$T^1_{(X,0)} = \underline{\mathcal{D}ef}_{(X,0)}(\mathbb{D}).$$

Let $(X,0)$ be a singularity defined by the ideal $I=(f_1,\dots,f_k)$ inside $(\mathbb{C}^n,0)$. Consider an infinitesimal deformation of $\widetilde{X}\to\mathbb{D}$ of $(X,0)$. By Definition 3.1.4, the total space $(\widetilde{X},0)$ is defined by equations $F_1,\dots,F_k\in\mathcal{O}_{(\mathbb{C}^n,0)\times\mathbb{D}}$. As $\mathcal{O}_{\mathbb{D}}=\mathbb{C}\{\varepsilon\}/\varepsilon^2$, these equations are of the form $F_i=f_i+f_i'\varepsilon$. Moreover, any relation $\sum_1^k r_if_i=0$ lift to a relation $\sum_1^k R_iF_i=0$ with $R_i=r_i+r_i'\varepsilon$. After substitution,

$$0=\sum_1^k R_iF_i=\sum_1^k(r_i+r_i'\varepsilon)(f_i+f_i'\varepsilon)=\sum_1^k r_if_i+\varepsilon\sum_1^k(r_if_i'+r_i'f_i)=\varepsilon\sum_1^k(r_if_i'+r_i'f_i),$$

using $\varepsilon^2=0$.

With these notations, we also have the resolution

$$\mathcal{O}^l_{(\mathbb{C}^n,0)\times\mathbb{D}}\xrightarrow{\underline{R}}\mathcal{O}^k_{(\mathbb{C}^n,0)\times\mathbb{D}}\xrightarrow{\underline{F}}\mathcal{O}_{(\mathbb{C}^n,0)\times\mathbb{D}}\longrightarrow\mathcal{O}_{(\widetilde{X},0)}\longrightarrow 0,$$

where $\underline{F}=(F_1,\dots,F_k)$ and $\underline{R}$ denote the vectors of lifted functions and relations, respectively.

Proposition 3.1.21. *The set of infinitesimal deformations $\mathcal{D}ef_{(X,0)}(\mathbb{D})$ carry an $\mathcal{O}_{X,0}$-module structure.*

Proof: The natural sum and scalar multiplication by $\mathcal{O}_{\mathbb{C}^n,0}$-elements are well behaved. This means that if two deformations satisfy some relations $(\underline{f}+\varepsilon\underline{f_j'})(\underline{r}+\varepsilon\underline{r_j'})=0$ for $j=1,2$, then their sum $\underline{f}+\varepsilon(\underline{f_1'}+\underline{f_2'})$ satisfies the respective relation

$$(\underline{f}+\varepsilon(\underline{f_1'}+\underline{f_2'}))(\underline{r}+\varepsilon(\underline{r_1'}+\underline{r_2'}))=0.$$

Similarly, for a scalar $\alpha\in\mathcal{O}_{\mathbb{C}^n,0}$, the product $\underline{f}+\alpha\underline{f}'\varepsilon$ also respects the given relation:

$$(\underline{f}+\alpha\underline{f}'\varepsilon)(\underline{r}+\alpha\underline{r}'\varepsilon)=\underline{f}\underline{r}+\varepsilon\alpha(\underline{r}\underline{f}'+\underline{r}'\underline{f})=0.$$

The last thing to check is that the scalar multiplication is well-defined, that is, multiplying by an scalar element $\alpha\in I^k$ – that vanish on $(X,0)$ – results in the trivial deformation. The fact $\alpha\underline{f}'\in I^k=(f_1,\dots,f_k)^k$ implies that we can express the two deformations in question with each other using a suitable $k\times k$ matrix M over $\mathcal{O}_{\mathbb{C}^n,0}$:

$$\underline{f}+\alpha\underline{f}'\varepsilon=\underline{f}(\mathrm{id}+\varepsilon M)$$

and

$$\underline{f}=\underline{f}(\mathrm{id}+\varepsilon M)(\mathrm{id}-\varepsilon M)=(\underline{f}+\alpha\underline{f}'\varepsilon)(\mathrm{id}-\varepsilon M).$$

This means that the scalar multiple $\underline{f}+\alpha\underline{f}'\varepsilon$ defines the same ideal in $\mathcal{O}_{(\mathbb{C}^n,0)\times\mathbb{D}}$ as $\underline{f}$, hence they induce the same, trivial deformation of $(X,0)$. □

Theorem 3.1.22. *The $\mathcal{O}_{X,0}$-module of infinitesimal deformations is isomorphic to the normal module of X at 0,*

$$\mathcal{D}ef_{(X,0)}(\mathbb{D}) \cong \mathcal{N}_{(X,0)} = \mathrm{Hom}_{\mathcal{O}_{X,0}}(I/I^2, \mathcal{O}_{X,0})$$

as $\mathcal{O}_{X,0}$-modules.

Proof: First, we want a map $\mathcal{D}ef_{(X,0)}(\mathbb{D}) \to \mathrm{Hom}_{\mathcal{O}_{X,0}}(I/I^2, \mathcal{O}_{X,0}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{C},0}}(I, \mathcal{O}_{X,0})$. Consider an infinitesimal deformation $\underline{f} + \underline{f}'\varepsilon$. This defines such a map by $f_i \mapsto f_i'$, $I \to \mathcal{O}_{X,0}$ that is well defined because $\underline{r}\underline{f}' + \underline{r}'\underline{f} = 0$ implies $\underline{r}\underline{f}' \in I$.

Similarly, given a homomorphism $\rho \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{C},0}}(I, \mathcal{O}_{X,0})$, it defines a map $\underline{f} + \varepsilon\rho(\underline{f})$. We can find an $\underline{r}'$ that makes the relation $\underline{r}\rho(\underline{f}) + \underline{r}'\underline{f} = 0$ true. □

An infinitesimal deformation is trivial, if it defines the same ideal as $\underline{f} \circ \varphi$, where $\varphi(x, \varepsilon) = (x + \varepsilon\psi(x), \varepsilon)$ is an automorphism of $(\mathbb{C}^n, 0) \times \mathbb{D}$. Differentiating by ε gives

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\underline{f} \circ \varphi(x, \varepsilon)|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\underline{f}(x + \varepsilon\psi(x), \varepsilon)|_{\varepsilon=0} = \sum_i \frac{\partial \underline{f}}{\partial x_i}\psi_i(x).$$

If we denote the module of vector field germs at the origin by Θ_n, we have a natural map

$$\Theta_n|_{(X,0)} = \Theta_n \otimes \mathcal{O}_{X,0} \to \mathcal{N}_{(X,0)} = \mathrm{Hom}_{\mathcal{O}_{\mathbb{C},0}}(I, \mathcal{O}_{X,0}), \quad v \mapsto (g \mapsto v(g)).$$

The computation above shows that trivial deformations are in the image of this map.

Corollary 3.1.23.

$$T^1_{X,0} = \mathcal{N}_{(X,0)} \Big/ \mathrm{im}\Big(\Theta_n|_{(X,0)} \to \mathcal{N}_{(X,0)}\Big) = \mathrm{coker}\Big(\Theta_n|_{(X,0)} \to \mathcal{N}_{(X,0)}\Big)$$

In particular, T^1_X is also an $\mathcal{O}_{X,0}$-module.

The dimension of $T^1_{(X,0)}$ as a $\mathbb{C}$-vector space is an important invariant.

Definition 3.1.24. *The* ***Tjurina number*** *of a singularity $(X, 0)$ is*

$$\tau(X, 0) = \mathrm{dim}_{\mathbb{C}} T^1_{(X,0)}.$$

Proposition 3.1.25. *The Tjurina number of a hypersurface singularity $(X, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$ equals*

$$\tau(V(f), 0) = \mathrm{dim}_{\mathbb{C}} \left(\mathcal{O}_{\mathbb{C}^n,0} \Big/ \left(f, \tfrac{\partial f}{\partial x_1}, \ldots, \tfrac{\partial f}{\partial x_n}\right) \right).$$

Proof: The normal module $\mathcal{N}_{(X,0)} = \mathrm{Hom}_{\mathcal{O}_{X,0}}(I/I^2, \mathcal{O}_{X,0})$ is a free $\mathcal{O}_{(X,0)}$-module with the homomorphism $f \mapsto 1$ as generator. $\square$

The Tjurina number can be regarded as the analytic sibling of the Milnor number (see Definition 1.6.7), whereas the latter turns out to be more topological.

The following result is due to Grauert [Gra71].

Theorem 3.1.26. *A singularity with finite Tjurina number has a semiuniversal deformation.*

The connection between $T^1_{X,0}$ and versality is even stronger. The module $T^1_{X,0}$ can also be viewed as the Zariski tangent space to the semiuniversal base space of $(X,0)$ when that exists.

Theorem 3.1.27. *Consider a deformation $\pi : (\widetilde{X}, 0) \to (S, 0)$ of the singularity $(X, 0)$. Then there is a linear map, the **Kodaira–Spencer map**:*

$$KS : T_{S,0} \to T^1_{(X,0)}.$$

If π is versal, then the map is surjective; if π is semiuniversal, then KS is bijective. Furthermore, if a smooth semiuniversal base exists, the bijectivity of KS implies semiuniversality of $(S, 0)$, too.

Remark 3.1.28. *It is a natural question to ask whether an infinitesimal deformation can be lifted to be a second order deformation, that is a deformation over the triple point with local ring $\mathbb{C}\{t\}/t^3$. As it turns out, there is an obstruction map $\mathrm{ob} : T^1_{X,0} \to T^2_{X,0}$, whose vanishing means that the particular class of T^1 can be lifted. The module T^2 is a cokernel of a more complicated map than T^1.*

For a singularity with finite Tjurina number, this lifting is the first step towards creating its formal versal base space. (See [Sch68] for details.)

3.2 Toric geometry

"Toric varieties provide a quite different yet elementary way to see many examples and phenomena in algebraic geometry. In the general classification scheme, these varieties are very special. . . . Nevertheless, toric varieties have provided a remarkably fertile testing ground for general theories."

[Ful93]

Toric geometry builds a bridge between the elementary combinatorial branch of geometry and algebraic geometry. As a byproduct, it gives us examples that are complicated enough to be worth studying, but also rather accessible and computable due to their combinatorial nature. We refer to [Dan78] as the most classic introduction to the subject along with [Ful93] and [Ewa96]. We also recommend [CLS11] as an extremely detailed and extensive handbook of (normal) toric varieties.

Here, we only summarize shortly the basics of normal toric varieties before we turn to the much less studied non-normal case.

A (normal) toric variety is a normal algebraic variety X that contains an algebraic torus $T \cong (\mathbb{C}^*)^n$ as a dense open subset together with an action $T \times X \to X$ of the torus on the variety that extends the natural action of the torus on itself. The torus action induces a combinatorial structure on the algebraic structure sheaf $\mathcal{O}_X^{alg}$ – by introducing weight constraints on the defining equations – involving semigroups, lattices and fans of cones. Sumihiro's theorem [Sum74] tells us that, for a (separated) normal variety, this combinatorial description is equivalent to the definition above. As we are interested in singularities, we concentrate on affine toric varieties – they serve as patches for the general case in a manifold-like fashion.

Let $M \cong \mathbb{Z}^n$ be an integer lattice and $\sigma \subset M_{\mathbb{R}} \cong M \otimes \mathbb{R}$ a convex rational polyhedral cone – that is, a convex cone generated by finitely many lattice vectors – with apex at the origin inside the real vector space corresponding to the lattice. (From now on, by 'cone' we will always mean such a cone). The lattice points inside the cone form a semigroup: $S_\sigma = M \cap \sigma$. The corresponding semigroup algebra $\mathbb{C}[S_\sigma]$ is generated by the elements x^a for $a \in S_\sigma$ as a vector space and the multiplication rule is $x^a \cdot x^b = x^{a+b}$. In fact, $\mathbb{C}[S_\sigma]$ is a subalgebra of the ring of algebraic functions $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}] \cong \mathbb{C}[M]$ on the torus T_M.

The toric variety defined by M and σ is the spectrum of the semigroup algebra generated by S_σ:

$$TV(\sigma) = X_\sigma = \text{Spec } \mathbb{C}[S_\sigma].$$

The lattice M is the **character lattice** of the torus T_M or it can also be viewed as the **lattice of monomials** on X_σ. Its dual lattice $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ is the **lattice of one-parameter subgroups**. The latter lattice is essential when we construct non-affine toric varieties from affine pieces.

We can obtain algebraic tori as the toric varieties corresponding to lattices:

$$\text{Spec } \mathbb{C}[\mathbb{Z}^n] = \text{Spec } \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}] = (\mathbb{C}^*)^n = T_{\mathbb{Z}^n}.$$

We will work with lattices $M, N \cong \mathbb{Z}^n$, that do not have distinguished bases. From a torus, we can get back the corresponding lattice as the lattice of characters: $\mathrm{Hom}_{gp}(T_M, \mathbb{C}^*) =$

M. On the other hand, the dual lattice N can be realized as the lattice of one parameter subgroups, that is, as $N = \mathrm{Hom}_{gp}(\mathbb{C}^*, T_M)$.

3.2.1 Non-normal toric varieties

There is much less literature on general, not necessarily normal toric varieties. We can refer to [Stu96, Chapter 13.] for different aspects of normality of toric varieties and to [GB09] for the theory of the relevant affine semigroups. We mainly rely on the paper [GT14] of González Pérez and Teissier. We restrict our attention to affine varieties.

The lattice M represents the Laurent monomial functions on the variety X_σ. In the normal setup, the cone σ encodes which monomials are holomorphic on X_σ. The normality of the variety is equivalent to the semigroup S_σ being saturated. Let us recall the definitions needed.

