Equivariant classes of coincident root strata and related invariants of varieties of tangent lines

by

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Declaration

I hereby declare

- that the dissertation contains no materials accepted for any other degrees in any other institution, and
- that the dissertation contains no materials previously written and/or published by another person except where appropriate acknowledgement is made in the form of bibliographic reference etc.

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Abstract

The thesis is centered around the investigation of two, strongly related classes of subsets. Our approach is to first investigate GL(2)-equivariant classes of coincident root strata (CRS), then use these to deduce invariants of varieties $\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n)$ consisting of certain tangent lines to a generic degree d hypersurface $Z_f \subset \mathbb{P}(\mathbb{C}^n)$.

Three chapters are dedicated to the study of three different classes of these subspaces, the fundamental class and the Chern-Schwartz-MacPherson class living in singular cohomology and the motivic Chern class living in K-theory. Each of these classes contains more information then the previous and, correspondigly, is more difficult to calculate.

The first serves to solve enumerative problems such as counting certain lines tangent to hypersurfaces. The answers to these problems generalize classical Plücker formulas counting bitangents and flexes of a degree d generic plane curve. The second can be further used to e.g. calculate the Euler characteristics of the varieties $\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n)$. From the third, for instance, we can infer the χ_y-genus of $\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n)$.

We give new recursive methods to calculate the above equivariant classes. These algorithms also help us to investigate a key feature of these classes: Using them we managed to prove that the d-dependence of the fundamental and the Chern-Schwartz-MacPherson class of CRS is polynomial, just like classical Plücker formulas are polynomials in d. These algorithms are also easy to implement and quite fast. Based on the vast amount of examples they provide we conjecture the polynomiality of the motivic Chern class of CRS.

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Contents

CHAPTER 1

Introduction

1.1. Coincident root strata

The vector space

$$\mathrm{Pol}^d(\mathbb{C}^2) := \{\text{homogeneous polynomials of degree } d \text{ in two variables}\}$$

admits a stratification into the so-called coincident root strata (CRS).

DEFINITION 1.1.1. Let $\lambda = (2^{e_2}, \dots, m^{e_m})$ be a partition without 1's and $d \geq |\lambda| = \sum_{i=1}^k \lambda_i$. Then the *coincident root stratum* of λ is

$$Y_\lambda(d) := \left\{ f \in \mathrm{Pol}^d(\mathbb{C}^2) : f = \prod_{i=1}^k \left(f_i^{\lambda_i}\right) \prod_{j=|\lambda|+1}^d (f_j) \right\},$$

where $0 \neq f_i, f_j : \mathbb{C}^2 \to \mathbb{C}$ are linear and no two of them are scalar multiples of each other.

Here we slightly changed the usual notation since we are interested in the d-dependence of these strata. If d is clear from the context we will also use the shorthand notation Y_λ. Note that the above definition includes for all d's the stratum $Y_\emptyset(d)$ corresponding to the empty partition $\emptyset$.

Their closure $\overline{Y}_\lambda(d)$, i.e. where we don't require that the f_i's and f_j's are different are algebraically closed subsets of $\mathrm{Pol}^d(\mathbb{C}^2)$. This means that the coincident root strata are smooth, locally closed subsets.

Throughout this thesis we refer to locally closed algebraic sets over the complex numbers simply as “varieties”. We use the word ”closed varieties“ for algebraically closed subsets.

The varieties $\{Y_\lambda(d)\}_{\{\lambda:|\lambda|\leq d\}}$ together with $\{0\}$ gives a stratification of $\mathrm{Pol}^d(\mathbb{C}^2)$. A key property of these strata is that they are invariant for the GL(2)-action on $\mathrm{Pol}^d(\mathbb{C}^2) \cong \mathrm{Sym}^d\left(\mathbb{C}^{2\vee}\right)$ coming from the standard representation of GL(2) on $\mathbb{C}^2$. It is intuitively clear that the codimension of $Y_\lambda(d)$ in $\mathrm{Pol}^d(\mathbb{C}^2)$ is $\sum_{i=1}^k(\lambda_i - 1) = \sum_{j=2}^m (j-1)e_j$ since every increase of the multiplicity of a root by one increases the codimension by one. For details, see e.g. [**FNR06**]. For this reason we introduce the partition

$$\tilde{\lambda} := (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_k - 1),$$

the *reduction of* λ. Then

$$\mathrm{codim}\left(Y_\lambda(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)\right) = |\tilde{\lambda}|.$$

The length of the partition λ we will denote by $l(\lambda)$; for the empty partition it is $l(\emptyset) = 0$.

1.2. Type λ tangent lines to hypersurfaces

Let $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ be a nonzero homogeneous polynomial of degree d in n variables. It defines a hypersurface $Z_f = (f = 0)$ in $\mathbb{P}(\mathbb{C}^n)$. Let

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k) = (2^{e_2}, \dots, m^{e_m})$$

be a partition without 1's and $d \geq |\lambda|$. A line in $\mathbb{P}(\mathbb{C}^n)$ is called a tangent line of type λ to Z_f if it has e_2 ordinary tangent points, e_3 flex points, etc. A formal definition can be given the following way. Projective lines $[V]$ in $\mathbb{P}(\mathbb{C}^n)$ correspond to affine planes V^2 in $\mathrm{Gr}_2(\mathbb{C}^n)$.

DEFINITION 1.2.1. The projective line $[V]$ is called a *tangent line of type λ to Z_f* (or *λ-line* for short) if

$$f|_V = \prod_{i=1}^{k} (f_i^{\lambda_i}) \prod_{j=|\lambda|+1}^{d} (f_j),$$

where $f_i, f_j : V \to \mathbb{C}$ are linear, and no two of them are scalar multiples of each other.

For a given polynomial $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ let us denote by

$$\mathcal{T}_\lambda Z_f := \{\text{tangent lines of type } \lambda \text{ to } Z_f\} \subset \mathrm{Gr}_2(\mathbb{C}^n),$$

the *variety of tangent lines of type λ to Z_f*.

Note here that although λ-lines are well-defined for the partition $\lambda = \emptyset$, those are not, in the usual sense, tangent to Z_f. Hopefully, this will not cause any confusion. Also, we will not examine $\mathcal{T}_\lambda Z_f$ for $\lambda = \emptyset$.

The variety $\mathcal{T}_\lambda Z_f$ can be identified as a coincident root locus of a certain section of a vector bundle:

Let $S \to \mathrm{Gr}_2(\mathbb{C}^n)$ denote the tautological rank two vector bundle over the Grassmannian. Consider the vector bundle $\mathrm{Pol}^d(S) \to \mathrm{Gr}_2(\mathbb{C}^n)$ with fiber $\mathrm{Pol}^d(V)$ above each $V \in \mathrm{Gr}_2(\mathbb{C}^n)$. To be more precise, $\mathrm{Pol}^d(S)$ can be defined as $P \times_{\mathrm{GL}(2)} \mathrm{Pol}^d(\mathbb{C}^2)$, a bundle associated to the principal $\mathrm{GL}(2)$-bundle $P \to \mathrm{Gr}_2(\mathbb{C}^n)$ the frame bundle of S via the $\mathrm{GL}(2)$-representation $\mathrm{Pol}^d(\mathbb{C}^2)$ as above. The $Y_\lambda(d)$-points of $\mathrm{Pol}^d(S)$ form a subbundle over $\mathrm{Gr}_2(\mathbb{C}^n)$,

$$Y_\lambda(d)\left(\mathrm{Pol}^d(S)\right) := P \times_{\mathrm{GL}(2)} Y_\lambda(d) \subset \mathrm{Pol}^d(S) = P \times_{\mathrm{GL}(2)} \mathrm{Pol}^d(\mathbb{C}^2).$$

We will also use the shorthand notation $Y_\lambda(\mathrm{Pol}^d(S))$ for $Y_\lambda(d)(\mathrm{Pol}^d(S))$.

Note that the exact same construction can and will later be applied to other rank two complex vector bundles $D \to B$ resulting in bundles $Y_\lambda(\mathrm{Pol}^d(D)) \subset \mathrm{Pol}^d(D)$.

Given a nonzero homogeneous polynomial $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ we can define a section $\sigma_f(V) := f|_V$ of the vector bundle $\mathrm{Pol}^d(S) \to \mathrm{Gr}_2(\mathbb{C}^n)$. Then, by definition,

$$\mathcal{T}_\lambda Z_f = \sigma_f^{-1}\big(Y_\lambda(d)(\mathrm{Pol}^d(S))\big).$$

This, among many things, implies that for a *generic* polynomial $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ the variety of λ-lines $\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n)$ is also $|\tilde{\lambda}|$-codimensional. The dimension of the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$—the space of projective lines in $\mathbb{P}(\mathbb{C}^n)$—is $2(n-2)$. If, for example, $|\tilde{\lambda}| = 2(n-2)$, then $\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n)$ is finite for a generic f, and we can ask its cardinality. This is the simplest enumerative problem we will solve.

Being generic is a premise crucial for our approach to work. In general, a property is called generic, if elements satisfying it form a nonempty open subset of the parameter space, here $\mathrm{Pol}^d(\mathbb{C}^2)$.

The reason for identifying $\mathcal{T}_\lambda Z_f$ as a coincident root loci is that $\mathrm{GL}(2)$-equivariant classes of $Y_\lambda(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)$ are universal. Indeed, this is a key aspect of equivariant cohomology (and K-theory): From a G-equivariant class of an invariant subset $X \subset V$ of a G-representation V, it is sometimes possible to deduce non-equivariant classes of X-loci of vector bundles. Details of this universal property will be provided for each of the three classes we investigate in the following chapters.

CHAPTER 2

Equivariant cohomology classes of coincident root strata and generalized Plücker formulas

2.1. Introducing generalized Plücker formulas

In this chapter we give a new method to calculate the GL(2)-equivariant cohomology classes of coincident root strata. We show a polynomial behavior of them, and apply this result to prove that generalized Plücker formulas are polynomials in the degree, just as classical Plücker formulas counting bitangents and flexes to a degree d generic plane curve. We calculate the leading term of these polynomials to determine the asymptotic behaviour of the Plücker formulas. We also explain how the equivariant method can be "translated" into the traditional non-equivariant method of resolutions.

Let us start by defining enumerative problems, we call the *Plücker formulas*, that can be deduced form GL(2)-equivariant cohomology classes of CRS. In fact, as we will see, Plücker formulas and these cohomology classes are equivalent in some sense.

2.1.1. Definition of Plücker numbers. We have already seen that for a generic $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ if $2(n-2) = |\tilde{\lambda}|$ holds for the partition $\lambda = (2^{e_2}, \dots, m^{e_m})$, then the variety $\mathcal{T}_\lambda Z_f$ is finite. This motivates the following.

Definition 2.1.1. Let $\lambda = (2^{e_2}, \dots, m^{e_m})$ be a partition without 1's such that $2(n_0-2) = |\tilde{\lambda}|$ for some n_0. Then the *Plücker numbers* $\mathrm{Pl}_\lambda(d)$ for $d \geq |\lambda|$ are defined as the number of type λ tangent lines to a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^{n_0})$.

Example 2.1.2. The Plücker numbers

$$\mathrm{Pl}_{2,2}(d) = \frac{1}{2}d(d-2)(d-3)(d+3), \quad \mathrm{Pl}_3(d) = 3d(d-2),$$

the number of bitangent lines and flex lines to a generic degre d plane curve were calculated by Plücker in the 1830's. His formulas also include the cases of singular curves, but we only study the generic case.

Note that, for typographical reasons, we omit the brackets from the indices.

If the dimension of $\mathcal{T}_\lambda Z_f$ is positive, we can obtain further numbers by adding linear conditions:

Definition 2.1.3. Let $\lambda = (2^{e_2}, \cdots, m^{e_m})$ be a partition without 1's. Choose n_0 and $0 \leq i \leq |\tilde{\lambda}|$ such that $|\tilde{\lambda}| + i = 2(n_0 - 2)$. We define the *Plücker number* $\mathrm{Pl}_{\lambda;i}(d)$ for $d \geq |\lambda|$ as the number of λ-lines of a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^{n_0})$ intersecting a generic $(i+1)$-codimensional projective subspace.

For $\mathrm{Pl}_{\lambda;0}(d)$ we recover the previous definition: $\mathrm{Pl}_{\lambda;0}(d) = \mathrm{Pl}_\lambda(d)$. We will use both notations.

Example 2.1.4. For tangent lines we have

$$\mathrm{Pl}_{2;1}(d) = d(d-1),$$

the number of lines in $\mathbb{P}(\mathbb{C}^3)$ through a given point and tangent to a generic degree d curve. In other words, the degree of the dual curve is $d(d-1)$.

EXAMPLE 2.1.5. For bitangent lines we also have

$$\mathrm{Pl}_{2,2;2}(d) = \frac{1}{2} d\,(d-1)\,(d-2)\,(d-3)\,,$$

the number of bitangent lines to a generic degree d surface in $\mathbb{P}(\mathbb{C}^4)$ going through a point.

EXAMPLE 2.1.6. For $\lambda = (4)$ (the 4-flexes) we also have two Plücker numbers:

$$\mathrm{Pl}_{4;1}(d) = 2d(3d-2)(d-3), \quad \mathrm{Pl}_{4;3}(d) = d(d-1)(d-2)(d-3),$$

where $\mathrm{Pl}_{4;1}(d)$ is the number of 4-flex lines to a generic degree d surface in $\mathbb{P}(\mathbb{C}^4)$ intersecting a line, and $\mathrm{Pl}_{4;3}(d)$ is the number of 4-flex lines to a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^5)$ going through a point.