Definition 3.2.1.

*(i) A commutative semigroup S is **affine** if it is the submonoid of some $\mathbb{Z}^n$.*

*(ii) An affine semigroup $S \subset \mathbb{Z}^n$ is **saturated** or **normal** if $k \cdot a \in S$ (for some $k \in \mathbb{N}, a \in \mathbb{Z}^n$) implies $a \in S$. The **saturation** or **normalization** of S in $\mathbb{Z}^n$ is*

$$\widehat{S} = \{a \in \mathbb{Z}^n : \exists k \in \mathbb{N} \text{ such that } ka \in S\}.$$

Remark 3.2.2. *A commutative semigroup S is affine if and only if it is finitely generated, cancellative (that is, $a+c = b+c \implies a = b$) and torsion-free (that is, $ka = kb \implies a = b$ for $k \in \mathbb{N}$).*

Remark 3.2.3. *Every saturated semigroup $S \subset \mathbb{Z}^n$ can be expressed in the form $S = \mathbb{Z}^n \cap \sigma$ for some cone $\sigma \subset \mathbb{R}^n$. In fact, $\sigma = \mathbb{R}_{\geq 0}\langle S\rangle$.*

When we drop the condition of normality from the definiton of toric varieties, we can consider non-saturated affine semigroups for S_σ. Such a semigroup cannot be defined as the intersection of a cone and the lattice.

However, we can normalize a non-normal semigroup S filling in the gaps, that is the elements of $\widehat{S} \setminus S$. The set of gaps have the following structure.

Proposition 3.2.4. *Let S be an affine monoid with normalization $\widehat{S} = \sigma \cap \mathbb{Z}^n$. Then the set of gaps can be decomposed into a finite disjoint union*

$$\widehat{S} \setminus S = \bigsqcup_{1}^{k} \Big(a_i + (S \cap \tau)\Big)$$

where $a_i \in \widehat{S}$ and $\tau \leq \sigma$ is a face.

See [GB09, Proposition 2.35. (b)].

A morphism of semigroups $\varphi : S \to S'$ induces a morphism of $\mathbb{C}$-algebras $\mathbb{C}[\varphi] : \mathbb{C}[S] \to \mathbb{C}[S']$. This map is injective (respectively surjective) if φ is that too.

We can obtain algebraic tori as the toric varieties corresponding to lattices:

$$\text{Spec } \mathbb{C}[\mathbb{Z}^n] = \text{Spec } \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}] = (\mathbb{C}^*)^n = T_{\mathbb{Z}^n}.$$

We will work with lattices $M, N \cong \mathbb{Z}^n$, that do not have distinguished bases. From a torus, we can get back the corresponding lattice as the lattice of characters: $\text{Hom}_{gp}(T_M, \mathbb{C}^*) = M$. On the other hand, the dual lattice N can be realized as the lattice of one parameter subgroups, that is, as $N = \text{Hom}_{gp}(\mathbb{C}^*, T_M)$.

Definition 3.2.5. *The (not necessarily normal)* ***affine toric variety*** *corresponding to the affine semigroup S is*

$$X_S = TV(S) = \text{Spec } \mathbb{C}[S].$$

The set of closed points of Spec $\mathbb{C}[S]$ are in a one-to-one correspondence with the semigroup homomorphisms $\text{Hom}_{sg}(S, \mathbb{C})$ with respect to the multiplicative structure of $\mathbb{C}$. We can think about this as giving values to each monomial x^s for $s \in S$. In particular, the **maximal torus** consists of those points that satisfy the open condition $x^{s_1} \dots x^{s_l} \neq 0$ for a generating set $\{s_1, \dots, s_l\}$ of S.

Another way to describe the maximal torus is the following. Let $M = M_S = \mathbb{Z}S$, that is the lattice generated by S. The embedding $S \hookrightarrow M$ induces the Zariski-open embedding $T_M \hookrightarrow X_S$. This is true because $\mathbb{C}[M] = (\mathbb{C}[S])_{x^a}$, the localization in x^a for some semigroup element that satisfies $M = S + \mathbb{N} \cdot (-a)$. Moreover, this implies that $\dim X_S = \dim T_M = \dim_{\mathbb{Z}} M$.

We want to describe X_S with equations. First, we choose a generating set $\{s_1, \dots, s_l\}$ for S. Then S is the image of $\mathbb{N}^l$ at the $\mathbb{Z}$-linear map $\beta : \mathbb{Z}^l \to M$, $e_i \mapsto s_i$. The corresponding $\mathbb{C}$-algebra morphism

$$\mathbb{C}[\beta|_{\mathbb{N}^l}] : \mathbb{C}[N^l] = \mathbb{C}[y_1, \dots, y_l] \to \mathbb{C}[M] \cong \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$$

is surjective onto the subalgebra $\mathbb{C}[S] \subset \mathbb{C}[M]$. Let us rewrite each element $a \in \mathbb{Z}^l$ as $a = a_+ - a_-$ where the two parts are non-negative and have disjoint supports. Then the map be written as $\beta(a) = \beta(a_+) - \beta(a_-)$, with both parts in S. The kernel $\ker(\beta)$ of the group morphism consists those elements a for which $\beta(a_+) - \beta(a_-) = 0$. In turn, the kernel of the $\mathbb{C}$-algebra morphism is

$$J_\beta = \ker(\mathbb{C}[\beta|_{\mathbb{N}^l}]) = \left(\left\{y^{a_+} - y^{a_-} : \ \beta(a_+) = \beta(a_-)\right\}\right) \subset \mathbb{C}[y_1, \dots, y_l].$$

This is called the **toric ideal** associated to the map β. These binomial equations describe the toric variety

$$X_S = V(J_\beta).$$

The combinatorics of the cones and faces induce geometric structure on the geometric level. First, we extend the notion of face to semigroups.

Definition 3.2.6. *We call such a subset $F = S \cap \tau \subset S$, for $\tau \leq \sigma$, a* ***face of*** *S. Faces are also characterized by the following condition: a subsemigroup $F \subset S$ is a face if and only if $a + b \in F$ for $a, b \in S$ implies $a, b \in F$.*

The latter condition is equivalent to the ideal

$$I_F = \mathbb{C}[S \setminus F] \subset \mathbb{C}[S]$$

corresponding to the face F being a prime ideal.

Note that $\mathbb{R}_{\geq 0}S = \sigma$ is the cone that corresponds to the normalization $\widehat{S} = \sigma \cap M$. Consider a face $\tau \leq \sigma$ of the cone and the corresponding face $\tau \cap S$ of the semigroup. Let us denote the smaller dimensional lattices they generate by $M(\tau) = \mathbb{Z}(\tau \cap M)$ and $M_S(\tau) = \mathbb{Z}(\tau \cap S)$. Then $M_S(\tau)$ is a sublattice of $M(\tau)$ of finite index $i(\tau)$.

Proposition 3.2.7. *The toric variety $TV(\tau \cap S)$ corresponding to the semigroup $\tau \cap S$ and inside the lattice $M(\tau)$ can be embedded into X_S as follows*

$$\iota : TV(\tau \cap S) \hookrightarrow X_S, \quad \iota(TV(\tau \cap S)) = V(I_{\tau \cap S})$$

as the subvariety defined by the vanishing of the monomials in $S \setminus F$.

This embedding map can also be described as $\iota : \mathrm{Hom}_{sg}(S \cap \tau, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{sg}(S, \mathbb{C})$ taking a semigroup morphism η of the source to the morphism $\iota(\eta)$ that is $\iota(\eta)(a) = \eta(a)$ for $a \in S \cap \tau$ and $\iota(\eta)(a) = 0$ otherwise.

The maximal torus of $TV(\tau \cap S)$ is $T_{M_S(\tau)}$. The images of these tori are the orbits of the action of T_M on X_S. Hence they subdivide the variety

$$X_S = \bigsqcup_{\tau \leq \sigma} \iota(T_{M_S(\tau)}).$$

For $\tau \leq \sigma$, the closed subvariety $V(I_{\tau \cap S})$ is the closure of the orbit $\iota(T_{M_S(\tau)})$, that contains all the orbits corresponding to faces $\upsilon \leq \tau$. This is called the **orbit–cone correspondance**. (Note that we differ from the standard notation, where the cones live in the dual space $N_\mathbb{R}$, reversing the inclusion order of them.)

The regularity of a normal affine toric variety can be read off from its defining cone.

Definition 3.2.8. *A cone $\sigma^\vee \subset N_\mathbb{R}$ is **regular** if its minimal lattice generators form a basis of the lattice N.*

Proposition 3.2.9. *An normal affine toric variety X_σ is smooth if and only if the cone $\sigma^\vee$ is regular.*

The non-normal case is slightly more complicated.

Proposition 3.2.10. *The non-singular locus of X_S is the union of those orbits that correspond to faces $\tau \leq \sigma$ with index $i(\tau) = 1$ and a regular dual cone $\tau^\perp \cap \sigma^\vee \subset N_\mathbb{R}$. In other words, the singular locus consists of the vanishing loci of ideals $I_{\tau\cap M}$ corresponding to the faces τ not satisfying the above conditions.*

See [GT02, Remark 4.11.].

Remark 3.2.11. *If we want to build a non-normal non-affine (for example, projective) toric variety, we need to glue affine pieces together. In the normal case, the gluing is encoded in a fan Σ of cones in $N_\mathbb{R}$. In our case, in addition to the fan Σ, we also need a family of affine semigroups $\{S_\sigma\}_{\sigma^\vee\in\Sigma}$, where $S_\sigma \subset M \cap \sigma$, that satisfies some additional compatibility conditions (see [GT14, Definition 4.1.]).*

3.2.2 Cyclic quotient surface singularities

Let us see the case of ffine two-dimensional normal toric varieties. We fix the two-dimensional lattices $M, N \cong \mathbb{Z}^2$ and a cone $\sigma \subset M_\mathbb{R}$.

Proposition 3.2.12. *We can choose a basis $\{e_1, e_2\}$ of M such that $\sigma = \mathbb{R}_{\geq 0}\langle e_2, pe_1{+}qe_2\rangle$ where $0 \leq q < p$ and $gcd(p, q) = 1$.*

For the proof of the equivalent statement for the dual cone in $N_\mathbb{R}$, see [CLS11, Proposition 10.1.1.].

Notation 3.2.13. *We denote the corresponding affine toric surface as*

$$X_{p,p-q} = X_\sigma = \mathrm{Span}\mathbb{C}[\sigma \cap M].$$

Two pairs of parameters, (p, q) and (p', q'), define the same cone up to a change of bases, and in turn isomorphic surfaces, if $p = p'$ and either $q = q'$ or $qq' \equiv 1 \pmod p$.

According to Proposition 3.2.9, $X_{p,p-q}$ is smooth if e_2 and $pe_1 + qe_2$ form a basis of M. If their generated lattice $M' = \mathbb{Z}\langle e_2, pe_1 + qe_2\rangle$ is a proper sublattice of M, then $X_{p,p-q}$

has an isolated singularity. The singular point is defined by the semigroup morphism $\mathrm{Hom}(S_\sigma, \mathbb{C})$, $a \mapsto 0$ – let us denote it by $0 \in X_{p,p-q}$ from now on.

The quotient M/M' is isomorphic to the cyclic group $\mathbb{Z}/p$ of order p. This group is also isomorphic to the group of pth root of unity $G_p = \{\xi \in \mathbb{C} : \ \xi^p = 1\}$.

Proposition 3.2.14. *The toric surface $X_{p,p-q}$ is isomorphic to the quotient $\mathbb{C}^2/G_p$ with respect to the action $\xi(x, y) = (\xi x, \xi^{p-q} y)$.*

The singularity $(X_{p,p-q}, 0)$ is called **cyclic quotient surface** singularity or **Hirzebruch–Jung singularity**. In fact, there are three more equivalent ways to describe this class of singularites.

(i) The singularity $X_{p,p-q}$ is isomorphic to the normalization of vanishing locus of the equation $xy^q = z^p$ in $\mathbb{C}^3$.

(ii) The minimal resolution graph of $X_{p,p-q}$ is a bamboo with $g_v = 0$ for every vertex v.

(iii) There is an analytic covering map $(X_{p,p-q}, 0) \to (\mathbb{C}^2, 0)$ whose branch locus is $\{xy = 0\}$ for suitable coordinates x, y of $(\mathbb{C}^2, 0)$.

For the proof of the equivalence and further reference, see [Ném22, Section 2.3.].

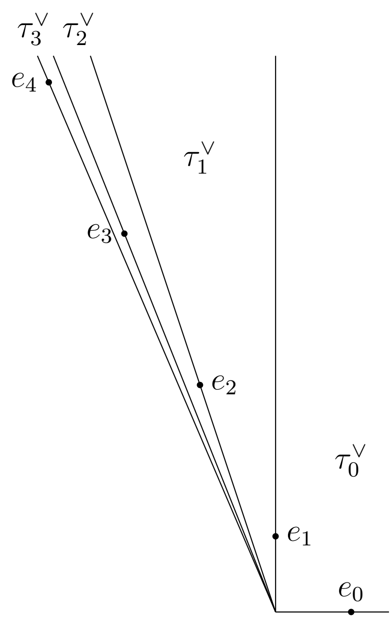
In later calculations, it will be crucial to understand this resolution graph, in particular the generic S^1-fibre of the plumbing over each component of the exceptional divisor. First, we consider the semigroup $\sigma^\vee \cap N$ of those one-parameter subgroups that can be extended to 0, and choose a minimal set of generators $e_0, \dots, e_{r+1}$ of it – the same way as in Proposition 3.3.1. From that proof it follows that adjacent generators e_i, e_{i+1} form a basis of N, or, equivalently, their cone $\tau_i^\vee = \mathbb{R}_{\geq 0}\langle e_i, e_{i+1}\rangle \subset N_\mathbb{R}$ is regular. Let us denote the fan of consisting of the subcones of the cones $\tau_i^\vee$ by Σ. Then the subdivision $\sigma^\vee = \bigcup_0^r \tau_i^\vee$ defines a toric map

$$\rho : X_\Sigma \to X_\sigma,$$

that is a toric resolution of the cyclic quotient singularity of X_σ.