REMARK 2.1.7. Notice that n_0 doesn't appear in our notation: $\mathrm{Pl}_{\lambda;i}(d)$ is defined as a number of certain λ-lines in $\mathbb{P}(\mathbb{C}^{n_0})$ for a specific n_0 that is determined by λ and i via $|\tilde{\lambda}| + i = 2(n_0 - 2)$. This also shows that the parity of admissible i's is fixed: $i = |\tilde{\lambda}|, |\tilde{\lambda}| - 2, \ldots$. In particular, $\mathrm{Pl}_{\lambda;0}(d) = \mathrm{Pl}_{\lambda}(d)$ is defined for a partition λ only if $|\tilde{\lambda}|$ is even.

Moreover, $\mathrm{Pl}_{\lambda;i}(d)$ solves a family of enumerative problems: Choose an $n \geq n_0$. Elementary geometric considerations imply that $\mathrm{Pl}_{\lambda;i}(d)$ is the number of λ-lines of a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^n)$ intersecting a generic $(n - n_0 + i + 1)$-codimensional projective subspace A and contained in a generic $(n_0 - 1)$-dimensional projective subspace B such that $A \subset B$.

2.1.2. Calculating Plücker numbers from classes of varieties of λ-lines. The key observation is that for a given λ, all the Plücker numbers $\mathrm{Pl}_{\lambda;i}(d)$ are encoded in the cohomology class

$$\big[\,\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\big] \in H^*\big(\,\mathrm{Gr}_2(\mathbb{C}^n)\big)$$

for any $n \geq |\tilde{\lambda}| + 2$ and $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ generic.

Let us add here that throughout this thesis cohomology will be understood with $\mathbb{Z}$ coefficients.

Let $s_{k,l}$ for $l \leq k \leq n-2$ denote the Schur polynomials, $s_1 = c_1$, $s_2 = c_1^2 - c_2$, $s_{1,1} = c_2$, $s_{2,1} = c_1 c_2$, etc., where c_1, c_2 are the Chern classes of $S^\vee \to \mathrm{Gr}_2(\mathbb{C}^n)$, the dual of the tautological rank two bundle over the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$. Then $\{s_{k,l} : l \leq k \leq n-2\}$ is a basis of $H^*(\mathrm{Gr}_2(\mathbb{C}^n))$ with dual basis $\{s_{n-2-l,n-2-k} : l \leq k \leq n-2\}$. The Schur polynomial $s_{k,l}$ is the cohomology class of the Schubert variety

$$\sigma_{k,l} = \big\{V \in \mathrm{Gr}_2(\mathbb{C}^n) : \dim(V \cap F_{n-k-1}) \geq 1 \text{ and } V \subset F_{n-l}\big\},$$

where $F_i \subset \mathbb{C}^n$ is the subspace spanned by the first i coordinate vectors. This implies that the Schur coefficients of $\big[\,\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\big]$ are solutions of enumerative problems: Standard transversality argument implies that if

$$\big[\,\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\big] = \sum a_{k,l} s_{k,l},$$

then $a_{k,l}$ is the number of λ-lines in $\sigma_{n-2-l,n-2-k}$. Setting $(k,l) = (|\tilde{\lambda}| - j, j)$, we see that being in $\sigma_{n-2-l,n-2-k}$ is equivalent to the linear conditions of Remark 2.1.7 for $i = |\tilde{\lambda}| - 2j$. This implies

PROPOSITION 2.1.8. *Let* $\lambda = (2^{e_2}, \dots, m^{e_m})$ *be a partition without* 1*'s,* $n \geq |\tilde{\lambda}| + 2$ *and* $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ *generic.*

$$\big[\,\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\big] = \sum_{j=0}^{t} \mathrm{Pl}_{\lambda;|\tilde{\lambda}|-2j}(d) s_{|\tilde{\lambda}|-j,j},$$

where $t = \lfloor |\tilde{\lambda}|/2 \rfloor$.

Consequently, for a given $\lambda = (2^{e_2}, \dots, m^{e_m})$ calculating all the Plücker numbers for λ is equivalent to calculating the cohomology class $\big[\,\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\big]$ for any $n \geq |\tilde{\lambda}| + 2$ and $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ generic.

Note that—corresponding to Remark 2.1.7—this cohomology class is stable in the sense that in the Schur basis it is independent of $n \geq |\tilde{\lambda}| + 2$. This justifies omitting n from Schur coefficients of $\big[\,\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\big]$ and writing $a_{k,l}$. Taking the formal limit $n \to \infty$, we obtain a unique polynomial in $\mathbb{Z}[c_1, c_2]$. The ring $\mathbb{Z}[c_1, c_2]$ can be identified with the GL(2)-equivariant cohomology ring of the point $H^*_{\mathrm{GL}(2)} := H^*(\mathrm{B}\,\mathrm{GL}(2))$, and this polynomial is the equivariant cohomology class $[\overline{Y}_\lambda(d)]_{\mathrm{GL}(2)}$ of the coincidence root stratum $\overline{Y}_\lambda(d)$, what we will define in Section 2.2.

This is the reason why in this chapter we first give an algorithm to calculate these equivariant cohomology classes, then we study the implications on the behaviour of the Plücker numbers.

2.1.3. A summary of results related to Plücker numbers. The key result of this chapter is that the Plücker numbers $\mathrm{Pl}_{\lambda;i}(d)$ are polynomials in d. This was our motivation for having d as a variable in our notation. We also give several structural results on these polynomials.

In Section 2.2 we explain how to calculate the Plücker numbers using the equivariant cohomology classes $\big[\overline{Y}_\lambda(d)\big]$ of the coincident root strata $Y_\lambda(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)$. This connection makes Theorem 2.2.5 the key technical result of this chapter: we give an inductive formula, where the induction is on the length of the partition λ. Several formulas were already known for the equivariant classes $\big[\overline{Y}_\lambda(d)\big]$, however those are less suited for our purposes. A detailed account of those formulas is given in Section 2.2.5.

Section 2.3 is devoted to the proof of Theorem 2.2.5. Most of the chapter is independent of this section, except parts of Sections 2.7 and 2.8.

In Section 2.4 we show—using Theorem 2.2.5—that the equivariant classes of $\overline{Y}_\lambda(d)$ are polynomials of degree $|\lambda|$ in d (Theorem 2.4.1) and, consequently, the Plücker numbers $\mathrm{Pl}_{\lambda;i}(d)$ are polynomials of degree at most $|\lambda|$ in d (Theorem 2.4.3). Furthermore, in Theorem 2.4.5 we calculate the leading term (the d-degree $|\lambda|$ part) of $\big[\overline{Y}_\lambda(d)\big]$. We also deduce a simple closed formula for a large class of Plücker numbers:

Theorem 2.4.7. *Let* $\lambda = (2^{e_2}, \cdots, m^{e_m})$ *be a partition without 1's. Then*

$$\mathrm{Pl}_{\lambda;|\tilde{\lambda}|} = \mathrm{coef}\big(s_{|\tilde{\lambda}|}, \big[\overline{Y_\lambda}(d)\big]\,\big) = \frac{1}{\prod_{i=2}^m e_i!} d(d-1)\cdots(d-|\lambda|+1),$$

in other words, for $n \geq |\tilde{\lambda}| + 2$ *we calculated the number of* λ*-lines for a generic degree* d *hypersurface in* $\mathbb{P}(\mathbb{C}^n)$ *through a generic point of* $\mathbb{P}(\mathbb{C}^n)$.

Finally, we state Theorem 2.4.8 that tells us the d-degrees of all the Plücker numbers.

In Section 2.5 we restrict our attention to $\lambda = (m)$ and give closed formulas for the Plücker numbers $\mathrm{Pl}_{m;i}(d)$. As a consequence, we get a closed formula for a classical problem:

Theorem 2.5.4. *A generic degree $d = 2n-3$ hypersurface in $\mathbb{P}(\mathbb{C}^n)$ possesses*

$$\sum_{u=1}^{n-1}(-1)^{u+n+1}\begin{bmatrix} d \\ u \end{bmatrix}\binom{d-u+1}{n-1}d^u$$

lines which intersect the hypersurface in a single point. Here $\begin{bmatrix} d \\ u \end{bmatrix}$ *denotes the Stirling number of the first kind.*

For $n=3$ it says that a generic cubic plane curve has 9 flexes. For $n=4$ we obtain the classical result that a generic quintic has 575 lines which intersect the hypersurface in a single point. In 2.5.2 we connect Plücker numbers $\mathrm{Pl}_{d;i}(d)$ to enumerative problems regarding the number of lines on degree d hypersurfaces. In particular, we give a new proof of Don Zagier's formula ([**GM08**]) on the number of lines on a degree $d = 2n-3$ hypersurface in $\mathbb{P}(\mathbb{C}^{n+1})$. This connection also implies that

Theorem 2.5.8. *The number of lines on a generic degree $d = 2n-3$ hypersurface in $\mathbb{P}(\mathbb{C}^{n+1})$ is d times the number of hyperflexes to a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^n)$.*

We expected this to be a classical result, but found no mention of it in the literature.

In Section 2.6 we calculate the coefficient of $d^{|\lambda|}$ in $\mathrm{Pl}_{\lambda;i}(d)$ by relating it to certain Kostka— and for special λ's to Catalan and Riordan—numbers. This coefficient informs us about the asymptotic behaviour of the Plücker number $\mathrm{Pl}_{\lambda;i}(d)$ as d tends to infinity, so we will call it the *asymptotic Plücker number* $\mathrm{aPl}_{\lambda;i}$. The main theorem of the section is

Theorem 2.6.1. *Let $\lambda = (2^{e_2}, \dots, m^{e_m})$ be a partition without 1's and $j \le \lfloor |\tilde{\lambda}|/2 \rfloor$ a nonnegative integer. Let $n = |\tilde{\lambda}| - j + 2$. Then*

$$\mathrm{aPl}_{\lambda;|\tilde{\lambda}|-2j} = \frac{K_{(n-2,j),\tilde{\lambda}}}{\prod_{i=2}^m e_i!},$$

where $K_{\mu,\nu}$ denote Kostka numbers.

In Section 2.7, using the example of m-flex lines, we compare our method with the classical non-equivariant approach. We introduce the notion of incidence varieties and we use them to formulate a non-equivariant version of Theorem 2.2.5. We try to convince the readers who are not familiar with the equivariant method, that it is a useful language which can be translated to classical terms.

In Section 2.8 we study variants of the Plücker numbers. In 2.8.1 we show how a substitution into the cohomology class $\left[\overline{Y}_\lambda(d)\right]$ calculates Plücker numbers for linear systems of hypersurfaces. A small example is the number of flex lines through a point to a pencil of degree d curves. In 2.8.2 we study the variety of m-flex points of λ-lines. To demonstrate the versatility of the method we give the details for computing the degree of the curve of flex points of the $(3,2)$-lines to a surface. In 2.8.3 we show that the previous two constructions can be combined without difficulty. As an example we calculate the degree of the curve of tangent points of bitangent lines to a pencil of degree d plane curves.

2.2. Cohomology classes of coincident root strata and coincident root loci

2.2.1. Equivariant cohomology classes of invariant subvarieties. Suppose that the algebraic Lie group G acts on an algebraic manifold M and $Y \subset M$ is a k-codimensional closed G-invariant subvariety. Then we can define the *G-equivariant cohomology class of Y*:

$$[Y \subset M]_G \in H_G^{2k}(M).$$

This class was defined by several people independently and by quite different methods. Our approach is the closest to [**Tot99**]. In this thesis G is always the product of general linear groups, in which case the construction of the G-equivariant cohomology class is simpler:

Suppose that $G = \mathrm{GL}(r)$. We define an *approximation of the universal bundle* $\mathrm{E}\,\mathrm{GL}(r) \to \mathrm{B}\,\mathrm{GL}(r)$ as $P \to \mathrm{Gr}_r(\mathbb{C}^N)$, where P is the frame bundle of the tautological bundle of the Grassmannian $\mathrm{Gr}_r(\mathbb{C}^N)$. Then $B := P \times_{\mathrm{GL}(r)} M$ approximates the Borel construction $\mathrm{B}_{\mathrm{GL}(r)} M$ in the sense that the map $\beta : B \to \mathrm{B}_{\mathrm{GL}(r)} M$—induced by the classifying map of P—induces an isomorphism $\beta^* : H^{2k}_{\mathrm{GL}(r)}(M) = H^{2k}(\mathrm{B}_{\mathrm{GL}(r)} M) \to H^{2k}(B)$ if N is bigger than k. Therefore we can define

$$[Y \subset M]_{\mathrm{GL}(r)} := (\beta^*)^{-1}[P \times_{\mathrm{GL}(r)} Y \subset P \times_{\mathrm{GL}(r)} M], \tag{1}$$

and it is not difficult to see that this definition is independent of the choice of $N > k$. For products of $\mathrm{GL}(r_i)$'s we can use the products of the approximations.

If it is clear from the context, we drop the group G or the ambient space from our notation and write $[Y]$ or $[Y]_G$ for $[Y \subset M]_G$. Similarly, we sometimes drop the group from equivariant characteristic classes of G-bundles $E \to B$, and write $c_i(E)$ for $c_i^G(E)$ and $e(E)$ for $e_G(E)$.

2.2.2. Universal property of equivariant cohomology classes. Most of the time our ambient manifold will be a complex vector space V with a linear G-action and a G-invariant closed (affine) subvariety. In such cases, as $H^*_G(V) \cong H^*_G$ canonically, $[Y \subset V]_G$ can be considered as a G-characteristic class.

Any principal G-bundle $P \to B$ over the algebraic manifold B is a pullback of $\mathrm{E}\,G \to \mathrm{B}\,G$ via a classifying map $\kappa : B \to \mathrm{B}\,G$. Consider the associated bundle $E = P \times_G V$ and its subbundle

$$Y(E) = P \times_G Y,$$

and denote by $\hat{\kappa}_V : E \to \mathrm{E}\,G \times_G V$ the lift of the classifying map κ to E. Then

$$[Y(E) \subset E] = \hat{\kappa}_V^* \, [Y \subset V]_G \, .$$

If, moreover, we have a section $\sigma : B \to E$ that is transversal to $Y(E)$, then we can take this correspondence further to get

$$\left[\sigma^{-1}(Y(E)) \subset B\right] = \kappa^* \, [Y \subset V]_G = [Y \subset V]_G \, (P),$$

the G-characteristic class evaluated at the bundle $P \to B$.