The affine charts of the resolution correspond to the cones $\tau_i^\vee$ of the subdivision. Each embedding $\tau_i^\vee \hookrightarrow \sigma^\vee$ yields a dual embedding $\sigma \hookrightarrow \tau_i \subset M_\mathbb{R}$ and, in turn, an embedding of semigroups $S_\sigma \hookrightarrow S_{\tau_i}$. The latter defines a morphism $X_{\tau_i} \to X_\sigma$ of toric varieties. Note that τ_i is also regular. Let us denote its two generators by $n_i, m_{i+1} \in M$ with $n_i \in e_i^\perp, m_{i+1} \in e_{i+1}^\perp$. Therefore the corresponding chart is smooth $X_{\tau_i} = \mathbb{C}\langle x^{n_i}, x^{m_{i+1}}\rangle \cong \mathbb{C}^2$ and the resolution on this chart can be given by expressing the semigroup generators $\{g_j\}_0^{s+1}$ of S_σ in terms of n_i, m_{i+1}. Let us remark that $n_i = -m_i$.

As an example, consider $\sigma^\vee = \mathbb{R}_{\geq 0}\langle(1,0),(-3,7)\rangle$. The picture on the right shows the decomposition and Figure 3.2.1 shows the semigroup embedding that corresponds to τ_2.



On each chart – except for the first and the last – two components of the exceptional divisor $E = \rho^{-1}(0)$ can be seen: $E_{i+1} = V(x^{n_i})$ and $E_i = V(x^{m_{i+1}})$. On the two extreme charts, there is only one component because $n_0 = g_0$ and $m_{r+1} = g_{s+1}$ so they must vanish on the exceptional locus. Under the orbit–cone correspondence, each component E_i belongs to a new one-dimensional cone $\mathbb{R}_{\geq 0}\langle e_i\rangle = \tau_{i-1}^\vee \cap \tau_i^\vee$ in the fan Σ, that we introduced during the resolution.

How can we express a generic fibre over E_i? This component is $V(x^{n_{i-1}})$ on one chart and $V(x^{m_{i+1}})$ on the other one, and it is parametrized by $x^{n_i} = (x^{m_i})^{-1}$ – with the equation holding on the overlap. Therefore $x^{n_i} = 1$, and respectively $x^{m_i} = 1$, define a generic fibre $F_i \subset X_\Sigma$. These equations imply that any two characters $a, a + kn_i \in M$ agree on $F_i \cap T_M$ as group morphisms $T_M \to \mathbb{C}^*$. Hence F_i is the closure of the image of the map of tori $T_{M/n_i} \to T_M$ that is defined by the quotient map of lattices $M \to M/\mathbb{Z}n_i$. In turn, this is equivalent to the image of the one-parameter subgroup $\lambda^{e_i} : \mathbb{C}^* \to T_M$ corresponding to the lattice element $e_i \in N$ that is perpendicular to n_i. We can express this map in the toric coordinates: $\lambda^{e_i}(t) = (t^{e_i \cdot g_j})_{j=0}^{s+1}$ where $e_i \cdot g_j$ means the scalar pairing between the dual lattices N and M.

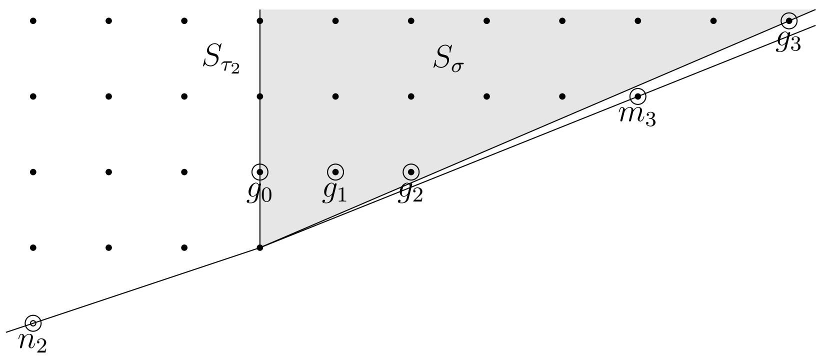


Figure 3.2.1: The semigroup incusion corresponding to a chart of the resolution.

As the resolution ρ is an isomorphism outside the exceptional locus, we get the following statement.

Proposition 3.2.15. *The one-parameter subgroup corresponding to the generator $e_i \in N$ defines a generic fibre $L_i = \lambda^{e_i}(\mathbb{C}) \cap S_\varepsilon$ of the plumbing of the link of X_σ, that belongs to the exceptional divisor E_i.*

3.2.3 Toric deformations

This subsection is dedicated mainly to the work of Altmann on the deformation theory of normal toric varieties. [Alt93; Alt94; Alt95a; Alt95b; Alt97; ACF22; ACF20]

In [Chr91], Christophersen observed that cyclic quotient surface singularities have versal deformations with toric total space over each component of the base space. This raises the question whether every toric sungularity has such a versal deformation. It turns out that a component of the versal base must belong to the 'negative part' of T^1 in order to have toric total space over it ([Alt95a, p. 8]).

In [Alt93; Alt95b], Altmann charecterized the toric deformations of a given normal affine toric variety Y with an isolated singularity over $\mathbb{C}^k$ as base. More precisely, he characterized deformations that fit into the commutative diagram

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "\pi"] \\ \{0\} \arrow[r, hook] & \mathbb{C}^k \end{tikzcd},$$

where i is a morphism of toric varieties, Y has an isolated singularity and π is flat.

Consider a lattice M and a top-dimensional cone $\sigma \subset M_\mathbb{R}$ and a k-dimensional sublattice $L \subset M$ such that M/L is torsion-free and $L \cap \sigma = 0$. (Note that we switch the roles of σ and $\sigma^\vee$ compared to Altmann's papers.) Let the sublattice L be generated by a set $\{q_1 - r_1, \dots, q_k - r_k\}$ of differences semigroup elements and let us 'squeeze' the cone σ to $\bar{\sigma} = \sigma/L_\mathbb{R}$. Then the toric ideal $(x^{q_1} - x^{r_1}, \dots, x^{q_k} - x^{r_k}) \subset \mathbb{C}[S_\sigma]$ defines a toric complete intersection subvariety $X_{\bar{\sigma}} X_\sigma$. This is isomorphic to $X_{\bar{\sigma}}$ defined by $\bar{\sigma}$ and the quotient lattice M/L if the faces of the dual cone $\sigma^\vee \subset N_\mathbb{R}$ and the generators of L satisfy some – rather technical – conditions $(*)$, for the deatils see [Alt93, (2.4) Theorem]. To summarize, we obtain a toric deformation of Y with the above construction if $Y \cong X_{\bar{\sigma}}$ and $(*)$ are satisfied. In fact, the conditions $(*)$ guarentee that the squeezed semigroup S_σ/L is saturated. We will actually be interested in the non-saturated case.

We may also hope for a nice combinatorial description of the infinitesimal deformations of a toric singularity. By construction, $T^1_{X_\sigma,0}$ is a $\mathbb{C}[S_\sigma]$-module that gives an M-grading on it. In [Alt94], Altmann gives a formula for the graded pieces $T^1_{X_\sigma,0}(-R)$ for $R \in M$. To state the theorem, we need to introduce some notations. Let the cone σ be the intersection of the half-spaces defined by the inequalities $(a_i, m) \geq 0$ for some $a_i \in N$ – that in turn generate the dual cone $\sigma^\vee = \mathbb{R}\langle a_1, \ldots, a_l\rangle$. Let E be the generating set of the semigroup S_σ and let us pick $R \in M$. Then let us denote the generators below R in the a_i direction by $E_i^R = \{e \in E : (a_i, e) < (a_i, R)\}$ and their union by $E^R = \bigcup_1^l E_i^R$.

Theorem 3.2.16. *[Alt94, (2.3)]*

$$T^1_{X_\sigma,0}(-R) = \left(\mathrm{Rel}(E^R) \middle/ \textstyle\sum_1^l \mathrm{Rel}(E_i^R)\right)^* \underset{\mathbb{Z}}{\otimes} \mathbb{C},$$

where $\mathrm{Rel}(F)$ *denotes the group of linear relations between the elements of* F*.*

In particular, we can deduce that each graded piece is finite dimensional.

In [Alt95a, Chapter 3.], he gave alternative T^1-formulas. Moreover, he deduced that, for normal affine toric varieties of dimension at least 3, the space T^1 is infinite-dimensional (see [Alt95a, p. 44]).

In [Alt97], he creates a deformation for toric Gorenstein singularites X_σ, that are smooth in codimension 2. The defining dual cone $\sigma^\vee$ of such a variety has a lattice polytope Q embedded in it as the hyperplane cut at height 1 in the Gorenstein direction. We can build a deformation from the different ways of splitting Q into Minkowski summands. This deformation is versal when $\dim T^1_{X_\sigma} < \infty$, in particular, when the singularity of X_σ is isolated. In [ACF22], Altmann, Constantinescu and Filip generalized this construction to the maximal deformations with prescribed tangent space $T^1(-R)$ of any toric singularity while also strengthening the statement about versality.

This theory also gives a new way of describing the versal deformation of cyclic quotient surface singularities.

3.3 A family of non-isolated toric surfaces

3.3.1 Semigroup generators

Take a two-dimensional lattice $M \cong \mathbb{Z}^2$ and a strongly convex rational polyhedral cone $\sigma \subset M_\mathbb{R} \cong \mathbb{R}^2$. Let us denote the corresponding semigroup by $S_\sigma = \sigma \cap M$. We create a normal affine toric surface with this data: $X_\sigma = \mathrm{Spec}\ (\mathbb{C}[S_\sigma])$. Note that $(X_\sigma, 0)$ is a cyclic quotient surface singularity.

Up to an integral linear coordinate change, we can assume that one extremal ray generator of S_σ is $(0, 1)$. We can also assume that the other extremal ray generator (p, q) is primitive and $p > q \geq 0$. Consider the primitive extremal lattice points of the convex hull $\mathrm{conv}(S_\sigma \setminus 0)$. They can be ordered naturally from $g_0 = (0, 1)$ to $g_{s+1} = (p, q)$.

Proposition 3.3.1. *The primitive extremal lattice points $g_0, g_1, \ldots, g_{s+1}$ of the convex hull form a Hilbert basis of the semigroup S_σ.*

Proof: First, we cut the semigroup S_σ into slices along the rays $\mathbb{R}_{\geq 0}\langle g_i \rangle$. We show that g_i and g_{i+1} generate the corresponding slice $S_i = \mathbb{R}_{\geq 0}\langle g_i, g_{i+1}\rangle \cap M$ of the semigroup S for all $0 \leq i \leq s$. Any lattice point $h \in S_i$ can be written as $h = \alpha g_i + \beta g_{i+1}$ with $\alpha, \beta \geq 0$. We want to show that α and β are integers. Let us shift h 'backwards' with a combination of g_i, g_{i+1} into the parallelogram $P_i = \mathrm{conv}(0, g_i, g_{i+1}, g_i + g_{i+1})$. The shifted copy is

$$h' = (\alpha - \lfloor \alpha \rfloor) g_i + (\beta - \lfloor \beta \rfloor) g_{i+1} =: \alpha' g_i + \beta' g_{i+1}.$$

Now $0 \leq \alpha', \beta' < 1$, and h' is still in $\sigma \cap M$.

Where can this h' be inside the parallelogram? We know that it cannot be inside the 'lower half' triangle formed by $0, g_i$ and g_{i+1} – only if it coincides with one of the vertices – as in that case h' would be extremal too. This means that $\alpha' + \beta' \geq 1$. However, being in the 'upper half' triangle – meaning $\alpha' + \beta' > 1$ – is not possible either for the following reason. The lattice M is centrally symmetric to the bisector $\frac{g_i + g_{i+1}}{2}$. This reflection interchanges the two triangles, taking $\alpha' g_i + \beta' g_{i+1}$ to $(1 - \alpha') g_i + (1 - \beta') g_{i+1}$. Finally, $\alpha' + \beta' = 1$ would mean that h' is on the segment connecting the two extremal points, making it, again, extremal. Therefore, h' must coincide with one of the vertices of the parallelogram, and according to the inequalities, that can only be 0. This means $\alpha' = \beta' = 0$, in turn making α and β integers. □

The assumption $p > q$ implies $(1, 1) \in \sigma$, yielding $g_1 = (1, 1)$ as this point must be extremal.

One can compare the neighbouring triples of these g_i generators resulting in a set of very nice relations.

Proposition 3.3.2. *The equations*

$$\begin{aligned} g_0 + g_2 &= k_1 \cdot g_1 \\ g_1 + g_3 &= k_2 \cdot g_2 \\ &\vdots \\ g_{s-1} + g_{s+1} &= k_s \cdot g_s \end{aligned} \tag{3.1}$$

hold for some $k_i \geq 2$ integers.

Proof: The sum $g_{i-1} + g_{i+1}$ obviously lies in the union $S_{i-1} \cup S_{i+1}$ of the two corresponding slices. Thus we can express the sum as a positive integer combination of either g_{i-1}, g_i or g_i, g_{i+1}. Assume $g_{i-1} + g_{i+1} = \alpha g_{i-1} + \beta g_i$. Then $\alpha = 0$ must hold, otherwise $g_{i+1} = (\alpha - 1)g_{i-1} + \beta g_i \in S_{i-1}$ would be a contradiction. The same happens for $g_{i-1} + g_{i+1} = \alpha g_i + \beta g_{i+1}$. Hence $g_{i-1} + g_{i+1} = k_i g_i$.

The coefficient k_i cannot be 1, as $g_{i-1} + g_{i+1}$ is not an extremal point of the convex hull. □

Example 3.3.3. *As a running example, let us consider* $(p, q) = (7, 3)$ *that gives* $X_\sigma = X_{7,4}$. *The generators of the semigroup are* $g_0 = (0, 1)$, $g_1 = (1, 1)$, $g_2 = (2, 1)$, $g_3 = (7, 3)$ *and they satisfy the equations* $g_0 + g_2 = 2g_1, g_1 + g_3 = 4g_2$. *(See Figure 3.3.1)*

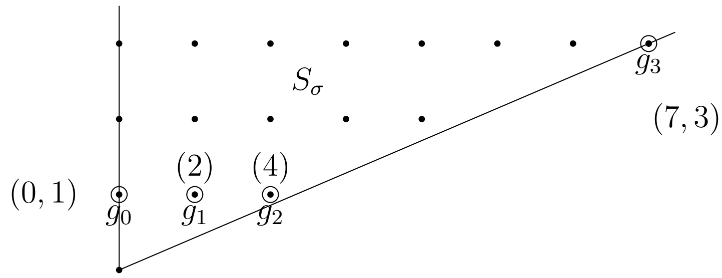


Figure 3.3.1: The generators of the semigroup S_σ corresponding to $X_{7,4}$ with their coefficients with respect to (3.1).

The coefficients $\{k_i\}$ also appear in the Hirzebruch–Jung – or from now on – negative continued fraction

$$\frac{p}{p-q} = [k_1, k_2, \dots k_s]. \tag{3.2}$$

For a clear and concise, yet slightly differently structured description of the generators, coefficients and the continued fraction, see [CLS11, §10.2.].

Let us denote the algebra elements corresponding to g_i by z_i. Then the semigroup equations (3.1) turn into

$$\begin{aligned} z_0 z_2 &= z_1^{k_1} \\ z_1 z_3 &= z_1^{k_2} \\ &\vdots \\ z_{s-1} z_{s+1} &= z_s^{k_s} \end{aligned} \tag{3.3}$$

Note that this is not the complete list of equations that generate the vanishing ideal of the variety. For that, see (3.8).

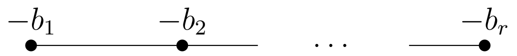
Dual picture and the plumbing graph

The dual cone of σ is $\sigma^\vee = \mathbb{R}_{\geq 0}\langle(1,0),(-q,p)\rangle \subset N_\mathbb{R}$. Up to integral change of bases $\sigma^\vee$ is equivalent to $\mathbb{R}_{\geq 0}\langle(1,0),(p-q,p)\rangle$.

One can read off the data from this dual picture that determines the topology of the singularity. The dual semigroup consists of the one-parameter subgroups $\mathbb{C}^* \to T_M$ that can be extended to $0 \in \mathbb{C}$. This semigroup is generated by $e_0, e_1, \ldots e_{r+1}$ (with $e_0 = (1,0)$) satisfying the relations analogous to (3.1) with the coefficients $b_1, b_2, \ldots b_r$. As above, these give the negative continued fraction

$$\frac{p}{q} = [b_1, b_2, \ldots b_r]. \tag{3.4}$$

What is more important, they describe a resolution of X_σ. The dual resolution graph of X_σ is



(3.5)

(In this sense the two continued fractions (3.2) and (3.4) are dual to each other.)

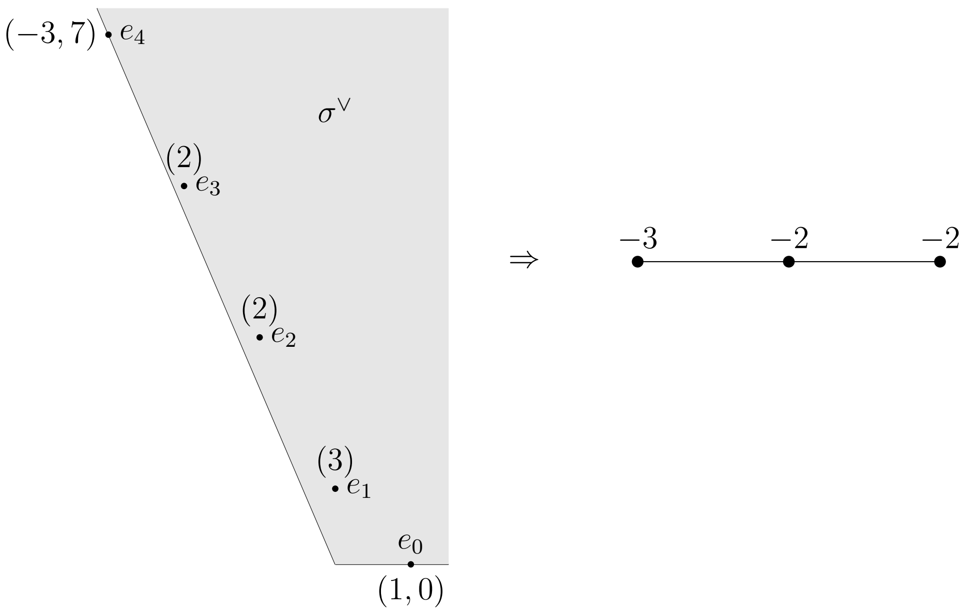


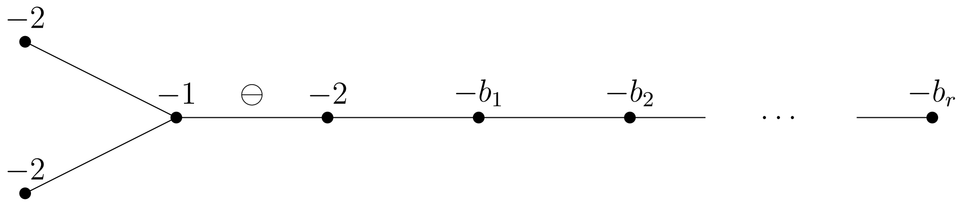
Figure 3.3.2: The dual cone and the resolution graph of $X_{7,4}$.

3.3.2 The theorem

Due to its normality, the affine surface X_σ can only have an isolated singularity at the origin. In order to create a non-isolated toric singularity, we remove some elements from the semigroup. In fact, we need to remove an infinite family of elements. Otherwise, it would be a non-normal germ whose δ-invariant equals the number of gaps. Hence, it would still have an isolated singularity. The normalization, defined by the saturation on the semigroup level, is described in Section 3.3.3.

As it turns out, the simplest case is when we remove a ‘mod 2’ family of elements. Let $S = S_\sigma \setminus \{(0,1),(0,3),(0,5),...(0,2k+1),...\}$. Let $X = \text{Spec }(\mathbb{C}[S])$ denote the corresponding toric variety.

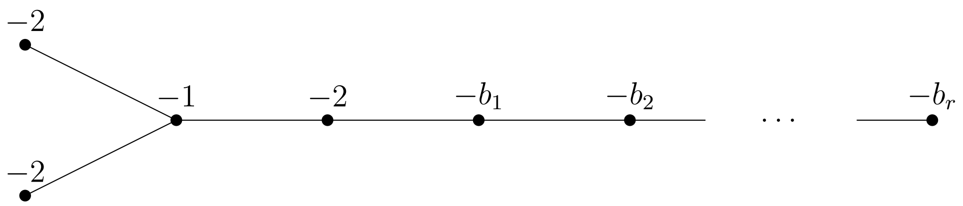
Theorem 3.3.4. *The boundary of the Milnor fibre of X with respect to the deformation described in Section 3.3.4 is a graph manifold with the following plumbing graph*



(3.6)

The rest of the paper is devoted to setting up and proving this result.

Remark 3.3.5. *If either side of a negative edge is a tree, then one can remove the minus sign. Indeed, we can simply remove the negative sign and change the orientation of the fibres and the bases – which is the R0. (a) move in [Neu81] – on one side resulting in the graph*



(3.7)

This type of graph belongs to Neumann’s N1 type according to his classification theorem in [Neu81].

3.3.3 Non-normal singularity

The normal variety X_σ can be embedded into $\mathbb{C}^{s+2} \cong \mathbb{C}\langle z_0, ...z_{s+1}\rangle$. Riemenschneider in [Rie81] tells us that the a set of functions generating the ideal defining X_σ has a nice compressed form

$$\begin{pmatrix} z_0 & & z_1 & & z_2 & \dots & & & z_s \\ & z_1^{k_1-2} & & z_2^{k_2-2} & & \dots & & z_s^{k_s-2} & \\ z_1 & & z_2 & & z_3 & \dots & & & z_{s+1} \end{pmatrix} \tag{3.8}$$

that means the following: for any $0 \leq i < j \leq s$

$$z_i z_{j+1} = z_{i+1} \Big(\prod_{l=i+1}^{j} z_l^{k_l-2} \Big) z_j. \tag{3.9}$$

Now, we want to find the equations defining X. First, we need a new generating set of the punctured semigroup S as we have lost the first generator $g_0 = (0, 1)$.

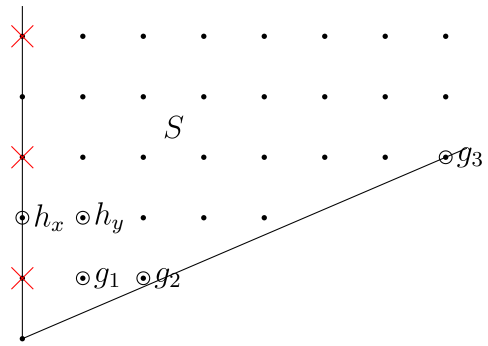


Figure 3.3.3: Removal of a 'mod 2' family from the semigroup and the new generators.

Proposition 3.3.6. *The semigroup S is generated by the elements $h_x = (0, 2)$, $h_y = (1, 2)$, g_1, $g_2, \dots g_{s+1}$.*

Proof: Let $s \in S \subset S_\sigma$. As an element of the latter, we can express s as

$$s = a_0 g_0 + a_1 g_1 + ... + a_{s+1} g_{s+1}$$

with $a_i \in \mathbb{Z}_{\geq 0}$. We will show that we can create an expression for s using the new generators. If a_0 is even, we can replace g_0 with $h_x = 2g_0$ in the following way:

$$s = \frac{a_0}{2} h_x + a_1 g_1 + ... + a_{s+1} g_{s+1}. \tag{3.10}$$

Now, assume a_0 is odd. We can take care of $(a_0 - 1) \cdot g_0$ as above leaving only one g_0. Our assumption implies that $s \neq a_0 g_0$ because odd multiples of g_0 are exactly the elements we threw out in (3.10). So, let $a_m > 0$. Then we can use the additive version of the equation (3.9) for $i = 0, j = m-1$. That is $g_0 + g_m = g_1 + \left(\sum_{l=1}^{m-1}(k_l - 2)\cdot g_l\right) + g_{m-1}$. The right hand side does not have g_0. Hence, we have the expression

$$\begin{aligned} s &= a_0 g_0 + a_1 g_1 + ... + a_m g_m + ... + a_{s+1} g_{s+1} \\ &= \tfrac{a_0-1}{2} h_x + g_1 + \textstyle\sum_{l=1}^{m-1}(k_l-2)\cdot g_l + g_{m-1} + a_1 g_1 + ... + (a_m - 1) g_m + ... + a_{s+1} g_{s+1}, \end{aligned}$$

that uses only the new generators. The only problem, that might occur, is if the only nonzero coefficient is a_1, because then $i = 0 = m-1 = j$ and we cannot use (3.9). But in that case, using $h_y = g_0 + g_1$, we can obtain $s = a_0 g_0 + a_1 g_1 = \frac{a_0-1}{2} h_x + h_y + (a_1 - 1) g_1$, which works, too. $\square$

In the sequel, we denote the algebra elements corresponding to h_x and h_y by x and y.

New relations

We need to find a set of equations which generates the ideal of X.

From the equations (3.8) we can obtain new ones by getting rid of z_0. Fortunately, this only appears in the top left corner. If we multiply the first column with any term, we will get a set of relations – that corresponds to adding a vector to the two sides of the additive relations involving g_0 and g_1.

If we multiply by z_0, as $z_0^2 = x$ and $z_0 z_1 = y$, we get

$$\begin{pmatrix} x & & z_1 & & z_2 & \cdots & & z_s \\ & z_1^{k_1-2} & & z_2^{k_2-2} & & \cdots & z_s^{k_s-2} & \\ y & & z_2 & & z_3 & \cdots & & z_{s+1} \end{pmatrix}. \tag{I}$$

Similarly, multiplying by z_1 results in

$$\begin{pmatrix} y & & z_1 & & z_2 & \cdots & & z_s \\ & z_1^{k_1-2} & & z_2^{k_2-2} & & \cdots & z_s^{k_s-2} & \\ z_1^2 & & z_2 & & z_3 & \cdots & & z_{s+1} \end{pmatrix}. \tag{II}$$

In both cases the rule to read off relations is the analogue of (3.9).