Note that complex closed subvarieties are naturally stratified submanifolds, a stratification of $Y \subset V$ induces a stratification of $Y(E) \subset E$, and that here we call a map to a stratified submanifold *transversal* if it is transversal to all the strata.

2.2.3. Cohomology classes of coincident root loci. As we have already described in Chapter 1, varieties of λ-lines can be identified as concident root loci $\mathcal{T}_\lambda Z_f = \sigma_f^{-1}\left(Y_\lambda\left(\mathrm{Pol}^d(\mathbb{C}^2)\right)\right)$. This will imply that for a generic polynomial $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ the cohomology class of its closure can be obtained from $\left[\overline{Y_\lambda}(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)\right]_{\mathrm{GL}(2)}$ by evaluating it at the frame bundle of the tautological bundle $S \to \mathrm{Gr}_2(\mathbb{C}^n)$.

Throughout this chapter, illustrating its effectiveness, we will use the above claim to calculate from $\left[\overline{Y_\lambda}(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)\right]_{\mathrm{GL}(2)}$ the class $\left[\overline{Y}_\lambda(\mathrm{Pol}^d(D)) \subset \mathrm{Pol}^d(D)\right]$ for other rank two vector bundles D as well.

In Section 2.1.2, we hinted that $H^*_{\mathrm{GL}(2)}$ can be identified with the polynomial ring $\mathbb{Z}\,[c_1, c_2]$. Evaulating $\left[\overline{Y_\lambda}(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)\right]_{\mathrm{GL}(2)}$ at the frame bundle of D amounts to substituting $c_i\,(D^\vee)$

into these variables c_i, thus

$$\left[\overline{Y}_\lambda(\mathrm{Pol}^d(D)) \subset \mathrm{Pol}^d(D)\right] = \left[\overline{Y}_\lambda(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)\right]_{\mathrm{GL}(2)} |_{c_i \mapsto c_i(D^\vee)}.$$

For more details on the choice and interpretation of the generators c_1, c_2 see Section 2.3.2.

For a generic homogeneous polynomial $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ the section σ_f is transversal to $\overline{Y}_\lambda(\mathrm{Pol}^d(S))$, see Section A.1.1, implying that

$$[\sigma_f^{-1}\left(\overline{Y}_\lambda(\mathrm{Pol}^d(S))\right) \subset \mathrm{Gr}_2(\mathbb{C}^n)] = \sigma_f^* \left[\overline{Y}_\lambda(\mathrm{Pol}^d(S)) \subset \mathrm{Pol}^d(S)\right],$$

where the pullback $\sigma_f^* : H^*(\mathrm{Pol}^d(S)) \to H^*(\mathrm{Gr}_2(\mathbb{C}^n))$ is an isomorphism. This isomorphism is independent of f, so we will not denote it in our formulas.

Corollary 2.2.1. *The cohomology class $\left[\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\right]$ is obtained from the equivariant class $\left[\overline{Y}_\lambda(d)\right]_{\mathrm{GL}(2)} \in \mathbb{Z}[c_1, c_2]$ by substituting $c_i(S^\vee)$ into c_i for $i = 1, 2$.*

Remark 2.2.2. There is a subtle detail about the preimage of the closure. Our definition for transversality is that σ_f has to be transversal to all strata $Y_\mu(\mathrm{Pol}^d(S))$. The vector bundle $\mathrm{Pol}^d(S)$ admits a Whitney stratification adapted to our situation ([**GM88**]): a Whitney stratification such that every $\overline{Y}_\lambda(\mathrm{Pol}^d(S))$ is a union of strata of this stratification. Transversality with respect to this Whitney stratification implies that

$$\sigma_f^{-1}\left(\overline{Y}_\lambda(\mathrm{Pol}^d(S))\right) = \overline{\sigma_f^{-1}\left(Y_\lambda(\mathrm{Pol}^d(S))\right)} = \overline{\mathcal{T}_\lambda Z_f}.$$

However, usage of the Whitney property is not needed. Let $f : M \to N$ be an algebraic map of smooth varieties and assume that f is transversal to the closed subvariety $X \subset N$, in the sense that it is transversal to some stratification $X = \coprod X_i$ with X_0 being the open stratum. Then it is possible that $\overline{f^{-1}(X_0)}$ is strictly smaller than $f^{-1}(X)$, but the difference is a union of components of smaller dimension, so the cohomology classes $\left[\overline{f^{-1}(X_0)}\right]$ and $[f^{-1}(X)] = f^*[X]$ agree.

Now we can rephrase Proposition 2.1.8:

Proposition 2.2.3. *Let $\lambda = (2^{e_2}, \dots, m^{e_m})$ be a partition without 1's. Then*

$$\left[\overline{Y}_\lambda(d)\right] = \sum_{j=0}^{t} \mathrm{Pl}_{\lambda; |\tilde{\lambda}|-2j}(d) s_{|\tilde{\lambda}|-j, j},$$

where $t = \lfloor |\tilde{\lambda}|/2 \rfloor$.

This connection motivates our calculation of the equivariant classes $\left[\overline{Y}_\lambda(d)\right]$.

Remark 2.2.4. As we have mentioned in the Section 2.1.2, we can avoid referring to equivariant cohomology here. First, observe that for the embedding $i : \mathrm{Gr}_2(\mathbb{C}^n) \to \mathrm{Gr}_2(\mathbb{C}^{n+1})$ the equality

$$i^* \left[\left(\overline{Y}_\lambda \left(\mathrm{Pol}^d(S)\right) \subset \mathrm{Pol}^d(S)\right) \to \mathrm{Gr}_2\left(\mathbb{C}^{n+1}\right)\right] = \left[\left(\overline{Y}_\lambda \left(\mathrm{Pol}^d(S)\right) \subset \mathrm{Pol}^d(S)\right) \to \mathrm{Gr}_2\left(\mathbb{C}^{n}\right)\right]$$

holds. Also notice that $H^*(\mathrm{Gr}_2(\mathbb{C}^n)) = \mathbb{Z}[c_1, c_2]/I_n$ where the degree of the generators of I_n tends to infinity with n. This implies the existence and uniqueness of a polynomial $\left[\overline{Y}_\lambda(d)\right]_{\mathrm{GL}(2)} \in \mathbb{Z}[c_1, c_2]$ with the property above.

For a general rank two vector bundle $D \to M$ over a projective algebraic manifold we can use the fact that any such bundle can be pulled back from $S \to \mathrm{Gr}_2(\mathbb{C}^n)$ for $n >> 0$. This argument can be generalized to obtain a general definition of the G-equivariant cohomology class of a G-invariant closed subvariety of a vector space V, where G is an algebraic group acting on V (see e. g. [**Tot99**]).

2.2.4. A recursive formula for $\left[\overline{Y}_\lambda(d)\right]$. The main result of this section is Theorem 2.2.5, which gives an algorithm to calculate the universal cohomology classes $\left[\overline{Y}_\lambda(d)\right]$.

The class $\left[\overline{Y}_\lambda(d)\right] \in \mathbb{Z}[c_1, c_2]$ can be expressed in the *Chern roots* a and b: substituting $c_1 \mapsto a+b$ and $c_2 \mapsto ab$, we obtain a polynomial symmetric in the variables a and b.

THEOREM 2.2.5. *Let $\lambda = (2^{e_2}, \dots, m^{e_m})$ be a partition without 1's and $d \geq |\lambda|$. Let λ' denote the partition $(2^{e_2}, \dots, m^{e_m - 1})$, where $e_m = 1$ is allowed. We also use the notation $d' = d - m$. Then*

$$\left[\overline{Y}_\lambda(d)\right] = \frac{1}{e_m}\partial\Big(\left[\overline{Y}_{\lambda'}(d')\right]_{m/d'} \prod_{i=0}^{m-1} \big(ia + (d-i)b\big)\Big),$$

where for a polynomial $\alpha \in \mathbb{Z}[a,b]$ and $q \in \mathbb{Q}$ we use the notation

$$\alpha_q(a,b) = \alpha(a + qa, b + qa)$$

and

$$\partial(\alpha)(a,b) = \frac{\alpha(a,b) - \alpha(b,a)}{b-a}$$

denotes the divided difference operation.

The notation $d' = d - m$ will be used throughout this thesis.

REMARK 2.2.6. For any given d the class $\left[\overline{Y}_\lambda(d)\right]$ is in $\mathbb{Z}[c_1, c_2]$, which is not obvious from the recursion formula because of the divisions.

EXAMPLE 2.2.7. For $\lambda = (m)$ we recover the formula of [**FNR06**, Ex. 3.7 (4)]:

$$\left[\overline{Y}_m(d)\right] = \partial\left(\prod_{i=0}^{m-1} \big(ia + (d-i)b\big)\right). \tag{2}$$

For example,

$$\begin{aligned}\left[\overline{Y}_2(d)\right] =& \partial\Big(\big(db\big)\big(a + (d-1)b\big)\Big) = d\frac{(ab + (d-1)b^2) - (ba + (d-1)a^2)}{b-a} \\ =& d(d-1)(a+b) = d(d-1)c_1 = d(d-1)s_1.\end{aligned}$$

and

$$\begin{aligned}\left[\overline{Y}_3(d))\right] =& d(d-1)(d-2)c_1^2 - d(d-2)(d-4)c_2 \\ =& d(d-2)(d-1)s_2 + 3d(d-2)s_{1,1}.\end{aligned}$$

EXAMPLE 2.2.8. For $\lambda = (2,2)$, $m = 2$, $\lambda' = (2)$ and $d' = d - 2$. Hence we have

$$\left[\overline{Y}_2(d-2)\right]_{2/d-2} = (d-2)(d-3)\left(a + \frac{2}{d-2}a + b + \frac{2}{d-2}a\right) = (d-3)\big((d+2)a + (d-2)b\big),$$

implying that

$$\begin{aligned}\left[\overline{Y}_{2,2}(d)\right] &= \frac{1}{2}\partial\Big((d-3)\big((d+2)a + (d-2)b\big)db\big(a + (d-1)b\big)\Big) \\ &= \frac{1}{2}d(d-3)\partial\Big(b(a + (d-1)b)\big((d+2)a + (d-2)b\big)\Big) \\ &= \frac{1}{2}d(d-3)(d-2)\Big((d-1)c_1^2 + 4c_2\Big) \\ &= \frac{1}{2}d\,(d-1)\,(d-2)\,(d-3)\,s_2 + \frac{1}{2}d\,(d-2)\,(d-3)\,(d+3)\,s_{1,1},\end{aligned} \tag{3}$$

which is a calculation still manageable by hand. Notice that the result is in agreement with Examples 2.1.2 and 2.1.5.

Also notice that we obtained these results for all d's at the same time, and the polynomial dependence is also obvious. This is true for any partition λ, which will be proved in Section 2.4.

The recursion formula to calculate these polynomials is easy to implement for example in Maple, and it is fast: for $|\lambda| < 40$ the results are immediate on a PC.

2.2.5. Earlier formulas. Using Kleiman's theory of multiple point formulas ([**Kle77, Kle81, Kle82**]), Le Barz in [**LB82**] and Colley in [**Col86**] calculated examples of Plücker numbers.

Kirwan gave formulas for the SL(2)-equivariant cohomology classes of coincident root strata in [**Kir84**]. The first formula for the GL(2)-equivariant cohomology classes $\left[\overline{Y}_\lambda(d)\right]$ was given in [**FNR06**]. Notice that the SL(2)-equivariant cohomology classes are obtained from the GL(2)-equivariant ones by substituting zero into c_1, therefore they do not determine the corresponding Plücker numbers. Soon after a different formula was calculated with different methods in [**K"03**]. These formulas don't seem to be useful for proving polynomiality in d. In 2006 in his unpublished paper [**Kaz06**] Kazarian deduced a formula in a form of a generating function from his theory of multisingularities of Morin maps based on Kleiman's theory of multiple point formulas. This formula shows the polynomial dependence but further properties doesn't seem to follow easily. He also calculated several Plücker numbers $\mathrm{Pl}_\lambda(d)$. The paper [**ST22**] of Spink and Tseng also develops a method to calculate the GL(2)-equivariant cohomology classes $\left[\overline{Y}_\lambda(d)\right]$. One of their main goals is to establish relations between these classes.

2.3. Proof of the recursion formula

The proof of Theorem 2.2.5 is based on the following fundamental property of the equivariant cohomology class:

LEMMA 2.3.1. *Let $f : M \to N$ be a proper G-equivariant map of smooth varieties with $\tilde{Y} \subset M$. Suppose that $f|_{\tilde{Y}}$ is generically k-to-1 to its image $Y \subset N$. Then*

$$[Y \subset N] = \frac{1}{k} f_! [\tilde{Y} \subset M].$$

We will apply Lemma 2.3.1 to the projection $\pi : \mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2) \to \mathrm{Pol}^d(\mathbb{C}^2)$ and $Y = \overline{Y}_\lambda(d)$.