Note that the relations coming from after the second column are repeated identically in the two sets.

In addition to the above relations, there is the relation formed by the two first columns of the above matrices.

$$\begin{pmatrix} x & y \\ y & z_1^2 \end{pmatrix} \tag{III}$$

We can patch these relations into one Riemenschneider matrix-like diagram

$$\begin{pmatrix} x & & y & & z_1 & & \dots & & z_s \\ & 1 & & z_1^{k_1-2} & & z_2^{k_2-2} & \dots & z_s^{k_s-2} & \\ y & & z_1^2 & & z_2 & & \dots & & z_{s+1} \end{pmatrix}. \tag{$*$}$$

Proposition 3.3.7. *The relations encoded in* $(*)$ *form a generating set of the ideal of the variety* $X \subset \mathbb{C}^{s+3}\langle x, y, z_1, ...z_{s+1}\rangle$.

Proof: Firstly, the discussion above shows that all these relations hold for the variables.

Secondly, we shall prove that these relations generate each relation that holds true. According to the toric construction, it is enough to show that any binomial relation can be generated by the relations of $(*)$. Consider a generic relation

$$x^{\alpha_x} y^{\alpha_y} \prod_1^{s+1} z_i^{\alpha_i} = x^{\beta_x} y^{\beta_y} \prod_1^{s+1} z_i^{\beta_i}.$$

We shall assume that each variable appears on one side only – otherwise the relation would be a multiple of a simpler one. Thus $\forall i \in \{x, y, 1, ..., s+1\}$ we have $\alpha_i = 0$ or $\beta_i = 0$. Assume $\alpha_x \neq 0$. Take the last variable with nonzero power on the left side: $z_j^{\alpha_j}$, and apply the appropriate multiple of the relation

$$xz_j = y \cdot 1 \cdot z_1^{k_1-2} \cdot ... \cdot z_{j-1}^{k_{j-1}-2} \cdot z_{j-1},$$

replacing the left hand side with

$$x^{\alpha_x-1} y^{\alpha_y+1} z_1^{\alpha_1+k_1-2} \cdot ... \cdot z_{j-1}^{\alpha_{j-1}+k_{j-1}-1} \cdot z_{j-1}^{\alpha_j-1}.$$

Note that this step decreases the power of x by one. In α_x steps we get rid of all of x. Then we can remove all the y from the side where they appear after simplification using similar multiples of the relation

$$yz_j = z_1^2 \cdot z_1^{k_1-2} \cdot ... \cdot z_{j-1}^{k_{j-1}-2} \cdot z_{j-1}.$$

This step decreases the power of y by one and does not reintroduce x. Once we manage to get rid of both x and y, we end up with a relation involving only z_i's.

Consider the cone $\bar{\sigma} = \mathbb{R}_{\geq 0}\langle g_1, g_{s+1}\rangle$ and the corresponding semigroup $S_{\bar{\sigma}}$. The arguments in 3.3.1 and 3.3.2 hold for $S_{\bar{\sigma}}$, hence this semigroup is generated by $\{g_1, g_2, \dots, g_{s+1}\}$. Similarly, the analogue of (3.8):

$$\begin{pmatrix} z_1 & & \dots & & z_s \\ & z_2^{k_2-2} & \dots & z_s^{k_s-2} & \\ z_2 & & \dots & & z_{s+1} \end{pmatrix} \tag{3.11}$$

gives the generating set of equations between the variables $z_1, \dots z_{s+1}$. On the other hand, this (3.11) is included in $(*)$, thus we are done.

The only question that remains is how this algorithm can get stuck. In what situation can we not reduce the power of x (or y respectively)? If there is any $\alpha_i \neq 0, i \geq 2$, we are fine. If there isn't any, but $\alpha_1 \geq 2$, then we can apply a multiple of (III): $xz_1^2 = y^2$, decreasing α_x further. The only problematic cases remaining are when $\alpha_i = 0$ for $i \geq 2$ and $\alpha_1 < 2$ or when $\alpha_i = 0$ for $i \geq 2, \alpha_x = 0$. That is we only have to take care of the following types of monomials:

$$x^{\alpha_x} y^{\alpha_y},\ x^{\alpha_x} y^{\alpha_y} z_1,\ y^{\alpha_y} z_1^{\alpha_1}.$$

Note that the these monomials cannot be nontrivially equal to a monomial involving only z_i's, because the former represent a lattice vector in $\mathbb{R}_{\geq 0}\langle h_x, g_1\rangle$, whereas the latter in $\mathbb{R}_{\geq 0}\langle g_1, g_{s+1}\rangle$. So, could the listed monomials equal each other? This would mean an additive lattice equation involving only h_x, h_y and g_1 with no redundancy on the two sides. For convexity, the only possibility is $\alpha_x h_x + g_1 = \beta_y h_y$. This, on the other hand, is not possible as in the canonical basis it turns into

$$\alpha_x \cdot (0,2) + (1,1) = (1, 2\alpha_x + 1) = (\beta_y, 2\beta_y) = \beta_y \cdot (1,2),$$

where the parity of the second coordinate does not match.

Thirdly, we need to show that there is no redundancy. That is, none of the relations of $(*)$ can be expressed with the others. This becomes obvious if we notice that for any of our relations, the monomial on the left hand side involves only two variables, and that pair characterises the particular relation.

$\square$

Example 3.3.8. *For the running example (3.3.3), the equations are the following. The surface $X_\sigma \subset \mathbb{C}^4$ is defined by the equations*

$$\begin{pmatrix} z_0 & & z_1 & & z_2 \\ & z_1^{2-2} & & z_2^{4-2} & \\ z_1 & & z_2 & & z_3 \end{pmatrix}.$$

For instance, the first equation is $z_0 z_2 = z_1 z_1^0 z_1$. The new equations for the non-normal X are

$$\begin{pmatrix} x & & y & & z_1 & & z_2 \\ & 1 & & z_1^{2-2} & & z_2^{4-2} & \\ y & & z_1^2 & & z_2 & & z_3 \end{pmatrix}.$$

Normalization and singular locus

From the functoriality of the toric variety construction, we can obtain the normalization map. First, consider the embedding

$$\begin{array}{rcll} S & \hookrightarrow & S_\sigma & \\ h_x & \mapsto & 2g_0 & \\ h_y & \mapsto & g_0 + g_1 & \\ g_i & \mapsto & g_i & \qquad \text{for } 0 < i \leq s+1. \end{array}$$

This defines an embedding of $\mathbb{C}$-algebras

$$\begin{array}{rcll} \mathbb{C}[S] & \hookrightarrow & \mathbb{C}[S_\sigma] & \\ x & \mapsto & z_0^2 & \\ y & \mapsto & z_0 z_1 & \\ z_i & \mapsto & z_i & \qquad \text{for } 0 < i \leq s+1, \end{array}$$

that in turn defines a map of varieties

$$n : X_\sigma \to X.$$

This is the normalization of X. The singular locus $\mathrm{Sing}(X)$ of X is the x-axis. The map n is a branched $2:1$ cover over $\mathrm{Sing}(X)$, that is covered by the z_0-axis. Outside the singular locus, n is an isomorphism.

3.3.4 Deformation

We want to deform the non-isolated singularity X.

Remark 3.3.9. *First, we tried to modify the deformations of X_σ, coming from the general theory, so that they give deformations of X (see Subsection 3.2.3 and [Alt95b]). Consider a 1-parameter toric deformation of X_σ. That consists of a cone $\rho \subset M_{\mathbb{R}} \cong \mathbb{R}^3$ and a 1-sublattice $L \subset M$ such that the squeezed semigroup $S_\rho / L \subset M/L$ is isomorphic to S_σ. We can easily adjust this to the non-saturated case: let $\widetilde{S}$ be the subsemigroup of S_ρ that we obtain as the preimage of $S \subset S_\sigma$ by the quotient map. However, the semigroup $\widetilde{S}$ needs too many generators compared to S that means that $X \subset X_{\widetilde{S}}$ is cut out by more than one equations, hence the result is not a 1-parameter family.*

On the other hand, we can try 'lifting' the generators of S to $\bar{S} \subset S_\rho$ in a way that $X \subset X_{\bar{S}}$ is defined by one toric equation. Then the subsemigroup $\bar{S} \subset S_\rho$ has too many gaps resulting in a flat family with non-isolated singularities in all the fibres.

Instead, we define a 3-dimensional normal toric variety whose semigroup becomes non-saturated after squeezing it, as described in Subsection 3.2.3. We define the total space of a deformation as a normal toric variety corresponding to a 3-dimensional cone. Let $\widetilde{M} = M \oplus \mathbb{Z}$ and $\widetilde{\sigma} = \mathbb{R}_{\geq 0}\langle (h_x, 1), (h_y, 1), (h_y, 0), (g_{s+1}, 0)\rangle \subset \widetilde{M}_{\mathbb{R}}$. Note that the linear extension of the projection $\widetilde{M} \twoheadrightarrow M$ maps $\widetilde{\sigma}$ onto σ.

Let $\widetilde{S} = \widetilde{\sigma} \cap \widetilde{M}$, the semigroup defining the deformation space.

When talking about the semigroup generators in the 3-lattice $M \oplus \mathbb{Z}$, we will abuse notation and use both of these formats: $(h_x, 1) = (0, 2, 1)$, $(h_y, 1) = (1, 2, 1)$, $(h_y, 0) = (1, 2, 0)$, $(g_1, 0) = (1, 1, 0)$.

Proposition 3.3.10. *The semigroup $\widetilde{S}$ is generated by the set* $\{(h_x, 1), (h_y, 1), (h_y, 0), (g_1, 0), (g_2, 0), ...(g_{s+1}, 0)\}$.

Proof: We want to show that the given vectors generate each vector in the semigroup. For this, we cut the cone $\widetilde{\sigma}$ into simplicial cones in the following, rather natural way, and generate the lattice vectors subcone by subcone.

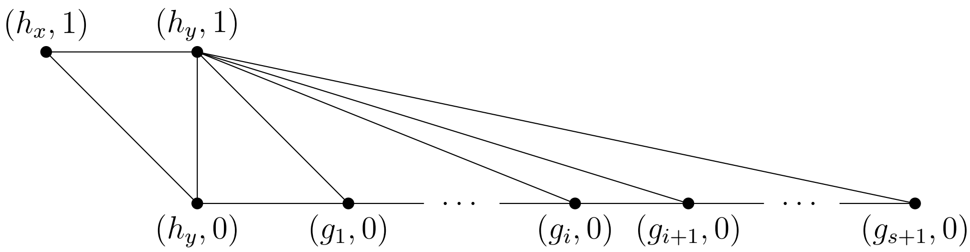


Figure 3.3.4: Simplicial subdivision.

All the lattice vectors in a simplicial subcone are generated by the three ray generators if (and only if) the ray generators form a basis of the lattice – we call these cones regular. An equivalent criterion of this is whether the 3×3 determinant of the generators equals ± 1.

We claim that all the cones besides the first one are regular. Take $\mathbb{R}_{\geq 0}\langle (h_y, 1), (h_y, 0), (g_1, 0)\rangle$. The corresponding determinant is

$$\begin{vmatrix} h_y & 1 \\ h_y & 0 \\ g_1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = -1.$$

For any $1 < i \leq s$, we have an inductive argument, claiming that they agree with the previous determinant.

$$\begin{vmatrix} h_y & 1 \\ g_i & 0 \\ g_{i+1} & 0 \end{vmatrix} = \begin{vmatrix} g_i \\ g_{i+1} \end{vmatrix} = \begin{vmatrix} g_i \\ k_i g_i - g_{i-1} \end{vmatrix} = - \begin{vmatrix} g_i \\ g_{i-1} \end{vmatrix} = \begin{vmatrix} g_{i-1} \\ g_i \end{vmatrix}$$

And the first step of the induction is

$$\begin{vmatrix} g_0 \\ g_1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1.$$

On the other hand the first cone is not regular, the corresponding generators only give a subgroup of order 2:

$$\begin{vmatrix} h_x & 1 \\ h_y & 1 \\ h_y & 0 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 2.$$

Fortunately, we are allowed to use the generators corresponding to other subcones too. The simplest sum $(h_x, 1) + (g_1, 0) = (1, 3, 1)$ landing in this cone luckily helps. Let us subdivide the first cone into three cones via this vector as the figure shows.

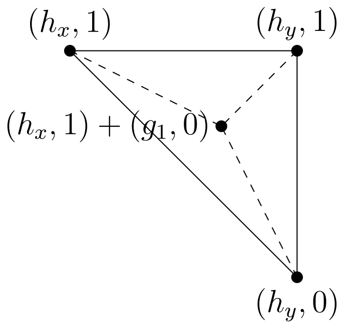


Figure 3.3.5: Subdivision into regular cones.

The three subcones are now regular:

$$\begin{vmatrix} h_x & 1 \\ h_y & 1 \\ h_x + g_1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} h_y & 1 \\ h_y & 0 \\ h_x + g_1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{vmatrix} = 1,$$

$$\begin{vmatrix} h_x & 1 \\ h_y & 0 \\ h_x + g_1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{vmatrix} = -1.$$

Hence, all the semigroup elements are generated by the given vectors. □

Let $\widetilde{X} = \text{Spec}\ (\mathbb{C}[\widetilde{S}])$. Let us denote the algebra generators in the following way. For each vector with 0 as last coordinate, we use the same variable name as in $\mathbb{C}[S]$: $y, z_1, \dots z_{s+1}$. In case of vectors with last coordinate 1, we introduce a '$\sim$' on top of them: $\widetilde{x}, \widetilde{y}$. This will help us to keep track of the relations and will not be such a big abuse of notation because of the deformation.

$\widetilde{X}$ fits into the commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \widetilde{X} \arrow[d, "t"] \\ 0 \arrow[r, hook] & \mathbb{C} \end{tikzcd}$$

with the function $t = \widetilde{y} - y$ parametrizing the family.

Proposition 3.3.11. *This is a deformation.*

Proof: According to Proposition 3.1.12 a t-family $\widetilde{X}$ is flat if and only if t is not a zero divisor in $\mathcal{O}_{\widetilde{X}}$. As $\widetilde{X}$ is a normal toric variety, its structure sheaf has no zero divisors. $\square$

Furthermore, we can lift all the defining equations $(*)$ of X obtaining

$$\begin{pmatrix} \widetilde{x} & & y & & z_1 & & \dots & & z_s \\ & 1 & & z_1^{k_1-2} & & z_2^{k_2-2} & \dots & z_s^{k_s-2} & \\ \widetilde{y} & & z_1^2 & & z_2 & & \dots & & z_{s+1} \end{pmatrix}. \tag{$**$}$$

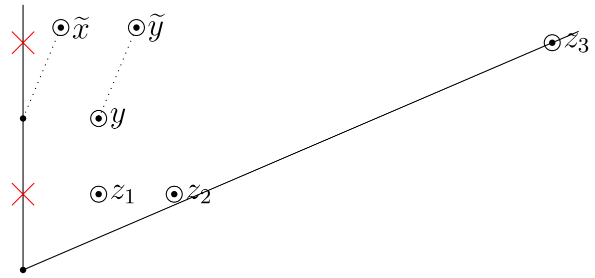
Proposition 3.3.12. *The relations* $(**)$ *above define the total deformation space* $\widetilde{X}$.

Proof: First, we check that these equations hold true. Dropping to the level of semigroup elements one only needs to check the third coordinates in the relations as the first two coordinates agree with those in $(*)$. There, only $\widetilde{x}$ and $\widetilde{y}$ (or $(h_x, 1), (h_y, 1)$ respectively) are nonzero, but they come as a pair in $(**)$.

We want to show that these equations are enough.

Each non-redundant equation of $\widetilde{X}$ is the unique lifting of an equation of X, because we only have a choice regarding the lifting of the generator y, and there the third coordinate of the lattice $\widetilde{M} = M \oplus \mathbb{Z}$ tells the right combination of y and $\widetilde{y}$ in the lifting. But as $(*)$ generate every equation of X, and $(**)$ generates a lifting of each of them, this gives all the equations for $\widetilde{X}$. $\square$

Example 3.3.13. *For the running example, the 3-dimensional cone corresponding to the total space, with the semigroup generators, looks like the following.*



This yields the equations

$$\begin{pmatrix} \widetilde{x} & & y & & z_1 & & z_2 \\ & 1 & & z_1^0 & & z_2^2 & \\ \widetilde{y} & & z_1^2 & & z_2 & & z_3 \end{pmatrix}.$$

3.3.5 Nearby singularities

Despite being a deformation, the family $t : \widetilde{X} \to \mathbb{C}$ is not a smoothing as the nearby fibres have singularities. However, we have the following.

Proposition 3.3.14. *Each nearby fibre has a single isolated singularity along the $\widetilde{y}$-axis.*

Proof: We want to find the singular locus of $\widetilde{X}$. In case of a normal toric variety such as $\widetilde{X}$, it can be read off the corresponding fan in the space $N_{\mathbb{R}}$. One only needs to check the regularity of the cones of the fan. In our case, the fan consists of all the subcones of $\tilde{\sigma}^\vee$.

The cone $\widetilde{\sigma}^\vee$ is generated by the following vectors:

$$(h_y, 0) \times (h_x, 1) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ \underline{i} & \underline{j} & \underline{k} \end{vmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \qquad (h_x, 1) \times (h_y, 1) = \begin{vmatrix} 0 & 2 & 1 \\ 1 & 2 & 1 \\ \underline{i} & \underline{j} & \underline{k} \end{vmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix},$$

$$(h_y, 1)\times(g_{s+1}, 0) = \begin{vmatrix} 1 & 2 & 1 \\ p & q & 0 \\ \underline{i} & \underline{j} & \underline{k} \end{vmatrix} = \begin{pmatrix} -q \\ p \\ q-2p \end{pmatrix}, \qquad (h_y, 0)\times(g_{s+1}, 0) = \begin{vmatrix} p & q & 0 \\ 1 & 2 & 0 \\ \underline{i} & \underline{j} & \underline{k} \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2p-q \end{pmatrix}.$$

We check each subcone. Firstly, the whole cone is not even simplicial, so it is singular. This results in a singularity at $0 \in \widetilde{X}$. Then we go through the four facets. Regularity of a 2-cone in 3-space is equivalent to the cross product of the corresponding two vectors being primitive.

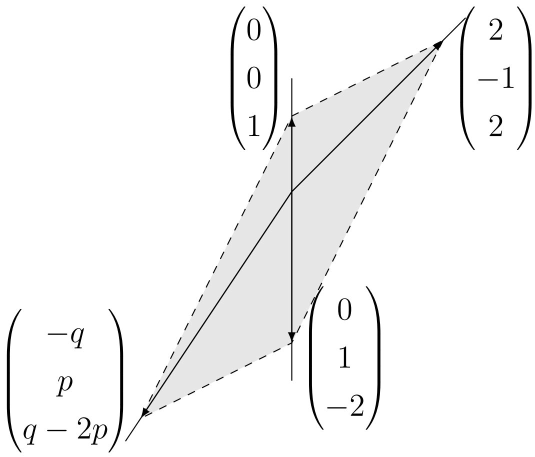


Figure 3.3.6: Dual cone $\widetilde{\sigma}^\vee$.

$$\begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}0\\1\\-2\end{pmatrix}=(0,4,2)=2\cdot(0,2,1),\qquad \begin{pmatrix}0\\1\\-2\end{pmatrix}\times\begin{pmatrix}-q\\p\\q-2p\end{pmatrix}=(q,2q,q)=q\cdot(1,2,1),$$

$$\begin{pmatrix}-q\\p\\q-2p\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}=(p,q,0),\qquad \begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}2\\-1\\2\end{pmatrix}=(1,2,0).$$

Hence, the two former facets listed are singular, whereas the latter two are regular.

Finally, the one-dimensional faces are always regular in the normal setup.

This means that along the axes corresponding to $(0,2,1)=(h_x,1)$ and $(1,2,1)=(h_y,1)$ lie the singular locus of the variety. In other words,

$$\operatorname{Sing}(\widetilde{X})=\{\widetilde{y}=y=z_1=\cdots=z_{s+1}=0\}\cup\{\widetilde{x}=y=z_1=\cdots=z_{s+1}=0\}=\widetilde{\Sigma}_1\cup\widetilde{\Sigma}_2. \tag{3.12}$$

Note that the $\widetilde{x}$-axis, $\widetilde{\Sigma}_1=\operatorname{Sing}(X)$, that is the first branch is contained in the fibre over 0. On the other hand, the $\widetilde{y}$-axis, $\widetilde{\Sigma}_2$ intersects each nearby fibre X_μ at one point $\{\widetilde{y}=\mu,\ \widetilde{x}=y=z_i=0\}$. See Figure 3.3.7.

Furthermore, this is the only singular point of a nearby fibre X_μ. To see this, we need to show that a smooth point of $\widetilde{X}$ is also a smooth point of the fibre containing it.

Let p be a smooth point of $\widetilde{X}$ with $t(p)=\mu\neq 0$. For p to be a singular point of X_μ, the differential of $t|_{\widetilde{X}}$ must vanish there. That is,

$$\mathrm{d}(t|_{\widetilde{X}})(p):T_p(\widetilde{X})\to T_\mu(\mathbb{C})$$

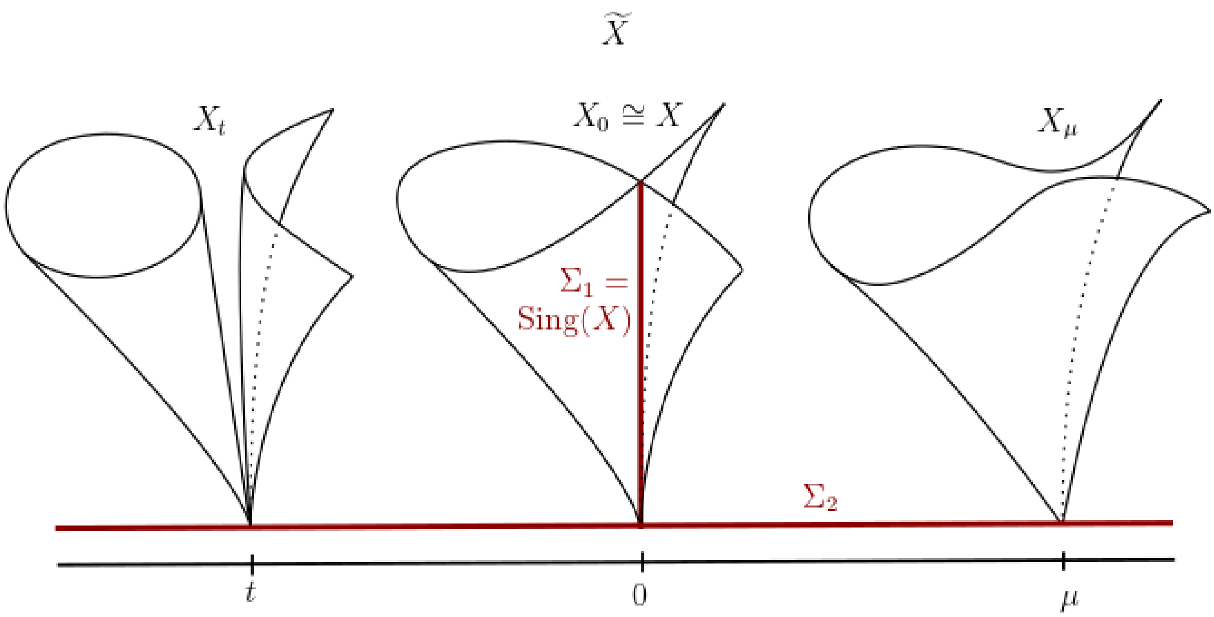


Figure 3.3.7: Singular locus of $\widetilde{X}$.

must be the zero map. This is a closed analytic criterion on p, so such points form a closed subvariety $\Delta \subset \widetilde{X}$.

The map $t|_\Delta : \Delta \to \mathbb{C}$ is a holomorphic function. The condition above means that it is locally constant.

Therefore $t(\Delta) \neq \mathbb{C}$. Then we can take a small neighbourhood $U \subset \mathbb{C}$ around 0 such that $U \cap t(\Delta) \subset \{0\}$. In this case, let us restrict our deformation to this neighbourhood U, where the proposition holds true. $\square$

Remark 3.3.15. *The nearby fibre $X_\mu \subset \mathbb{C}^{s+3}\langle \tilde{x}, y, z_1, \dots z_{s+1}\rangle$ for $\mu \neq 0$ is defined by the equations*

$$\begin{pmatrix} \tilde{x} & & y & & z_1 & & \dots & & z_s \\ & 1 & & z_1^{k_1-2} & & z_2^{k_2-2} & \dots & z_s^{k_s-2} & \\ y+\mu & & z_1^2 & & z_2 & & \dots & & z_{s+1} \end{pmatrix}. \tag{3.13}$$

It has an isolated singularity at $(0, 0, 0, 0, \dots, 0)$.

What kind of singularity is this?

Proposition 3.3.16. *The isolated singularity of a nearby fibre has the dual resolution graph*

$$\overset{-b_2}{\bullet} \!\!-\!\!-\!\!-\!\!-\!\!-\!\! \overset{-b_3}{\bullet} \!\!-\!\!-\!\!-\!\! \quad \dots \quad \!\!-\!\!-\!\!-\!\! \overset{-b_r}{\bullet} \tag{3.14}$$

Proof: We are only interested in the germ $(X_\mu, 0)$, thus we can replace the invertible bottom-left term $y+\mu$ with 1 in (3.13). The obtained set of relations is

$$\begin{pmatrix} \widetilde{x} & & y & & z_1 & & \dots & & z_s \\ & 1 & & z_1^{k_1-2} & & z_2^{k_2-2} & \dots & z_s^{k_s-2} & \\ 1 & & z_1^2 & & z_2 & & \dots & & z_{s+1} \end{pmatrix}. \tag{3.15}$$

This can be further simplified by replacing y with $\widetilde{x}z_1^2$ from its first equation:

$$\begin{pmatrix} \widetilde{x} & & \widetilde{x}z_1^2 & & z_1 & & \dots & & z_s \\ & 1 & & z_1^{k_1-2} & & z_2^{k_2-2} & \dots & z_s^{k_s-2} & \\ 1 & & z_1^2 & & z_2 & & \dots & & z_{s+1} \end{pmatrix}. \tag{3.16}$$

Note that the third column is the z_1^2-multiple of the first one, and they have a 1 between them. Hence we can delete the second and third column without losing any equations, getting

$$\begin{pmatrix} \widetilde{x} & & z_1 & & \dots & & z_s \\ & z_1^{k_1-2} & & z_2^{k_2-2} & \dots & z_s^{k_s-2} & \\ 1 & & z_2 & & \dots & & z_{s+1} \end{pmatrix}. \tag{3.17}$$

If $k_1 > 2$, we jump straight to (3.20) with $i = 1$. If $k_1 = 2$, (3.17) becomes

$$\begin{pmatrix} \widetilde{x} & & z_1 & & \dots & & z_s \\ & 1 & & z_2^{k_2-2} & \dots & z_s^{k_s-2} & \\ 1 & & z_2 & & \dots & & z_{s+1} \end{pmatrix}. \tag{3.18}$$

Using the first equation $\widetilde{x}z_2 = z_1$, we can substitute $\widetilde{x}z_2$ for z_1, which makes the third column redundant – being the z_2 multiple of the first one. After deleting it, we have

$$\begin{pmatrix} \widetilde{x} & & z_2 & & \dots & & z_s \\ & z_2^{k_2-2} & & z_3^{k_3-2} & \dots & z_s^{k_s-2} & \\ 1 & & z_3 & & \dots & & z_{s+1} \end{pmatrix}, \tag{3.19}$$

that is similar to (3.17), but with one less variables. We repeat this process until we reach a variable z_i with $k_i > 2$. Then we can move one z_i to the bottom-left obtaining

$$\begin{pmatrix} \widetilde{x} & & z_i & & \dots & & z_s \\ & z_i^{k_i-3} & & z_{i+1}^{k_{i+1}-2} & \dots & z_s^{k_s-2} & \\ z_i & & z_{i+1} & & \dots & & z_{s+1} \end{pmatrix}. \tag{3.20}$$

This set of equations belong to another cyclic quotient surface singularity. Now, we show the corresponding semigroup.