REMARK 2.3.2. To motivate the following construction of $\tilde{Y}$, let us look at a projective version: we construct an e_m-fold branched covering of $\mathbb{P}\overline{Y}_\lambda \subset \mathbb{P}\,\mathrm{Pol}^d(\mathbb{C}^2) \cong \mathbb{P}^d$. Consider the map

$$f : \mathbb{P}^1 \times \mathbb{P}^{d'} \to \mathbb{P}^d,$$

where $f = \mu \circ \left(v \times \mathrm{Id}_{\mathbb{P}^{d'}}\right)$, $v : \mathbb{P}^1 \to \mathbb{P}^m$ is the Veronese map and $\mu : \mathbb{P}^m \times \mathbb{P}^{d'} \to \mathbb{P}^d$ is the projectivization of the multiplication map $\mathrm{Pol}^m(\mathbb{C}^2) \times \mathrm{Pol}^{d'}(\mathbb{C}^2) \to \mathrm{Pol}^d(\mathbb{C}^2)$. Then it is not difficult to see that $f|_{\mathbb{P}^1 \times \mathbb{P}\overline{Y}_{\lambda'}}$ is generically e_m-to-1 to its image $\mathbb{P}\overline{Y}_\lambda(d)$.

Indeed, $\mathbb{P}^d$ can be identified with the space of unordered d-tuples of points of $\mathbb{P}^1$ with multiplicities. The map f corresponds to adding an extra point with multiplicity m to a d'-tuple of points, and a d-tuple of multiplicity λ has e_m preimages, depending on which point of multiplicity m comes from the $\mathbb{P}^1$ factor.

To obtain our $\tilde{Y}$ we need to "deprojectivize" this construction. It is possible to use this projective construction to prove the recursion formula, but the expressions for the equivariant cohomology rings and the pushforward maps are more complicated.

2.3.1. The construction of the covering space: twisting with a line bundle.

DEFINITION 2.3.3. For any representation $\rho : G \to \mathrm{GL}(V)$ of a Lie group G on a vector space V, we define its *scalar extension* $\tilde{\rho} : G \times \mathrm{GL}(1) \to \mathrm{GL}(V)$ as the tensor product of ρ and $\mathrm{Id}_{\mathrm{GL}(1)}$ with $V \otimes \mathbb{C} \cong V$ identified canonically.

Let $Y \subset V$ be a ρ-invariant subvariety, not necessarily closed. If Y is a *cone* i.e. invariant for the scalar $\mathrm{GL}(1)$-action on V, then it is also $\tilde{\rho}$-invariant.

Now, if $A = P \times_\rho V \to M$ is a vector bundle associated to P and $L \to M$ is any line bundle, then using its frame bundle $L^\times = \mathrm{Inj}(\mathbb{C}, L)$ (consisting of nonzero elements), we can obtain $A \otimes L$ as a bundle associated to the principal $G \times \mathrm{GL}(1)$ bundle $P \times_M L^\times \to M$:

$$A \otimes L = (P \times_M L^\times) \times_{\rho \otimes \mathrm{Id}_{\mathrm{GL}(1)}} (V \otimes \mathbb{C}) \cong (P \times_M L^\times) \times_{\tilde{\rho}} V.$$

This description allows us to define a subvariety of $A \otimes L$

$$\begin{aligned} Y(A \otimes L) :=& \big(P \times_M L^\times\big) \times_{\tilde{\rho}} Y \\ =& \left\{ e_m \otimes l_m : e_m \in (P \times_\rho Y)_m, l_m \in L_m \setminus \{0\}, m \in M \right\}. \end{aligned}$$

Notice that $\overline{Y}_{\lambda'}(d') \subset \mathrm{Pol}^{d'}(\mathbb{C}^2)$ is a cone, hence we can define

$$\tilde{Y} := \overline{Y}_{\lambda'}(d') \left(\mathrm{Pol}^{d'}(\mathbb{C}^2) \otimes \mathrm{Pol}^m(\mathbb{C}^2/\gamma) \right),$$

a subvariety of $\mathrm{Pol}^{d'}(\mathbb{C}^2) \otimes \mathrm{Pol}^m(\mathbb{C}^2/\gamma)$, where γ is the tautological line bundle over $\mathbb{P}(\mathbb{C}^2)$.

We have an injective map

$$j : \mathrm{Pol}^{d'}(\mathbb{C}^2) \otimes \mathrm{Pol}^m(\mathbb{C}^2/\gamma) \to \mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2),$$

induced by the multiplication of polynomials:

$$j(f \otimes g)(v) := \big(V, f(v) \cdot g(v + V)\big),$$

where $f \in \mathrm{Pol}^{d'}(\mathbb{C}^2)$ and $V < \mathbb{C}^2$ is the one-dimensional subspace such that $g \in \mathrm{Pol}^m(\mathbb{C}^2/V)$. Therefore we consider $E := \mathrm{Pol}^{d'}(\mathbb{C}^2) \otimes \mathrm{Pol}^m(\mathbb{C}^2/\gamma)$ and $\tilde{Y}$ to be subspaces of $\mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)$. The projection $\pi : \mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2) \to \mathrm{Pol}^d(\mathbb{C}^2)$ restricted to $\tilde{Y}$ is generically e_m-to-1 to its image $\overline{Y}_\lambda(d)$, implying that

$$\left[\overline{Y}_\lambda(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)\right] = \frac{1}{e_m} \pi_! \left[\tilde{Y} \subset \mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)\right]. \tag{4}$$

Notice that all the maps above are $\mathrm{GL}(2)$-equivariant, so we consider all these cohomology classes and the pushforward equivariantly.

REMARK 2.3.4. Notice that all these varieties admit compatible $\mathrm{GL}(1)$-actions induced by the scalar multiplication. Omitting the zeros sections and factoring out by this $\mathrm{GL}(1)$-action, we recover the construction of Remark 2.3.2.

An easy argument gives that

LEMMA 2.3.5. *Let $E \to M$ be a subbundle of the vector bundle $\hat{E} \to M$. Then the pushforward map induced by the inclusion $i : E \to \hat{E}$ is given by*

$$i_! z = z \cdot e(\hat{E}/E),$$

where we did not denote the isomorphisms $i^* : H^*(\hat{E}) \cong H^*(E)$ and $H^*(\hat{E}) \cong H^*(M)$. Equivariant versions of the statement also hold.

The lemma implies

$$\left[\tilde{Y} \subset \mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)\right] = e\left(\left(\mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)\right)/E\right) \cdot [\tilde{Y} \subset E]. \tag{5}$$

The key step in the proof of Theorem 2.2.5 is the calculation of $[\tilde{Y} \subset E]$, which we will do in the next sections.

2.3.2. Conventions. To be able to make these calculations we need to fix generators of the cohomology rings involved. Most calculations of the thesis happen in $H^*(\mathrm{B}\,\mathrm{GL}(2))$, the $\mathrm{GL}(2)$-equivariant cohomology ring of the point.

Our goal is to obtain "positive" expressions, so we choose $c_i = c_i(S^\vee)$ as generators of $H^*(\mathrm{B}\,\mathrm{GL}(2)) = \mathbb{Z}[c_1, c_2]$, where $S = E\,\mathrm{GL}(2) \times_{\mathrm{GL}(2)} \mathbb{C}^2$ is the tautological rank two bundle over the infinite Grassmannian $\mathrm{Gr}_2(\mathbb{C}^\infty) \simeq \mathrm{B}\,\mathrm{GL}(2)$. We will also use the "Chern roots": Let $\mathbb{T}$ denote the subgroup of diagonal matrices in $\mathrm{GL}(2)$. The complex torus $\mathbb{T}$ is isomorphic to $\mathrm{GL}(1)^2$. The inclusion $i : \mathbb{T} \to \mathrm{GL}(2)$ induces an injective homomorphism $\mathrm{B}\,i^* : H^*(\mathrm{B}\,\mathrm{GL}(2)) \to H^*(\mathrm{B}\,\mathbb{T}) \cong \mathbb{Z}[a, b]$ with image the symmetric polynomials in the variables a and b. Let $\pi_i : \mathbb{T} \to \mathrm{GL}(1)$ denote the projection to the i-th factor and $L_i := E\,\mathbb{T} \times_{\pi_i} \mathbb{C}$ denote the tautological line bundles over the factors of $\mathrm{B}\,\mathbb{T} \simeq \mathbb{P}(\mathbb{C}^\infty) \times \mathbb{P}(\mathbb{C}^\infty)$. To be consistent with our first choice we use the notation $a := c_1(L_1^\vee)$ and $b := c_1(L_2^\vee)$, so $\mathrm{B}\,i^*(c_1) = a + b$ and $\mathrm{B}\,i^*(c_2) = ab$.

For equivariant cohomology we need to specify (left) group actions on the spaces we are interested in. Our convention is that the $\mathrm{GL}(2)$-action on $\mathbb{C}^2$ is the standard action. This induces a $\mathrm{GL}(2)$-action on $\mathrm{Pol}^d(\mathbb{C}^2)$ via $(gp)(v) := p(g^{-1}v)$. We obtain $\mathbb{T}$-actions by restriction. These choices imply that the $\mathbb{T}$-equivariant Chern class of $\mathbb{C}^2$ is $c^{\mathbb{T}}(\mathbb{C}^2) = (1-a)(1-b)$, i.e. the *weights* of $\mathbb{C}^2$ are $-a$ and $-b$. Also $c^{\mathbb{T}}\big(\mathrm{Pol}^d(\mathbb{C}^2)\big) = \prod_{i=0}^d \big(1 + ia + (d-i)b\big)$, i.e. the weights of $\mathrm{Pol}^d(\mathbb{C}^2)$ are $db, a + (d-1)b, \dots, da$.

The standard action of $\mathrm{GL}(2)$ on $\mathbb{C}^2$ induces an action on $\mathbb{P}(\mathbb{C}^2)$. Its restriction to $\mathbb{T}$ has fixed points $\langle e_1 \rangle$ and $\langle e_2 \rangle$ for $\mathbb{C}^2 = \langle e_1, e_2 \rangle$. We will need the equivariant Euler classes of the tangent spaces of these fixed points:

$$e^{\mathbb{T}}\left(T_{\langle e_1 \rangle}\mathbb{P}(\mathbb{C}^2)\right) = e^{\mathbb{T}}\big(\mathrm{Hom}(\langle e_1 \rangle, \langle e_2 \rangle)\big) = (-b) - (-a) = a - b, \qquad e^{\mathbb{T}}\left(T_{\langle e_2 \rangle}\mathbb{P}(\mathbb{C}^2)\right) = b - a.$$

With these choices the formulas are nicer. We pay the price in the proof of Theorem 2.2.5, where the signs will change several times.

2.3.3. The twisted class. The results of this section are based on [**FNR05**, §6.]. A special case of the twisted class appeared earlier in [**HT84**] under the name of squaring principle.

As $\tilde{Y} = \overline{Y}_{\lambda'}(d')(\mathrm{Pol}^{d'}(\mathbb{C}^2) \otimes \mathrm{Pol}^m(\mathbb{C}^2/\gamma))$ can be defined as a bundle associated to a principal $\mathrm{GL}(2) \times \mathrm{GL}(1)$-bundle using the $\tilde{\rho}$-action on $\overline{Y}_{\lambda'}(d')$, we can—using the universal property of the equivariant class—compute its cohomology class $[\tilde{Y} \subset E]$ from $[\overline{Y}_{\lambda'}(d')]_{\mathrm{GL}(2)\times\mathrm{GL}(1)}$.

For our representation $\rho : \mathrm{GL}(2) \to \mathrm{GL}\left(\mathrm{Pol}^{d'}(\mathbb{C}^2)\right)$ and invariant subvariety $Y_{\lambda'}(d') \subset \mathrm{Pol}^{d'}(\mathbb{C}^2)$ it is possible to calculate $[\overline{Y}_{\lambda'}(d')]_{\mathrm{GL}(2)\times\mathrm{GL}(1)}$ from $[\overline{Y}_{\lambda'}(d')]_{\mathrm{GL}(2)}$.

More generally, let us say that a representation $\rho : G \to \mathrm{GL}(V)$ of a complex reductive group G *contains the scalars* if there is a homomorphism $\varphi : \mathrm{GL}(1) \to G$ and a positive integer d such that

$$\rho\left(\varphi(s)\right) = s^d\,\mathrm{Id}_V\,. \tag{6}$$

For such φ, G has a maximal complex torus $\mathbb{T}^r \subset G$ with $\mathrm{Im}(\varphi) \subset \mathbb{T}^r$ and $\varphi : \mathrm{GL}(1) \to \mathbb{T}^r \cong \mathrm{GL}(1)^r$ can be written as $\varphi(s) = (s^{w_1}, \dots, s^{w_r})$ for some integers w_i. For our representation $\mathrm{Pol}^{d'}(\mathbb{C}^2)$, we can choose $w_1 = w_2 = -1$ and $d = d'$.

In this thesis we are only concerned with group actions of the general linear group $G = \mathrm{GL}(r)$, in which case we can—without limiting generality—restrict those actions to a maximal complex torus $i : \mathbb{T}^r \cong \mathrm{GL}(1)^r \hookrightarrow G$. We choose $a_i := c_1\big(E\mathbb{T}^r \times_{\pi_i} \mathbb{C}\big)$ as generators of $H^*(\mathrm{B}\,\mathbb{T}^r)$, where the homomorphism $\pi_i : \mathbb{T}^r \to \mathrm{GL}(1)$ is the projection to the i-th factor. This is the most common choice for generators. Compared with the conventions of Section 2.3.2, we have $a = -a_1$ and $b = -a_2$.

By the splitting principle the induced map $i^* : H^*(\mathrm{B}\,\mathrm{GL}(r);\mathbb{Z}) \to H^*(\mathrm{B}\,\mathbb{T}^r;\mathbb{Z})$ is an isomorphism onto its image $H^*(B\mathbb{T}^r;\mathbb{Z})^{S_r} \cong \mathbb{Z}[a_1,\dots,a_r]^{S_r}$ such that $i^*[Y]_G = [Y]_{\mathbb{T}}$ for any G-invariant subset $Y \subset V$.

For a general connected Lie group G—by the Borel theorem—the analogous isomorphism holds with rational coefficients onto $H^*(BT,\mathbb{Q})^{\mathcal{W}}$ where T is a real maximal torus of G and $\mathcal{W}$ is the Weyl group of G. This implies that the results of this section can be easily generalized to connected Lie groups.