Consider the cone $\sigma' = \mathbb{R}_{\geq 0}\langle(-1,0),(p,q)\rangle \subset M_{\mathbb{R}}$ – that we obtained by replacing $g_0 = (0,1)$ with $g = (-1,0)$ as the first ray generator for σ. We want to find the generating set, described in Proposition 3.3.1, of $S_{\sigma'} = \sigma' \cap M$. Recall that $g_0 = (0,1), g_1 = (1,1), \ldots, g_{s+1} = (p,q)$ form the old set of generators of S_σ. As $g_0 = g + g_1$, we do not need g_0. If $k_1 = 2$, then $2g_1 = g_0 + g_2 = g + g_1 + g_2$, thus $g_1 = g + g_2$. In fact, $g_0, \ldots, g_{i-1}$ can all be expressed with g and g_i, hence they are not part of the generating set of $S_{\sigma'}$. The first nontrivial relation between neighbouring g_i's is $k_i g_i = g_{i-1} + g_{i+1} = g + g_i + g_{i+1}$. This implies $(k_i - 1)g_i = g + g_{i+1}$. The rest of the relations $k_j g_j = g_{j-1} + g_{j+1}$ remain unchanged for $j > i$.

Therefore, $\{g, g_i, g_{i+1}, \ldots, g_{s+1}\}$ is the desired generating set of $S_{\sigma'}$. This is also the set of primitive extremal lattice points of the convex hull of $S_{\sigma'} \setminus 0$. The corresponding continued fraction is $[k_i - 1, k_{i+1}, \ldots, k_s]$. The corresponding set of toric equations, defining $X_{\sigma'}$, is (3.20).

The dual cone is $\sigma'^{\vee} = \langle(0,1),(-q,p)\rangle \subset N_{\mathbb{R}}$, that we compare to $\sigma^{\vee} = \langle(1,0),(-q,p)\rangle$. As $(0,1) = e_1$ is the second generator of the one-parameter semigroup $\sigma^{\vee} \cap N$, our new semigroup is generated by $\{e_1, \ldots, e_{r+1}\}$. The corresponding continued fraction is $[b_1, \ldots, b_r]$, therefore we have proven the proposition. □

3.3.6 Boundary of Milnor fibre

We decompose the boundary of the singular Milnor fibre following [Sie91], where this technique was introduces for singularites defined by $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with 1-dimensional singular locus. We are inspired by [MP16] by Michel and Pichon and [NS12] by Némethi and Szilárd.

Let $X_\mu = \widetilde{X} \cap \{\widetilde{y} - y = \mu\}$ be a nearby fibre of our singularity X with some $|\mu| \ll \varepsilon$. The boundary $\partial X_\mu = X_\mu \cap S_\varepsilon$ of X_μ can be decomposed into two parts relative to the singular locus $\mathrm{Sing}(X)$ of X: the trunk and the vanishing zone:

$$\partial F = \partial_1 F \cup \partial_2 F. \tag{3.21}$$

We fix the triple $|\mu| \ll \delta \ll \varepsilon$, where δ will be the radius of the tubular neighbourhood separating the two parts.

Trunk: $\partial_1 X_\mu$

Now we treat both X and X_μ as subspaces of $\mathbb{C}^{s+3}\langle \widetilde{x}, y, z_1, \ldots, z_{s+1}\rangle$. Let us denote the link of $\mathrm{Sing}(X)$ in S^{2n+5} by $K = \mathrm{Sing}(X) \cap S^{2s+5}_\varepsilon$. The boundary of the Milnor fibre is

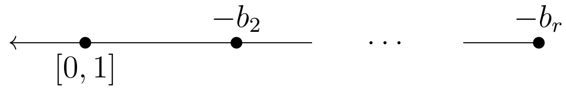
$\partial X_\mu = X_\mu \cap S_\varepsilon$.

We take a tubular δ-neighbourhood $T_\delta(K)$ of K in S_ε and we throw away the part of the boundary that is included in this: $\partial_1 X_\mu = \partial X_\mu \setminus T_\delta(K)$. We think about this as the part away from the singular locus. We claim that this is isomorphic to the link of the normalization with the neighbourhood of the double points removed from it:

$$\partial_1 X_\mu = \partial X_\mu \setminus T_\delta(K) \cong \partial X \setminus T_\delta(K) \xleftarrow[n]{\cong} \partial X_\sigma \setminus n^{-1}(T_\delta(K)), \tag{3.22}$$

where the variety X_σ is the cyclic quotient singularity defined by the cone $\sigma = \mathbb{R}_{\geq 0}\langle (0,1),(p,q)\rangle$. The link of X_σ is a plumbed 3-manifold with the bamboo (3.5) as dual graph.

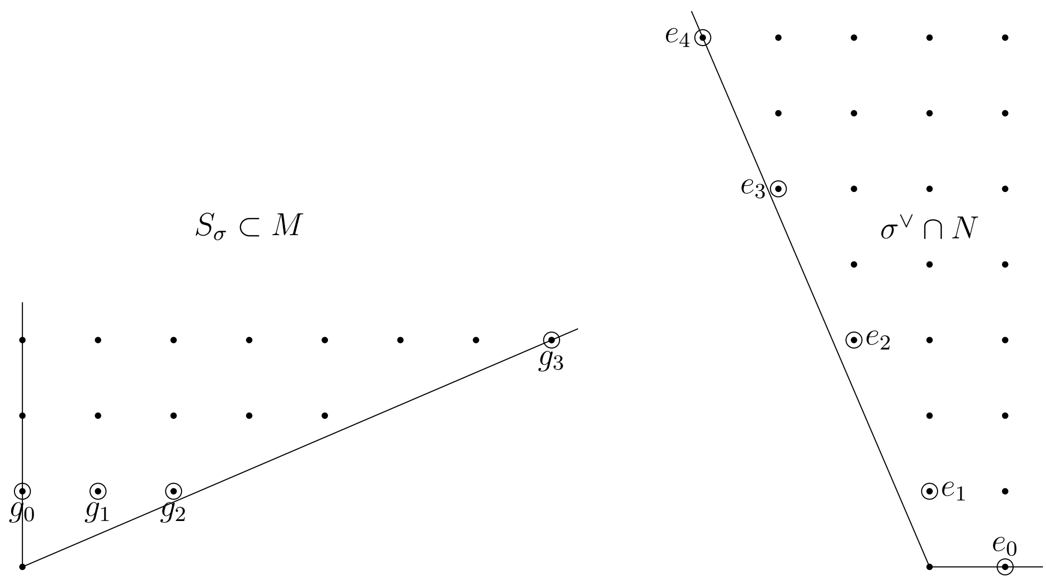
The pull-back of the tubular neighbourhood $n^{-1}(T_\delta(K)) \cap \partial X_\sigma$ that we remove from the link ∂X_σ of the normalized singularity X_σ is diffeomorphic to the tubular neighbourhood around $n^{-1}(K)$ in ∂X_σ. The pull-back of K is $n^{-1}(K) = \{z_0\text{-axis}\} \cap n^{-1}(S_\varepsilon)$, that is homeomorphic to S^1. Let us denote this S^1 with the parametrization $t \mapsto \varepsilon \cdot (t, 0, \dots, 0)$ as L_K. The trunk is a plumbed 3-manifold with boundary corresponding to the graph



(3.23)

where the label $[0,1]$ means that tha corresponding base is of genus 0 and 1 disk is removed from it.

Example 3.3.17. *As an orientation, let us remind ourselves to the generators of the dual semigroups of the running example 3.3.3.*



In order to be able to glue the trunk and the vanishing zone, we want to parametrize the boundary torus $\partial(\partial_1 X_\mu)$ by picking a longitude and a meridian.

In case of the longitude, we have to make a choice, but in our case L_1 is a natural choice. Due to Proposition 3.2.15 the fibre over the $(-b_1)$-indexed base in the graph manifold ∂X_σ can be parametrized as

$$L_1 : S^1 \to X_\sigma, \quad t \mapsto \varepsilon \cdot (t^{e_1 \cdot g_j})_j. \tag{3.24}$$

Note that $e_1 = (0,1)$ according to the normal form we chose in the beginning, thus $e_1 \cdot g_0 = 1$, $e_1 \cdot g_1 = 1$. There is a homotopy in X_σ

$$H : [0,1] \times S^1 \to X_\sigma, \quad (s,t) \mapsto \varepsilon \cdot (t, st, st^{e_1 \cdot g_2}, \dots, st^{e_1 \cdot g_{s+1}})$$

between L_K and L_1 that corresponds to moving in the base E_1 and it that can be pushed into the adequate link $X_\sigma \cap n^{-1}(S_\varepsilon)$. Therefore L_1 is indeed isotopic to a longitude of the boundary of the tubular neighbourhood $T_\delta(L_K) \subset X_\sigma$.

The meridian can be given as the boundary of a transversal section of the torus $T_\delta(L_K)$. Let us cut at $(\varepsilon, 0, 0, \dots, 0) \in L_K$ with $\{z_0 = \varepsilon\}$. The boundary of the (approximately-) δ-neighbourhood of $(1, 0, 0, \dots, 0)$ in $\{z_0 = \varepsilon\} \cap X_\sigma$ is the circle

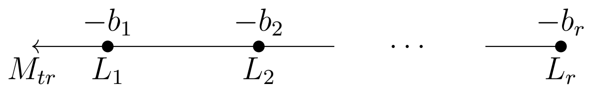
$$M_{tr} : S^1 \to X_\sigma \setminus T_\delta(L_K), \quad t \mapsto (\varepsilon, \delta t, \dots, \varepsilon^{g_i \cdot e_1}(t\delta/\varepsilon)^{g_i \cdot e_0}, \dots), \tag{3.25}$$

where the coordinate functions are computed from the first two coordinates using the toric monomial lattice structure to satisfy the equations (3.8) defining X_σ. This is, in fact homotopic to the one-parameter subgroup $M_{tr} \sim \lambda^{e_0}$ corresponding to e_0.

The relation of one-parameter subgroups $\lambda^{e_0}\lambda^{e_2} = b_1 \lambda^{e_1}$ yields an equation

$$M_{tr} + L_2 - b_1 L_1 = 0 \tag{3.26}$$

in the fundamental group of the link of X_σ. The homotopy can avoid the thrown out L_K, thus it is also an equation in the fundamental group of the trunk. Therefore $\partial_1(X_\mu)$ is the plumbed 3-manifold with boundary corresponding to the graph



(3.27)

where the arrow represents the fibre L_K whose neighbourhood we remove. We parametrized the boundary torus with the meridian-longitude pair (M_{tr}, L_1).

Vanishing zone: $\partial_2 X_\mu$

In this subsection, we are following some of the leading principles of [MP16], and [NP18] in crucial details.

The vanishing zone is the part of the deformed surface inside the tubular neighbourhood of the original singular locus.

The upshot of building the vanishing zone is the following. At a singular point of the link K we can cut the link ∂X transversally, getting a singular curve. In our case this is of equisingularity type A_1. When we deform X, we deform this transversal curve, obtaining the transversal type of the vanishing zone. Then we have to put together a fibration of these deformed curves over K with some potential vertical monodromy.

Recall, that the nearby fibre X_μ is defined by the equations (3.13):

$$\begin{pmatrix} \widetilde{x} & & y & & z_1 & & \dots & & z_s \\ & 1 & & z_1^{k_1-2} & & z_2^{k_2-2} & \dots & z_s^{k_s-2} & \\ y+\mu & & z_1^2 & & z_2 & & \dots & & z_{s+1} \end{pmatrix}. \tag{3.28}$$

When we take the link of this nearby fibre, we take the intersection $X_\mu \cap S_\varepsilon$, where $S_\varepsilon = \{|(\widetilde{x}, y, z_1, \dots, z_{s+1})| = \varepsilon\}$. Although, now we are only interested in a δ -tubular neighbourhood ($\delta \ll \varepsilon$) of the $\widetilde{x}$-axis, which is the singular locus. Thus, we can replace the condition $|(\widetilde{x}, y, z_1, \dots, z_{s+1})| = \varepsilon$ with $|\widetilde{x}| = \varepsilon$ in the tubular neighbourhood.

$$\pi_2 : \quad X_\mu \to \mathbb{C}^2, \quad (\widetilde{x}, y, z_1, \dots, z_{s+1}) \mapsto (\widetilde{x}, y).$$

Also let

$$\partial_2 X_\mu = \pi_2^{-1}(\varepsilon \cdot S^1 \times B^2(0, \delta)), \tag{3.29}$$

with $|\mu| \ll \delta \ll \varepsilon$.