For the following discussion it will be convenient to keep track of not only the groups acting but the actions themselves. For this reason, the G-equivariant class of a closed subvariety $Y \subset V$ invariant under the G-action $\rho : G \to \mathrm{GL}(V)$ will be denoted by $[Y]_\rho$.

Proposition 2.3.6. *Suppose that the representation $\rho : \mathrm{GL}(r) \to \mathrm{GL}(V)$ contains the scalars as above. If $Y \subset V$ is a ρ-invariant closed subvariety, then it is also invariant for the scalar extension $\tilde\rho$ (see Definition 2.3.3), and*

$$[Y]_{\tilde\rho}(a_1,\dots,a_r,x) = [Y]_\rho(a_1 + \frac{w_1}{d}x,\dots,a_r + \frac{w_r}{d}x),$$

where $[Y]_\rho \in H^*_{\mathrm{GL}(r)} \cong \mathbb{Z}[a_1,\dots,a_r]^{S_r}$ and $H^*_{\mathrm{GL}(1)} \cong \mathbb{Z}[x]$ such that $x = c_1\big(E\,\mathrm{GL}(1) \times_{1_{\mathrm{GL}(1)}} \mathbb{C}\big)$.

Proof. We can restrict the $\mathrm{GL}(r)$-action to the maximal torus $\mathbb{T}^r$ without losing information. We use the same notation ρ for the restriction to $\mathbb{T}^r$.

Let $\sigma : \mathbb{T}^r \times \mathrm{GL}(1) \to \mathbb{T}^r$ and $\psi : \mathbb{T}^r \times \mathrm{GL}(1) \to \mathbb{T}^r \times \mathrm{GL}(1)$ denote the homomorphisms

$$\sigma(t_1,\dots,t_r,s) = \varphi(s)\cdot(t_1^d,\dots,t_r^d) = (s^{w_1}t_1^d,\dots,s^{w_r}t_r^d),$$

and

$$\psi(t_1,\dots,t_r,s) = (t_1^d,\dots,t_r^d,s^d).$$

Then (6) and the definition of $\tilde\rho$ imply that $\rho\circ\sigma = \tilde\rho\circ\psi$. Equivariant cohomology is functorial in te G variable. This means that for any ρ-invariant closed subvariety $Y \subset V$ we have $\psi^*[Y]_{\tilde\rho} = \sigma^*[Y]_\rho$.
Since $\sigma^*(a_i) = da_i + w_i x$, $\psi^*(a_i) = da_i$ and $\psi^*(x) = dx$ with a_i, x chosen as above, for $[Y]_{\tilde\rho} \in H^*_{\mathbb{T}^r\times\mathrm{GL}(1)} \cong \mathbb{Z}[a_1,\dots,a_r,x]$ we have

$$[Y]_{\tilde\rho}(da_1,\dots,da_r,dx) = [Y]_\rho(da_1 + w_1x,\dots,da_r + w_rx).$$

Since $[Y]_{\tilde\rho}$ and $[Y]_\rho$ are homogeneous polynomials of the same degree—the codimension c of $Y \subset V$—, we can divide by d^c, which implies the proposition. ☐

The universal property of the equivariant class $[Y]_{\tilde\rho}$ immediately implies

Corollary 2.3.7. *Let $A = P\times_\rho V \to M$ for some principal $\mathrm{GL}(r)$-bundle $P \to M$, suppose that the representation $\rho : \mathrm{GL}(r) \to \mathrm{GL}(V)$ contains the scalars as above and $Y \subset V$ is a ρ-invariant closed subvariety. Let $L \to M$ a line bundle. Then for the subvariety $Y(A\otimes L)$ defined in Definition 2.3.3 we have*

$$[Y(A\otimes L) \subset A\otimes L] = [Y]_\rho(\alpha_1 + \frac{w_1}{d}\xi,\dots,\alpha_r + \frac{w_r}{d}\xi),$$

where $[Y]_\rho \in H^*_{\mathrm{GL}(r)} \cong \mathbb{Z}[a_1,\dots,a_r]^{S_r}$, $\alpha_1,\dots,\alpha_r$ and ξ are the Chern roots of P and L.

PROOF. Using the splitting principle, we can replace P with a principal $\mathbb{T}^r$-bundle. Indeed, we have the splitting manifold $\hat{M} := \mathrm{Fl}(P \times_{\mathrm{GL}(r)} \mathbb{C}^r)$ with the property that the projection $p : \hat{M} \to M$ induces an injective homomorphism $p^* : H^*(M) \to H^*(\hat{M})$ and p^*P can be reduced to a principal $\mathbb{T}^r$-bundle. We can also pull back the bundles A and L to $\hat{M}$. Therefore, in the rest of the proof let $P \to M$ denote a principal $\mathbb{T}^r$-bundle.

As $Y(A\otimes L) = (P\times_M L^\times)\times_{\tilde{\rho}} Y$—by the universal property of $[Y]_{\tilde{\rho}}$ and Proposition 2.3.6—we have

$$[Y(A \otimes L) \subset A \otimes L] = \tilde{\kappa}^* \, [Y]_{\tilde{\rho}} = \tilde{\kappa}^* \left([Y]_\rho \big(a_1 + \frac{w_1}{d}x, \dots, a_r + \frac{w_r}{d}x\big) \right),$$

where $\tilde{\kappa} : M \to \mathrm{B}\,(\mathbb{T}^r \times \mathrm{GL}(1)) = \mathrm{B}\,\mathbb{T}^r \times \mathrm{B}\,\mathrm{GL}(1)$ is the classifying map of $P \times_M L^\times$. We complete the proof by noticing that $\tilde{\kappa}^* a_i = \alpha_i$ and $\tilde{\kappa}^* x = \xi$. $\square$

COROLLARY 2.3.8. *Let $\rho : \mathbb{T} \to \mathrm{GL}(V)$ be a representation of the torus $\mathbb{T} = \mathbb{T}^r$ that contains the scalars, and $Y \subset V$ be a ρ-invariant closed subvariety. Let $L' \to M'$ be a $\mathbb{T}$-line bundle over the $\mathbb{T}$-space M'. Then $V \otimes L' \to M'$ is also a $\mathbb{T}$-vector bundle via the diagonal action δ. Then for the $\mathbb{T}$-invariant subvariety $Y(V \otimes L')$ we have*

$$[Y(V \otimes L') \subset V \otimes L']_\delta = [Y]_\rho \left(a_1 + \frac{w_i}{d} c_1^{\mathbb{T}}(L'), \dots, a_r + \frac{w_r}{d} c_1^{\mathbb{T}}(L') \right),$$

where $[Y]_\rho \in H^*_{\mathbb{T}} \cong \mathbb{Z}[a_1, \dots, a_r]$ and $c_1^{\mathbb{T}}(L') \in H^*_{\mathbb{T}}(M')$ is the $\mathbb{T}$-equivariant first Chern class of L'.

PROOF. Recall from Section 2.2.1 that we can approximate $E\,\mathbb{T} \to \mathrm{B}\,\mathbb{T}$ with

$$P := (\mathbb{C}^N \setminus 0)^r \to \mathbb{P}(\mathbb{C}^N)^r,$$

and for $\beta : P \times_{\mathbb{T}} M' \to \mathrm{B}_{\mathbb{T}} M'$ we have

$$\beta^*[Y(V \otimes L') \subset V \otimes L']_\delta = [P \times_\delta \big(Y(V \otimes L')\big) \subset P \times_\delta (V \otimes L')].$$

For $M := P \times_{\mathbb{T}} M'$, $A := p^*(P \times_{\mathbb{T}} V)$ for the fibration $p : M \to \mathbb{P}(\mathbb{C}^N)^r$, and $L := P \times_{\mathbb{T}} L'$ Corollary 2.3.7 implies that

$$[P \times_\delta \big(Y(V \otimes L')\big) \subset P \times_\delta (V \otimes L')] = [Y(A \otimes L) \subset A \otimes L] = [Y]_\rho(\alpha_1 + \frac{w_1}{d}\xi, \dots, \alpha_r + \frac{w_r}{d}\xi),$$

where the α_i's are Chern roots of p^*P and $\xi = c_1(L)$. Using that $\beta^* a_i = \alpha_i$, $\beta^* c_1^{\mathbb{T}}(L') = \xi$ and that β^* is injective, we obtain the result. $\square$

Now, we are able to compute $[\tilde{Y} \subset E]$: Set $w_1 = w_2 = -1$ and $d = d'$ corresponding to our $\mathrm{GL}(2)$-representation $\mathrm{Pol}^{d'}(\mathbb{C}^2)$. In accordance with our conventions $a_1 = -a$ and $a_2 = -b$,

$$c_1(\mathrm{Pol}^m(\mathbb{C}^2/\gamma)) = -m\,(-(a+b) - c_1(\gamma)) = m\,(a+b+c_1(\gamma))\,. \tag{7}$$

Then, by Corollary 2.3.8, $[\tilde{Y} \subset E] = [\overline{Y}_{\lambda'}(d')(\mathrm{Pol}^{d'}(\mathbb{C}^2)\otimes\mathrm{Pol}^m(\mathbb{C}^2/\gamma)) \subset \mathrm{Pol}^{d'}(\mathbb{C}^2)\otimes\mathrm{Pol}^m(\mathbb{C}^2/\gamma)]$ can be obtained from $[\overline{Y}_{\lambda'}(d')]$ by substituting

$$-a \mapsto -a + \frac{-1}{d'}\big(m(a+b+c_1(\gamma))\big) \quad \text{and} \quad -b \mapsto -b + \frac{-1}{d'}\big(m(a+b+c_1(\gamma))\big),$$

or, multiplying by -1, by substituting

$$a \mapsto a + \frac{m}{d'}\big(a+b+c_1(\gamma)\big) \quad \text{and} \quad b \mapsto b + \frac{m}{d'}\big(a+b+c_1(\gamma)\big). \tag{8}$$

2.3.4. The pushforward map $\pi_!$. According to the Atiyah-Bott-Berline-Vergne (ABBV) integral formula, for a cohomology class $\alpha \in H^*_{\mathbb{T}^2}\big(\mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)\big)$ its pushforward along $\pi : \mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2) \to \mathrm{Pol}^d(\mathbb{C}^2)$ is

$$\pi_!\alpha = \int_{\mathbb{P}^1} \alpha = \frac{\alpha|_{\langle e_1\rangle}}{e(T_{\langle e_1\rangle}\mathbb{P}^1)} + \frac{\alpha|_{\langle e_2\rangle}}{e(T_{\langle e_2\rangle}\mathbb{P}^1)} = \frac{\alpha|_{\langle e_1\rangle}}{a-b} + \frac{\alpha|_{\langle e_2\rangle}}{b-a},$$

where $\langle e_1\rangle$ and $\langle e_2\rangle$ are the fixed points of the torus $\mathbb{T}^2$ of diagonal matrices acting on $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$, and $-a$, $-b$ are the weights of $\langle e_1\rangle$ and $\langle e_2\rangle$, according to the conventions of Section 2.3.2.

If $\alpha \in H^*_{\mathrm{GL}(2)}(\mathbb{P}^1)$ and $q(a,b) := \alpha|_{\langle e_2\rangle}$, then $\alpha|_{\langle e_1\rangle} = q(b,a)$, therefore

$$\int_{\mathbb{P}^1} \alpha = \partial(q). \tag{9}$$

PROOF OF THEOREM 2.2.5. By (4) and (5),

$$[\overline{Y}_\lambda(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)] = \frac{1}{e_m}\pi_!\Big[\tilde{Y} \subset \mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)\Big] = \frac{1}{e_m} e\left(\big(\mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)\big)/E\right) \cdot [\tilde{Y} \subset E],$$

where $E = \mathrm{Pol}^{d'}(\mathbb{C}^2) \otimes \mathrm{Pol}^m(\mathbb{C}^2/\gamma) \to \mathbb{P}^1$. The pushforward can be computed as above; therefore, to complete the proof we have to determine the restriction of

$$\alpha = e\left(\big(\mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)\big)/E\right) \cdot [\tilde{Y} \subset E]$$

to $\langle e_2\rangle$. Restricting to $\langle e_2\rangle$ amounts to substituting $c_1(\gamma) \mapsto -b$, hence using (7), we have

$$e(\mathrm{Pol}^d(\mathbb{C}^2)/E)|_{\langle e_2\rangle} = \left.\frac{\prod_{i=0}^{d}\big(ia + (d-i)b\big)}{\prod_{i=0}^{d'}\big(m(c_1(\gamma)+a+b) + ia + (d'-i)b\big)}\right|_{c_1(\gamma)\mapsto -b} = \prod_{i=0}^{m-1}\big(ia + (d-i)b\big),$$

and by (8),

$$[\tilde{Y} \subset E]|_{\langle e_2\rangle} = \left[\overline{Y}_{\lambda'}(d')\right]_{m/d'}, \tag{10}$$

where we used the notation of Theorem 2.2.5. $\square$

REMARK 2.3.9. Note that the left hand side of (10) can be also be interpreted as the equivariant cohomology class of the $\mathbb{T}$-invariant subvariety $x^m\overline{Y}_{\lambda'}(d')$ of $x^m\,\mathrm{Pol}^{d'}(\mathbb{C}^2)$:

$$\Big[\tilde{Y} \subset E\Big]\Big|_{\langle e_2\rangle} = \Big[x^m\,\overline{Y}_{\lambda'}(d') \subset x^m\,\mathrm{Pol}^{d'}(\mathbb{C}^2)\Big]_{\mathbb{T}},$$

where $y, x \in \mathrm{Pol}^1(\mathbb{C}^2)$ denotes the dual basis of e_1, e_2.