To make computations less painful and actually doable, we reduce the number of coordinates we have to care about. Consider the projection

$$\pi_3 : \quad X_\mu \to \mathbb{C}^3, \quad (\widetilde{x}, y, z_1, \dots, z_{s+1}) \mapsto (\widetilde{x}, y, z_1).$$

On $\partial_2 X_\mu$, this is a real analytic isomorphism. In fact, provided that $\widetilde{X} \neq 0$, π_3 is an isomoprphism of complex spaces with the inverse – defined inductively:

$$\mathbb{C}^3 \to X_\mu, \quad z_{i+1} = \widetilde{x}^{-1} \cdot (y+\mu) \cdot z_1^{k_1-2} \cdot \dots \cdot z_i^{k_i-2} \cdot z_i. \tag{3.30}$$

This is well defined by definition. We only need to check that the target is indeed X_μ. For this we show that the coordinates we define satisfy the equations (3.28) above.

Equations involving $\widetilde{x}$ are satisfied by definition. Equations starting with y of the form $yz_{i+1} = z_1^{k_1} z_2^{k_2-2} \dots z_i^{k_i-2} z_i$ become multiples of the equation of $\widetilde{x}^{-1}(y+\mu)y = z_1^2$ after we substitute the expression in (3.30) for z_{i+1}. Finally, equations of the form $z_j z_{i+1} = z_{j+1} z_{j+1}^{k_{j+1}-2} \dots z_i^{k_i-2} z_i$ for $j < i$ become trivial after substituting for both z_{i+1} and z_{j+1}.

All in all, this means that, in computations concerning $\partial_2 X_\mu$, it suffices to calculate $\widetilde{x}, y, z_1$. Note that when we project further $\pi_{3,2} : \pi_3(\partial_2 X_\mu) \to \pi_2(\partial_2 X_\mu) = \varepsilon S^1 \times B^2(0,\delta)$, we get a double cover of 2-tori $\pi_3(\partial(\partial_2 X_\mu)) \to \varepsilon S^1 \times \delta S^1$ as $z_1^2 = \widetilde{x}^{-1}(y+\mu)y$.

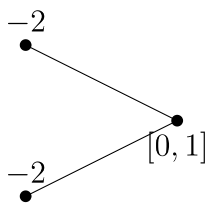
Proposition 3.3.18. *The vanishing zone $\partial_2 X_\mu$ is a Seifert manifold over a 2-disk with two special fibres of type* $(2,1)$.

Proof: Let us see what are the fibres of the projection π_2, or equivalently that of $\pi_{3,2}$ over the whole $\varepsilon S^1 \times B^2(0,\delta)$. Fix a pair of values $\widetilde{x}, y \in \mathbb{C}$ satisfying $|\widetilde{x}| = \varepsilon$ and $|y| \leq \delta$. We want to see $\pi_{3,2}^{-1}(\widetilde{x}, y)$. For $y = 0$ and $y = -\mu$, there is only one solution z_1 of $z_1^2 = \widetilde{x}^{-1}(y+\mu)y$, for any other value of y, we have two – as $\widetilde{x} \neq 0$. The rest of the variables are uniquely determined after these, and can be computed inductively:

Therefore the projection to the y-coordinate $\pi_y = \pi_{2,y} \circ \pi_2 : \partial_2 X_\mu \to S^1 \times B^2 \to B^2$ is a Seifert fibration with two special fibres over 0 and $-\mu$.

The two special fibres are of type $(2,1)$ because if we approach 0 or μ in the base $B(0,\delta)$ and follow the S^1 fibres, their limit is two times the respective special fibre. □

With the notation of [NP18] $\partial_2 X_\mu \cong Y$. This Seifert fibre space with boundary can be described as a graph manifold too, with the following graph.



(3.31)

The two special fibres are the fibres over the −2-bases on the left. They can be parametrized in the following way:

$$\begin{aligned} \Lambda_1 : \quad & S^1 \to \pi_3(\partial_2 X_\mu), \quad t \mapsto (\varepsilon t, 0, 0) \\ \Lambda_2 : \quad & S^1 \to \pi_3(\partial_2 X_\mu), \quad t \mapsto (\varepsilon t, -\mu, 0) \end{aligned} \tag{3.32}$$

Later, when we will have to identify these loops in the fundamental group. It will be useful to have homotopic curves $\Lambda_i' \sim \Lambda_i$ that are closed only after connecting them to the base point.

$$
\begin{aligned}
\Lambda'_1: \quad & [0,2\pi] \to \pi_3(\partial_2 X_\mu), \quad \tau \mapsto (\varepsilon e^{i\tau},\ \mu^2 e^{i\tau},\ \approx \mu^{3/2}\varepsilon^{-1/2}) \\
\Lambda'_2: \quad & [0,2\pi] \to \pi_3(\partial_2 X_\mu), \quad \tau \mapsto (\varepsilon e^{i\tau},\ -\mu + \mu^2 e^{i\tau},\ \approx \mu^{3/2}\varepsilon^{-1/2})
\end{aligned}
\tag{3.33}
$$

The third, z_1-coordinate of these functions are approximately constant. The approximations through the rest of the section are of magnitude $\frac{\delta\mu}{\varepsilon}$.

The boundary of the vanishing zone $\partial_2 X_\mu$ is a 2-torus, that we parametrize with two loops as in the case of the trunk. The longitude L_{VZ} shall be a generic fibre of the Seifert fibration. For the meridian M_{VZ}, we choose one component of the boundary of a transversal section of the vanishing zone.

$$
\begin{aligned}
L_{VZ}: \quad & S^1 \to \pi_3(\partial_2 X_\mu), \quad t \mapsto (\varepsilon t^2,\ \delta,\ \approx \delta\varepsilon^{-1/2} t^{-1}) \\
M_{VZ}: \quad & S^1 \to \pi_3(\partial_2 X_\mu), \quad t \mapsto (\varepsilon,\ \delta t,\ \approx \delta\varepsilon^{-1/2} t)
\end{aligned}
\tag{3.34}
$$

From the Seifert structure, we have the following equations in the fundamental group of $(\partial_2 X_\mu)$:

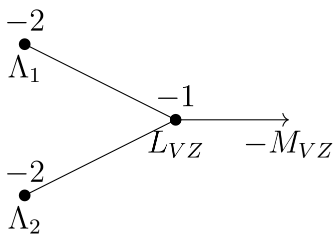
$$L_{VZ} = 2\cdot\Lambda_1, \quad L_{VZ} = 2\cdot\Lambda_2. \tag{3.35}$$

For a third equation, consider the sum $\Lambda'_1 + \Lambda'_2 - L_{VZ}$. It is homotopic to a loop that is constant in $\widetilde{x}$ and goes around the two special fibres in y. That makes it homotopic to M_{VZ}. Therefore

$$\Lambda_1 + \Lambda_2 - L_{VZ} = M_{VZ} \tag{3.36}$$

in the fundamental group.

The above equations imply that the Euler number of the middle vertex is -1 with this framing. The trunk is homeomorphic to the graph-3-manifold corresponding to the following graph – due to [Neu81].



(3.37)

The boundary torus is parametrized by L_{VZ} and M_{VZ}.

Gluing

As described in [NP18], when we glue the trunk $\partial_1 X_\mu$ and the vanishing zone $\partial_2 X_\mu$, the result is, again, a plumbed 3-manifold whose graph consists of those of the two parts connected by a vertex that replaces the two arrowheads. The boundary between the trunk and the vanishing zone is a 2-torus $(S^1)^2 \cong \partial(\partial_2 X_\mu) = \pi_2^{-1}(\varepsilon S^1 \times \delta S^1) \cong \pi_{3,2}^{-1}(\varepsilon S^1 \times \delta S^1)$, that is parametrized by a meridian-longitude basis from each side separately. We want to find the transition matrix between the two bases.

On the trunk side, the meridian, M_{tr} and the longitude, L_1 were defined in (3.24), and in (3.25) respectively, as loops in the normalization. In order to compare them with the other side, we compose them with the normalization map, obtaining

$$\begin{aligned} n \circ L_1 \circ \pi_3 : \quad & S^1 \to \partial X_\mu \setminus T_\delta(K), \quad t \mapsto (\varepsilon^2 t^2, \varepsilon^2 t^2, \varepsilon t) \\ n \circ M_{tr} \circ \pi_3 : \quad & S^1 \to \partial X_\mu \setminus T_\delta(K), \quad t \mapsto (\varepsilon^2, \varepsilon\delta t, \delta t). \end{aligned}$$

We can choose a homeomorphism $\partial_1 X_\mu = \partial X_\mu \setminus T_\delta(K) \cong \partial X \setminus T_\delta(K)$ such that they become

$$\begin{aligned} L_1' : \quad & S^1 \to \partial(\partial_1 X_\mu, \quad t \mapsto (\varepsilon t^2,\ \delta t^2,\ \approx \delta\varepsilon^{-1/2} t) \\ M_{tr}' : \quad & S^1 \to \partial(\partial_1 X_\mu), \quad t \mapsto (\varepsilon,\ \delta t,\ \approx \delta\varepsilon^{-1/2} t). \end{aligned}$$

In the vanishing zone, we have M_{VZ} and L_{VZ} defined in (3.34).

We notice that

$$M_{tr}' \equiv M_{VZ}. \tag{3.38}$$

We want to relate the longitudes, too. For that, we need

$$(2M_{VZ})(t) = (\varepsilon, \delta t^2, \approx \delta\varepsilon^{-1/2} t^2)$$

by doubling the 't-speed' of the loop. Similarly, $(L_{VZ} + 2M_{VZ})(t) = (\widetilde{x}_{VZ}, y_{VZ}, z_{1,VZ})$ is homotopic to $(\varepsilon t^2,\ \delta t^2,\ \approx \delta\varepsilon^{-1/2} t) = (\widetilde{x}_{tr}, y_{tr}, z_{1,tr}) = L_1'(t)$. To create such a homotopy, note that each coordinate function above maps S^1 to a suitably rescaled copy of $S^1 \subset \mathbb{C}$, that is the projection of the boundary torus. As the powers of t in the respective coordinate functions match, we can create such a homotopy by giving two homotopies between $\widetilde{x}_{tr}$ and $\widetilde{x}_{VZ}$, and between y_{tr} and y_{VZ} respectively, then compute the third coordinate of the homotopy from the equation of the torus, that makes the homotopy stay in the torus. Therefore, in the fundamental group of the boundary torus, we have

$$L_{VZ} + 2M_{VZ} = L_1'. \tag{3.39}$$

According to [Neu81], the above relations, (3.38) and (3.39), ensure that the two graph manifolds with boundary, (3.27) and (3.37), can be connected with a new vertex with Euler number -2. The negative edge between the new vertex and the vanishing zone is due to the minus sign in the rearranged (3.36): $-L_{VZ} + \Lambda_1 + \Lambda_2 - M_{VZ} = 0$ and (3.39): $-2M_{VZ} + L_1' - L_{VZ} = 0$. See the schematic picture of the plumbing in Figure 3.3.8.

This concludes the proof of Theorem 3.3.4.

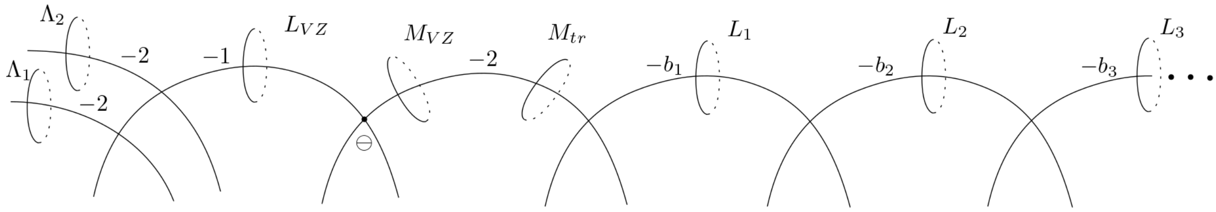
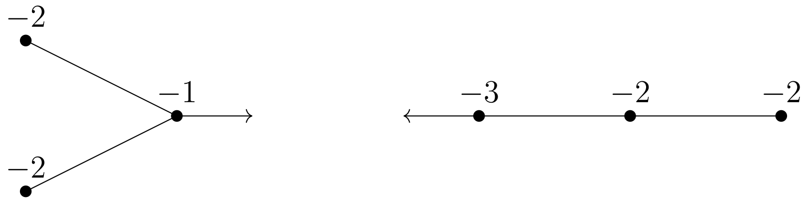
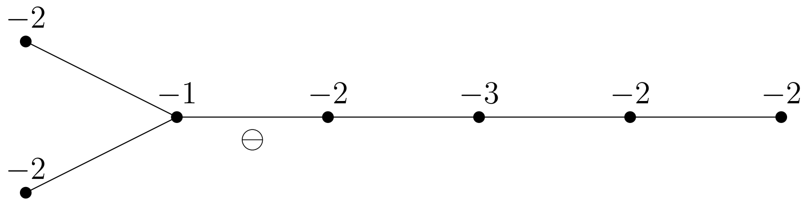


Figure 3.3.8: Graph manifold structure of ∂X_μ with generic fibres and Euler numbers of bases.

Example 3.3.19. *The trunk and the vanishing zone of the running example have the following plumbing graphs.*



We glue them together to obtain the graph of the Milnor fibre boundary.



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Symbols

List of symbols

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