2.4. Polynomiality of $[\overline{Y}_\lambda(d)]$

THEOREM 2.4.1. *The classes $[\overline{Y}_\lambda(d)]$ are polynomials in d: $[\overline{Y}_\lambda(d)] \in \mathbb{Q}[c_1, c_2, d]$.*

PROOF. We use the following well-known statement:

LEMMA 2.4.2. *Suppose that $q(x)$ is a rational function, such that $q(d)$ is an integer for all $d >> 0$ integers. Then $q(x)$ is a polynomial.*

The lemma can be proved by induction on the degree of $q(x)$: Notice that $q(x+1)-q(x)$ has the same property but smaller degree than $q(x)$.

We prove the theorem by induction on the length of the partition λ using Theorem 2.2.5. Suppose that we already know that $\left[\overline{Y}_{\lambda'}(d')\right]$ is a polynomial in $d'=d-m$. Then all the coefficients (of the monomials a^ib^j) of $\left[\overline{Y}_{\lambda'}(d-m)\right]_{m/(d-m)}$ are rational functions. These coefficients are also integers for $d>>0$, since, by (10), $\left[\overline{Y}_{\lambda'}(d-m)\right]_{m/(d-m)}$ is an equivariant cohomology class of an invariant subvariety. Then Lemma 2.4.2 implies that the class $\left[\overline{Y}_{\lambda'}(d-m)\right]_{m/(d-m)}$ is a polynomial in d. The class $e(\mathrm{Pol}^d(\mathbb{C}^2)/E)|_{\langle e_2\rangle}=\prod_{i=0}^{m-1}\big(ia+(d-i)b\big)$ is clearly a polynomial in d, and the divided difference operator preserves polynomiality in d. □

Using Proposition 2.2.3, we see that Theorem 2.4.1 is equivalent to

THEOREM 2.4.3. *The Plücker numbers* $\mathrm{Pl}_{\lambda;i}(d)$ *for* $0\le i\le|\tilde\lambda|$ *and* $i\equiv|\tilde\lambda| \pmod 2$ *are polynomials in d: there is a unique polynomial* $p(d)$ *such that* $\mathrm{Pl}_{\lambda;i}(d)=p(d)$ *for* $d\ge|\lambda|$.

REMARK 2.4.4. By definition, these polynomials have integer values for $d>>0$, therefore for every integer.

2.4.1. The leading term of $\left[\overline{Y}_\lambda(d)\right]$. The asymptotic behaviour of the classes $\left[\overline{Y}_\lambda(d)\right]$ is determined by the largest d-degree parts. Using the same inductive argument, we can find out what these d-leading terms are.

THEOREM 2.4.5. *For any* $\lambda=(2^{e_2},\cdots,m^{e_m})$, *the top d-degree part of* $\left[\overline{Y}_\lambda(d)\right]$ *is*

$$\frac{1}{\prod_{i=2}^m e_i!}h_{\tilde\lambda}d^{|\lambda|},$$

where h_ν is the complete symmetric polynomial *corresponding to the partition* $\nu=(\nu_1,\dots,\nu_k)$*:* $h_\nu=\prod h_{\nu_i}$ *with h_i the i-th complete symmetric polynomial in* $\{a,b\}$.

PROOF. First, notice that $\left[\overline{Y}_\emptyset(d)\right]=1$, proving the codimension 0 case.

Now, let $\lambda=(2^{e_2},\cdots,m^{e_m})$ and consider the divided difference formula of Theorem 2.2.5. Using the notations from the previous proof, the induction hypothesis says that the d-leading term of $\left[\overline{Y}_{\lambda'}(d)\right]$ is

$$\frac{1}{\prod_{i=2}^{m-1}(e_i!)\,(e_m-1)!}h_{\tilde\lambda'}d^{|\lambda|-m}.$$

The shifted class $\left[\overline{Y}_{\lambda'}(d-m)\right]_{m/(d-m)}$ has the same d-leading term, in particular, it remains symmetric. The largest d-degree part of $e(\mathrm{Pol}^d(\mathbb{C}^2)/E)|_{\langle e_2\rangle}=\prod_{i=0}^{m-1}\big(ia+(d-i)b\big)$ is d^mb^m, hence the d-leading term of $\left[\overline{Y}_\lambda(d)\right]$ is

$$\frac{1}{e_m}\frac{1}{\prod_{i=2}^{m-1}(e_i!)\,(e_m-1)!}h_{\tilde\lambda'}d^{|\lambda|-m}\frac{d^m(b^m-a^m)}{b-a}=\frac{1}{\prod_{i=2}^m e_i!}h_{\tilde\lambda'}h_{m-1}d^{|\lambda|}=\frac{1}{\prod_{i=2}^m e_i!}h_{\tilde\lambda}d^{|\lambda|}.$$

□

This description of the leading term immediately implies that

THEOREM 2.4.6. *The polynomial* $\left[\overline{Y}_\lambda(d)\right]\in\mathbb{Q}[c_1,c_2][d]$ *has d-degree* $|\lambda|$.

2.4.2. $\mathrm{Pl}_{\lambda;|\tilde\lambda|}$ **and** d**-degrees of Plücker numbers.** The recursive argument in Theorem 2.2.5 can also be used to calculate $\mathrm{Pl}_{\lambda;|\tilde\lambda|}$ for an arbitrary partition λ:

THEOREM 2.4.7. *Let $\lambda = (2^{e_2}, \cdots, m^{e_m})$ be a partition without 1's. Then*

$$\mathrm{Pl}_{\lambda;|\tilde\lambda|} = \mathrm{coef}\big(s_{|\tilde\lambda|}, \left[\overline{Y}_\lambda(d)\right]\big) = \frac{1}{\prod_{i=2}^m e_i!} d(d-1)\cdots(d-|\lambda|+1),$$

in other words, for $n \geq |\tilde\lambda| + 2$ we calculated the number of λ-lines for a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^n)$ through a generic point of $\mathbb{P}(\mathbb{C}^n)$.

PROOF. We use induction on $|\lambda|$. Write the restriction of the twisted class as

$$\left[\overline{Y}_{\lambda'}(d')\right]_{m/d'} = \sum_{t=0}^{|\widetilde{\lambda'}|} \left(\frac{m}{d'}a\right)^t q_t(a,b,d'), \tag{11}$$

where $q_t \in \mathbb{Q}[a,b,d']$ is symmetric in a, b. Note that $q_0 = [Y_{\lambda'}(d')]$.

As $m \geq 2$

$$\prod_{i=0}^{m-1} \big(ia + (d-i)b\big) = d(d-1)\dots(d-m+1)b^m + ab\cdot p(a,b,d)$$

for some $p \in \mathbb{Z}[a,b,d]$. Then

$$\left[\overline{Y}_\lambda(d)\right] = \frac{1}{e_m}\partial\left(\left[\overline{Y}_{\lambda'}(d')\right]_{m/d'} \prod_{i=0}^{m-1}\big(ia+(d-i)b\big)\right) =$$

$$\frac{1}{e_m}\partial\left(\sum_{t=0}^{|\widetilde{\lambda'}|}\left(\left(\frac{m}{d'}a\right)^t q_t(a,b,d')\right)\big(d(d-1)\dots(d-m+1)b^m + ab\cdot p(a,b,d)\big)\right) =$$

$$\frac{1}{e_m}\partial\big(q_0(a,b,d')\cdot d(d-1)\dots(d-m+1)b^m + ab\cdot r\big) =$$

$$\frac{1}{e_m}\left[\overline{Y}_{\lambda'}(d')\right] d(d-1)\dots(d-m+1)\partial(b^m) + \frac{1}{e_m}ab\cdot\partial\big(r\big)$$

for some $r \in \mathbb{Q}[a,b,d]$. Using the induction hypothesis,

$$\left[\overline{Y}_{\lambda'}(d')\right] = \frac{1}{\prod_{i=0}^{m-1}(e_i!)(e_m-1)!}(d-m)(d-m-1)\dots(d-m-|\lambda'|+1)s_{|\widetilde{\lambda'}|} + \dots,$$

and that $s_{|\widetilde{\lambda'}|}\partial(b^m) = s_{|\widetilde{\lambda'}|}s_{m-1} = s_{|\tilde\lambda|}$ is the only Schur polynomial not divisible by ab, we get the result. □

Note in particular, that the d-degree of $\mathrm{Pl}_{\lambda,|\tilde\lambda|}$ reaches $|\lambda|$, the highest possible by Theorem 2.4.6. The idea of writing $\left[\overline{Y}_{\lambda'}(d')\right]_{m/d'}$ as in (11) can be carried further to calculate the exact d-degrees of all Plücker numbers:

THEOREM 2.4.8. *Let λ_1 be the largest number in the partition λ. Then*

$$\deg\big(\mathrm{Pl}_{\lambda;|\tilde\lambda|-2j}(d)\big) = \begin{cases} |\lambda| & \text{if } j \leq |\tilde\lambda| - \lambda_1 + 1 \\ |\lambda| - (j - |\tilde\lambda| + \lambda_1 - 1) & \text{if } j > |\tilde\lambda| - \lambda_1 + 1 \end{cases}.$$

In other words, $\mathrm{Pl}_{\lambda;|\tilde\lambda|-2j}(d)$ has degree $|\lambda|$ for $j = 0, \dots, |\tilde\lambda| - \lambda_1 + 1$, then by increasing j by one, the degree drops by one.

The proof—which is straightforward but laborious—can be found is Appendix A.2.

EXAMPLE 2.4.9. For $\lambda = (10,2,2)$ we have $|\lambda| = 14$, $|\tilde{\lambda}| = 11$, $\lambda_1 = 10$ and $|\tilde{\lambda}| - \lambda_1 + 1 = 2$, implying

$$\deg\left(\mathrm{Pl}_{10,2,2;11}(d)\right) = \deg\left(\mathrm{Pl}_{10,2,2;9}(d)\right) = \deg\left(\mathrm{Pl}_{10,2,2;7}(d)\right) = 14,$$

and

$$\deg\left(\mathrm{Pl}_{10,2,2;5}(d)\right) = 13, \quad \deg\left(\mathrm{Pl}_{10,2,2;3}(d)\right) = 12, \quad \deg\left(\mathrm{Pl}_{10,2,2;1}(d)\right) = 11.$$

If λ_1 is not much bigger than the other λ_i, exactly if $\lambda_1 \leq \lceil |\tilde{\lambda}|/2 \rceil + 1$, then all Plücker numbers have degree $|\lambda|$. We saw this in Example 2.1.2 and 2.1.5 for the bitangents: both $\mathrm{Pl}_{2,2;0}$ and $\mathrm{Pl}_{2,2;2}$ have degree $|\lambda| = 4$. A slightly bigger example is $\lambda = (4,3,2)$ where all Plücker numbers have degree $|\lambda| = 9$.

2.5. The class of m-flexes

This is a family of Plücker problems where closed formulas can be given. The formula for the equivariant classes $\left[\overline{Y}_m(d)\right]$ were already computed in **[Kir84]**, and more explicitely in [**FNR06**, Ex. 3.7 (4)]. However, deduction of the corresponding Plücker numbers given below is new.

Using (2), we can write $\left[\overline{Y}_m(d)\right] = \partial\left(\prod_{i=0}^{m-1}\left(ia + (d-i)b\right)\right)$ in Schur basis:

$$\prod_{i=0}^{m-1}\left(ia + (d-i)b\right) = \prod_{i=0}^{m-1}\left(i(a-b) + db\right) = (db)^m \prod_{i=0}^{m-1}\left(1 + i\frac{a-b}{db}\right)$$
$$= \sum_{k=0}^{m-1} \sigma_k(1,2,\ldots,m-1)(a-b)^k (db)^{m-k},$$

where σ_k denotes the k-th elementary symmetric polynomial. By definition, we have a connection with the Stirling numbers of the first kind:

$$\sigma_k(1,2,\ldots,m-1) = \begin{bmatrix} m \\ m-k \end{bmatrix}.$$

Almost by definition, we have

$$\partial\left(a^i b^{m-i}\right) = s_{m-i-1,i},$$

where we use Schur polynomials indexed by vectors of integers. Using the straightening law, we can restrict ourself to Schur polynomials indexed by partitions:

$$\partial\left(a^i b^{m-i}\right) = \begin{cases} s_{m-i-1,i} & \text{if } \ 2i < m \\ -s_{i-1,m-i} & \text{if } \ 2i > m \end{cases},$$

and $\partial\left(a^i b^i\right) = 0$.

This implies that

THEOREM 2.5.1. *For $m \geq 2i+1$ the coefficient of $d^{m-k}s_{m-i-1,i}$ in $\left[\overline{Y}_m(d)\right]$ is*

$$\operatorname{coef}\left(d^{m-k}s_{m-i-1,i}, \left[\overline{Y}_m(d)\right]\right) =$$
$$\begin{cases} (-1)^{k+i}\binom{k}{i}\begin{bmatrix} m \\ m-k \end{bmatrix} & \text{if } i \leq k < m-i \\ \left((-1)^{k+i}\binom{k}{i} - (-1)^{k+m-i}\binom{k}{m-i}\right)\begin{bmatrix} m \\ m-k \end{bmatrix} & \text{if } m-i \leq k < m. \end{cases}$$

Note that adding up the above coefficients for $i=0$,

$$\mathrm{Pl}_{m;m-1}=\mathrm{coef}(s_{m-1},\left[\overline{Y}_m(d)\right])=\sum_{k=0}^{m-1}(-1)^k\begin{bmatrix}m\\ m-k\end{bmatrix}d^{m-k}=d(d-1)\dots(d-m+1),$$

we get back Theorem 2.4.7 in the $\lambda=(m)$ special case. In other words,

PROPOSITION 2.5.2. *For a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^n)$, the number of $(n-1)$-flex lines through a generic point of $\mathbb{P}(\mathbb{C}^n)$ is $d(d-1)\cdots(d-n+2)$.*

2.5.1. Enumerative consequences. Specializing Theorem 2.5.1 to $m=2i+1$, we obtain

$$\mathrm{coef}\left(d^{m-k}s_{i,i},\left[\overline{Y}_m(d)\right]\right)=(-1)^{k+i}\binom{k+1}{i+1}\begin{bmatrix}2i+1\\ 2i+1-k\end{bmatrix}$$

for $k=i,i+1,\dots,m-1$, using the identity $\binom{k}{i}+\binom{k}{i+1}=\binom{k+1}{i+1}$. In other words,

PROPOSITION 2.5.3. *The number of $m=2n-3$-flexes to a degree d hypersurface in $\mathbb{P}(\mathbb{C}^n)$ is*

$$\mathrm{Pl}_m(d)=\sum_{u=1}^{n-1}(-1)^{u+n+1}\begin{bmatrix}m\\ u\end{bmatrix}\binom{m-u+1}{n-1}d^u.$$

In particular, for $d=m$ we obtain

THEOREM 2.5.4. *A generic degree $d=2n-3$ hypersurface in $\mathbb{P}(\mathbb{C}^n)$ possesses*

$$\sum_{u=1}^{n-1}(-1)^{u+n+1}\begin{bmatrix}d\\ u\end{bmatrix}\binom{d-u+1}{n-1}d^u$$

lines which intersect the hypersurface in a single point.

For $n=3$ it says that a generic cubic plane curve has 9 flexes. For $n=4$ we obtain the classical result that a generic quintic has 575 lines which intersect the hypersurface in a single point (see e.g. [**EH16**, Thm. 11.1]). For $n=5,6,7,8,9$ we get

99715, 33899229, 19134579541, 16213602794675, 19275975908850375.

2.5.2. Lines on a hypersurface. After discovering Theorem 2.5.4, we found a formula of Don Zagier [**GM08**, p. 26] on the classical problem of counting lines on hypersurfaces. Comparing his formula with Theorem 2.5.4 shows a surprisingly simple connection between the two problems. In this section we prove this connection directly and we also generalize it. As a byproduct, we obtain a new proof of Zagier's result.

The *Fano variety F_f of lines* on a degree d hypersurface $Z_f\subset\mathbb{P}(\mathbb{C}^{n+1})$ is the zero locus of the section $\sigma_f:\mathrm{Gr}_2(\mathbb{C}^{n+1})\to\mathrm{Pol}^d(S)$. Therefore, for a generic f we have

$$\left[F_f\subset\mathrm{Gr}_2(\mathbb{C}^{n+1})\right]=e\left(\mathrm{Pol}^d(S)\right).$$

For $d=2n-3$ we have finitely many lines, and to find their number we need to calculate the coefficient of $s_{n-1,n-1}$ in $e\left(\mathrm{Pol}^d(S)\right)=\prod_{i=0}^d\left(ia+(d-i)b\right)$.

To establish the promised connection we need the following

PROPOSITION 2.5.5. *Written in Chern roots we have*

$$e\left(\mathrm{Pol}^d(\mathbb{C}^2)\right)=dab\left[\overline{Y}_d(d)\right].$$

PROOF. By basic properties of the divided difference, we have

$$[\overline{Y}_d(d)] = \partial\left(\prod_{i=0}^{d-1}\big(ia+(d-i)b\big)\right) = \prod_{i=1}^{d-1}\big(ia+(d-i)b\big)\partial(db) = d\prod_{i=1}^{d-1}\big(ia+(d-i)b\big), \tag{12}$$

implying that

$$e\big(\mathrm{Pol}^d(\mathbb{C}^2)\big) = dab\left[\overline{Y}_d(d)\right].$$

□

In retrospect, this connection could have been known to the authors: The first appearance of the factorized form (12) may be in [**Kő3**]. It can also be obtained by the classical resolution method.

OBSERVATION 2.5.6. For Schur polynomials in variables a, b we have

$$s_{i+1,j+1} = abs_{i,j}$$

for all $i \geq j$.

This observation together with Proposition 2.5.5 immediately implies

COROLLARY 2.5.7. *Expressing $e\big(\mathrm{Pol}^d(\mathbb{C}^2)\big)$ and $\left[\overline{Y}_d(d)\right]$ in Schur basis:*

$$e\big(\mathrm{Pol}^d(\mathbb{C}^2)\big) = \sum_{j=0}^{\lfloor (d+1)/2\rfloor} u_j s_{d+1-j,j} \quad \textit{and} \quad \left[\overline{Y}_d(d)\right] = \sum_{j=0}^{\lfloor (d-1)/2\rfloor} v_j s_{d-1-j,j}$$

we have the identities $u_{j+1} = dv_j$ for $j = 0, \ldots, \lfloor (d-1)/2\rfloor$ and $u_0 = 0$.

If $d = 2n-3$, the case $u_{n-1} = dv_{n-2}$ implies

THEOREM 2.5.8. *The number of lines on a generic degree $d = 2n-3$ hypersurface in $\mathbb{P}(\mathbb{C}^{n+1})$ is d times the number of hyperflexes to a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^n)$.*

REMARK 2.5.9. If would be nice to have a geometric explanation of this connection. Igor Dolgachev recommended to use cyclic coverings: let $f(x_1, \ldots, x_n) \in \mathrm{Pol}^d(\mathbb{C}^n)$ be generic and

$$\tilde{f}(x_1, \ldots, x_n, x_{n+1}) := f(x_1, \ldots, x_n) + x_{n+1}^d$$

with $d = 2n-3$. Then the projection $\pi : Z_{\tilde{f}} \to \mathbb{P}(\mathbb{C}^n)$ has the following property: The preimage of a hyperflex to Z_f is the union of d lines on $Z_{\tilde{f}}$. The case of $n = 3$ is explained in [**Dol12**, Ex 9.1.24]. It remains to be shown that for generic f the section $\sigma_{\tilde{f}} : \mathrm{Gr}_2(\mathbb{C}^{n+1}) \to \mathrm{Pol}^d(S)$ is transversal to the zero section and have no other zeroes. We will not pursue this approach further.

COROLLARY 2.5.10 (Zagier's formula). *The number of lines on a generic degree $d = 2n-3$ hypersurface in $\mathbb{P}(\mathbb{C}^{n+1})$ is*

$$\sum_{u=1}^{n-1}(-1)^{u+n+1}\begin{bmatrix} d \\ u \end{bmatrix}\binom{d-u+1}{n-1} d^{u+1}.$$

The identities $u_{j+1} = dv_j$ imply the following generalization of Theorem 2.5.8:

THEOREM 2.5.11. *Let d be any degree and choose n and $0 \leq i \leq d-1$ such that $d-1+i = 2(n-2)$. Then the number of lines on a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^{n+1})$ intersecting a generic $(i+1)$-codimensional projective subspace is d times the number of hyperflexes to a generic degree d hypersurface in $\mathbb{P}(\mathbb{C}^n)$ intersecting a generic $(i+1)$-codimensional projective subspace.*

2.6. Asymptotic behaviour of the Plücker number $\mathrm{Pl}_{\lambda;i}(d)$

Theorem 2.4.6 implies that the Plücker numbers $\mathrm{Pl}_{\lambda;i}(d)$ are polynomials in d and have degree at most $|\lambda|$. In this section we calculate the coefficient of $d^{|\lambda|}$ in $\mathrm{Pl}_{\lambda;i}(d)$ by relating it to certain Kostka numbers.

This coefficient informs us about the asymptotic behaviour of the Plücker number $\mathrm{Pl}_{\lambda;i}(d)$ as d tends to infinity, so we will call it the *asymptotic Plücker number* $\mathrm{aPl}_{\lambda;i}$. Theorem 2.4.8 shows that for some i the polynomial $\mathrm{Pl}_{\lambda;i}$ can have degree less than $|\lambda|$. In these cases $\mathrm{aPl}_{\lambda;i} = 0$. For example, the number of flexes is $\mathrm{Pl}_3(d) = 3d(d-2)$, so the coefficient of d^3, aPl_3 is zero. More generally, Proposition 2.5.3 shows that $\deg_d(\mathrm{Pl}_m(d)) = (m+1)/2$, so the degree of $\mathrm{Pl}_\lambda(d)$ can be much lower than $|\lambda|$.

Recall that Kostka numbers can be defined as coefficients of the Schur expansion of the complete symmetric polynomials:

$$h_\nu = \sum K_{\mu,\nu} s_\mu.$$

Then the leading term formula of Theorem 2.4.5 immediately implies

THEOREM 2.6.1. *Let $\lambda = (2^{e_2}, \dots, m^{e_m})$ be a partition without 1's and $j \leq \lfloor|\tilde\lambda|/2\rfloor$ a nonnegative integer. Let $n = |\tilde\lambda| - j + 2$. Then*

$$\mathrm{aPl}_{\lambda;|\tilde\lambda|-2j} = \frac{K_{(n-2,j),\tilde\lambda}}{\prod_{i=2}^m e_i!}.$$

In particular, from basic properties of the Kostka numbers (see e.g. [**FH91**, Exercise A.11.]) we obtain that

COROLLARY 2.6.2. *The asymptotic Plücker number* aPl_λ *is zero if and only if* $\lambda_1 \geq |\tilde\lambda|/2+2$.

Note that this can also been easily deduced from Theorem 2.4.8.

REMARK 2.6.3. Kostka numbers have an interpretation as solutions to linear Schubert problems: Let $n = |\mu| - j + 2$, then the Kostka number $K_{(n-2,j),\mu}$ is the number of lines in $\mathbb{P}(\mathbb{C}^n)$ intersecting generic subspaces of codimension $\mu_1+1, \dots, \mu_k+1$ and $n-j-1$.

For example, the number of 4-tangents is

$$\mathrm{Pl}_{2^4}(d) = \frac{1}{12} d(d-7)(d-6)(d-5)(d-4)(d^3+6d^2+7d-30),$$

therefore $\mathrm{aPl}_{2^4} = \frac{1}{12} = \frac{2}{4!}$, and this 2 can be interpreted as the solution of the famous Schubert problem: how many lines intersect four generic lines in $\mathbb{P}^3$?

More generally, for $\lambda = 2^{2(n-2)}$, we have

$$K_{(n-2)^2,1^{2(n-2)}} = C(n-2),$$

where $C(n)$ denotes the n-th Catalan number, e.g. the number of standard Young tableaux for the 2-by-n rectangle. This implies

PROPOSITION 2.6.4.

$$\mathrm{aPl}_{2^{2(n-2)}} = \frac{C(n-2)}{(2n-4)!} = \frac{1}{(n-2)!(n-1)!}.$$

Similarly, for $\lambda = 3^{n-2}$—as $K_{(n-2)^2,2^{n-2}} = R(n-2)$—we have

PROPOSITION 2.6.5.

$$\mathrm{aPl}_{3^{n-2}} = \frac{R(n-2)}{(n-2)!},$$

where $R(n)$ is the n-th Riordan number ($R(3) = 1,\ R(4) = 3,\ R(5) = 6,\ R(6) = 15$, see OEIS `https://oeis.org/A005043`*).*

On the other hand, for $\lambda = (n-1, n-1)$—as $K_{(n-2)^2,(n-2)^2} = 1$—we have

PROPOSITION 2.6.6.

$$\mathrm{aPl}_{n-1,n-1} = \frac{1}{2},$$

2.7. Comparison with the classical non-equivariant method

The goal of this section is to build a bridge between the classical method of solving enumerative problems and the equivariant one.

2.7.1. The general setup. The classical method for computing the cohomology class of a closed subvariety $Z \subset X$ is to give a resolution $\varphi : \tilde{Z} \to X$ of Z and compute $\varphi_! 1$. For a general $\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)$, to find a resolution for which we can calculate this pushforward is difficult.

Instead, we use equivariant cohomology classes that can be computed using equivariant methods, such as localization and the ABBV integral formula. Then we write $Z \subset X$ as a locus of a sufficiently transversal section σ in

$$\begin{tikzcd} P \times_G Y \arrow[r, phantom, "\subseteq"] & A = P \times_G V \arrow[d, shift right] \\ Z = \sigma^{-1}(P \times_G Y) \arrow[r, phantom, "\subseteq"] & X \arrow[u, shift right, bend right, "\sigma"'] \end{tikzcd}$$

so we can use the universal property of the equivariant class $[Y \subset V]_G$ to compute $[Z = \sigma^{-1}(P \times_G Y) \subset X]$.

However, if

$$\begin{tikzcd} E \arrow[r, hook, "j"'] \arrow[rr, bend left, "\varphi"] \arrow[dr] & M \times V \arrow[r, "\pi"'] \arrow[d] & V \\ & M & \end{tikzcd} \tag{13}$$

is an equivariant fibered resolution of $Y \subset V$, then, by Lemma 2.3.5, not only $[Y \subset V]_G = \pi_! e_G(V/E)$ but also we can avoid using equivariant theory as from (13) we can derive a resolution of $Z \subset X$ as follows. All the maps in diagram (13) are G-equivariant, so we can associate it to the principal G-bundle $P \to X$. Complete the resulting diagram with the associated vector bundle $p : \mathbf{M} = P \times_G M \to X$ to get

$$\begin{tikzcd} \mathbf{E} = P \times_G E \arrow[r, hook, "\boldsymbol{j}"] \arrow[dr] & \mathbf{V} = P \times_G (M \times V) = p^* A \arrow[r, "\boldsymbol{\pi}"] \arrow[d, shift right] & A = P \times_G V \arrow[d, shift right] \\ & \mathbf{M} = P \times_G M \arrow[r, "p"] \arrow[u, shift right, bend right, "\bar{\sigma} = p^*\sigma"'] & X \arrow[u, shift right, bend right, "\sigma"'] \end{tikzcd}.$$

It follows from the construction that $\bar{\sigma}$ is transversal to $\mathbf{E}$, hence p restricted to $\bar{\sigma}^{-1}(\mathbf{E})$ gives a resolution of $Z = \sigma^{-1}(P \times_G Y)$.

By Lemma 2.3.5, we get that

$$[Z \subset X] = p_! e(\mathbf{V}/\mathbf{E}).$$

Note that, to keep our formulas shorter, we use the same notation for a bundle and its injective image.

2.7.2. The case of m-flex lines. Our recursive method provides an equivariant fibered resolution such as (13) for $\overline{Y}_\lambda \subset \mathrm{Pol}^d(\mathbb{C}^2)$ when $\lambda = (m)$. In what follows, we work out the details of the above general method for this case. For some m's calculations are described in [**EH16**, Ch. 11].

We recall the construction from Section 2.2.3: $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ defines a hypersurface $Z_f \subset \mathbb{P}(\mathbb{C}^n)$. It also induces a section

$$\sigma_f : X = \mathrm{Gr}_2(\mathbb{C}^n) \to A = \mathrm{Pol}^d(S),\ \sigma_f(V) := f|_V,$$

where $S \to \mathrm{Gr}_2(\mathbb{C}^n)$ is the tautological bundle. We identified the variety of tangent lines of type λ as

$$\mathcal{T}_\lambda Z_f = \sigma_f^{-1}(Y_\lambda(d)).$$

Once we have $\left[\overline{Y}_\lambda(d)\right]$, we can get $\left[\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\right]$ by substituting $c_i \mapsto c_i(S^\vee)$ as in Section 2.2.

The covering map, described in Section 2.3.1, becomes a fibered resolution for $\lambda = (m)$:

$$\begin{tikzcd}
E = \mathrm{Pol}^{d'}(\mathbb{C}^2) \otimes \mathrm{Pol}^m(\mathbb{C}^2/\gamma) \arrow[r, hook, "j"'] \arrow[rrr, bend left, "\varphi"] \arrow[drr] & & \mathbb{P}(\mathbb{C}^2) \times \mathrm{Pol}^d(\mathbb{C}^2) \arrow[r, "\pi"'] \arrow[d] & \mathrm{Pol}^d(\mathbb{C}^2) \\
& & \mathbb{P}(\mathbb{C}^2) &
\end{tikzcd} \tag{14}$$

Associate diagram (14) to the frame bundle $P = \mathrm{Inj}(\mathbb{C}^2, S) \to X = \mathrm{Gr}_2(\mathbb{C}^n)$ to get:

$$\begin{tikzcd}
\mathbf{E} = \mathrm{Pol}^{d'}(S^2) \otimes \mathrm{Pol}^m(S^2/S^1) \arrow[r, hook, "\boldsymbol{j}"] \arrow[dr] & \mathbf{V} = \mathrm{Pol}^d(S^2) = p^* \mathrm{Pol}^d(S) \arrow[r, "\boldsymbol{\pi}"] \arrow[d, shift right] & A = \mathrm{Pol}^d(S) \arrow[d, shift right] \\
& \mathbf{M} = \mathrm{Fl}_{1,2}(\mathbb{C}^n) \arrow[r, "p"] \arrow[u, shift right, bend right, "\bar{\sigma}_f = p^*\sigma_f"'] & \mathrm{Gr}_2(\mathbb{C}^n) \arrow[u, shift right, bend right, "\sigma_f"']
\end{tikzcd}, \tag{15}$$

where S^i is the tautological bundle of rank i over the flag manifold $\mathrm{Fl}_{1,2}(\mathbb{C}^n)$ and $d' = d - m$. ([**EH16**, Ch. 11] uses the notation $\mathbb{G}(1, n-1)$ for the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ and Ψ for the flag manifold $\mathrm{Fl}_{1,2}(\mathbb{C}^n)$.)

As we explained in the previous section, if f is generic, then $\left[\overline{\mathcal{T}_m Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\right] = p_! e(\mathbf{V}/\mathbf{E})$, where $p : \mathrm{Fl}_{1,2}(\mathbb{C}^n) \to \mathrm{Gr}_2(\mathbb{C}^n)$ denotes the projection. To calculate the pushforward we use

PROPOSITION 2.7.1. [**EH16**, Prop. 10.3] *Let $D \to X$ be a rank k bundle over a smooth X, and $p : \mathbb{P}(D) \to X$ its projectivization. Then all $\alpha \in H^*(\mathbb{P}(D))$ is of the form $\alpha = \sum \beta^i p^* m_i$, where $\beta = c_1(\gamma^\vee \to \mathbb{P}(D))$ with $\gamma \to \mathbb{P}(D)$ the tautological line bundle of $\mathbb{P}(D)$ and*

$$p_! \alpha = \sum s_{i-k+1} m_i,$$

where $1/c(D) = s(D) = 1 + s_1 + \cdots$ is the Segre class of D (or, equivalently, $s_i = s_i(c_j(D^\vee))$, the Schur polynomial in the Chern classes of the dual bundle $D^\vee$).

Namely, recall that $\mathrm{Fl}_{1,2}(\mathbb{C}^n)$ is the total space of the projective bundle $p : \mathbb{P}(S) \to \mathrm{Gr}_2(\mathbb{C}^n)$, and $S^1 \to \mathrm{Fl}_{1,2}(\mathbb{C}^n)$ is the tautological line bundle of $\mathbb{P}(S)$.

To stay close to the notation of [**EH16**, Ch. 11], we choose generators of $H^*(\mathrm{Fl}_{1,2}(\mathbb{C}^n))$

$$c_1((S^2)^\vee) = \sigma_1, \quad c_1((S^1)^\vee) = \zeta :$$

S^1 is a subbundle of S^2 so S^2/S^1 is also a line bundle over $\mathrm{Fl}_{1,2}(\mathbb{C}^n)$ with $c_1((S^2/S^1)^\vee) = \eta = \sigma_1 - \zeta$. Using these generators, we can apply Proposition 2.7.1 and calculate the pushforward $p_!$ as

$$p_!(\zeta^a \sigma_1^b) = s_{a-1}\sigma_1^b, \text{ where } s = 1 + \sigma_1 + \sigma_1^2 - \sigma_2 + \ldots$$

is the Segre class of S. Note that we don't distinguish between cohomology classes and their pullbacks and hence, in the remainder of this Section, denote by σ_i the i-th Chern class of $S^\vee \to \mathrm{Gr}_2\,\mathbb{C}^n$. Be aware that in the earlier Sections we used c_i for $c_i(S^\vee)$.

The Chern classes of the "bold" bundles can be obtained by substituting the corresponding Chern roots:

$$c(\mathbf{V}) = \prod_{i=0}^{d}\big(1+(d-i)\zeta+i\eta\big) = \prod_{i=0}^{d}\big(1+(d-2i)\zeta+i\sigma_1\big).$$

Similarly,

$$c(\mathbf{E}) = \prod_{i=0}^{d-m}\big(1+m(\sigma_1-\zeta)+(d-m-i)\zeta+i(\sigma_1-\zeta)\big),$$

implying that

$$c(\mathbf{V}/\mathbf{E}) = \prod_{i=0}^{m-1}\big(1+(d-i)\zeta+i\eta\big) = \prod_{i=0}^{m-1}\big(1+(d-2i)\zeta+i\sigma_1\big)$$

and

$$e(\mathbf{V}/\mathbf{E}) = c_m(\mathbf{V}/\mathbf{E}) = \prod_{i=0}^{m-1}\big((d-i)\zeta+i\eta\big) = \prod_{i=0}^{m-1}\big((d-2i)\zeta+i\sigma_1\big).$$

The case $m=2$:

$$p_!c_2(\mathbf{V}/\mathbf{E}) = p_!\big(d\zeta((d-2)\zeta+\sigma_1)\big) = d(d-1)\sigma_1 = d(d-1)s_1,$$

which is equivalent to the Plücker formula $\mathrm{Pl}_{2;1}$ of Example 2.1.4.

The case $m=3$:

$$\begin{aligned}
p_!c_3(\mathbf{V}/\mathbf{E}) =& p_!\big(d\zeta((d-2)\zeta+\sigma_1)((d-4)\zeta+2\sigma_1)\big)\\
=& p_!\big(2d\zeta\sigma_1^2+d(3d-8)\zeta^2\sigma_1+d(d-2)(d-4)\zeta^3\big)\\
=& d(d-1)(d-2)\sigma_1^2-d(d-2)(d-4)\sigma_2 = d(d-1)(d-2)s_2+3d(d-2)s_{1,1},
\end{aligned}$$

whose coefficients are the Plücker numbers $\mathrm{Pl}_{3;2}$ and Pl_3 of Example 2.1.2.

For a specific n, calculations can be simplified by the observation that $\zeta^n=0$, since S^1 is the pullback of the tautological bundle of $\mathbb{P}(\mathbb{C}^n)$.

Remark 2.7.2. A different formula can be given for the pushforward map in the special case of $\mathbb{P}^1$-bundles $p:\mathbb{P}(D)\to X$ like $\mathrm{Fl}_{1,2}(\mathbb{C}^n)\to\mathrm{Gr}_2(\mathbb{C}^n)$. Let ζ,η denote the Chern roots of the rank two bundle $D^\vee\to X$: $\zeta := c_1(\gamma^\vee\to\mathbb{P}(D))$ and $\eta := c_1((p^*D/\gamma)^\vee\to\mathbb{P}(D))$. Then for any polynomial $q(\zeta,\eta)\in H^*(\mathbb{P}(D))$ its pushforward along $p:\mathbb{P}(D)\to X$ is given by

$$p_!q(\zeta,\eta) = \partial q.$$

This is easy to prove directly, but also follows from the equivariant pushforward formula (9). The calculations above become simpler if we use the variables ζ,η and this pushforward formula.

2.7.3. Incidence varieties. It is instructive to describe the section $\bar\sigma_f$ and the subvariety $\bar\sigma_f^{-1}(\mathbf{E}) \subset \mathrm{Fl}_{1,2}(\mathbb{C}^n)$ of (15).

$$\bar\sigma_f\left((W,L)\right) = f|_W \in \mathbf{V}_{(W,L)} = \mathrm{Pol}^d(W),$$

hence

$$\bar\sigma_f\left((W,L)\right) \in \mathbf{E}_{(W,L)} = \mathrm{Pol}^{d'}(W) \otimes \mathrm{Pol}^m(W/L)$$

is equivalent to having a basis x, y of $W^\vee$ such that $x(L)=0$ and $f|_W = x^m p(x,y)$ for some polynomial p of degree $d' = d-m$. Therefore

$$I_m := \bar\sigma_f^{-1}(\mathbf{E}) = \{(W,L) : [W] \text{ and } Z_f \text{ has a point of contact of order at least } m \text{ at } [L]\},$$

which is the usual resolution of $\overline{\mathcal{T}_m Z_f}$. We will call it the *incidence variety*.

Notice that the bundle $\mathcal{E}$ used in [**EH16**, Ch. 11] looks different than our $\mathbf{V}/\mathbf{E}$. They have the same Chern classes, so the calculations are the same. They are probably also isomorphic.

Remark 2.7.3. The construction of the incidence variety can be generalized. Let m be an element of λ, not necessarily equal to λ_1. Denote by λ' the partition λ minus m. Similarly to what we had for the covering map constructed in Section 2.3.1, $\mathbf{E}$ has a subbundle

$$Y_{\lambda'}(\mathbf{E}) = Y_{\lambda'}\left(\mathrm{Pol}^{d'}(S^2) \otimes \mathrm{Pol}^m(S^2/S^1)\right)$$

corresponding to invariant subvariety $Y_{\lambda'}(d') \subset \mathrm{Pol}^{d'}(\mathbb{C}^2)$.

Choose a generic $f \in \mathrm{Pol}^d(\mathbb{C}^n)$. It induces a section $\bar\sigma_f : \mathrm{Fl}_{1,2}(\mathbb{C}^n) \to \mathbf{V} = \mathrm{Pol}^d(S^2)$. Then $\bar\sigma_f^{-1}\left(\overline{Y}_{\lambda'}(\mathbf{E})\right)$ can be identified with the *incidence variety $I_{\lambda';m}$ of m-flex points and λ-lines for f*. Therefore, by Lemma 2.3.5 and Corollary 2.3.7,

$$[I_{\lambda';m} \subset \mathrm{Fl}_{1,2}] = e(\mathbf{V}/\mathbf{E}) \cdot \left[\overline{Y}_{\lambda'}(\mathbf{E}) \subset \mathbf{E}\right]$$

and

$$\left[\overline{Y}_{\lambda'}(\mathbf{E}) \subset \mathbf{E}\right] = \left[\overline{Y}_{\lambda'}(d')\right]\left(\eta + \frac{m}{d'}\eta, \zeta + \frac{m}{d'}\eta\right),$$

where we substitute into the equivariant class $\left[\overline{Y}_{\lambda'}(d')\right]$ expressed in Chern roots a, b, see Section 2.3.2.

Since $p|_{I_{\lambda';m}}$ is an e_m to 1 branched covering of $\overline{\mathcal{T}_\lambda Z_f}$, we can calculate its cohomology class:

$$\left[\overline{\mathcal{T}_\lambda Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\right] = \frac{1}{e_m} p_!\left(e(\mathbf{V}/\mathbf{E}) \cdot \left[\overline{Y}_{\lambda'}(\mathbf{E}) \subset \mathbf{E}\right]\right).$$

This is the non-equivariant version of the proof of Theorem 2.2.5. Notice that $I_{\lambda';m}$ is not smooth in general.

Example 2.7.4. For $\lambda = (2,2)$ the class of the incidence variety $I_{2;2}$ of bitangent points and bitangents is

$$\begin{aligned}
[I_{2;2} \subset \mathrm{Fl}_{1,2}(\mathbb{C}^n)] =& e\left(\mathbf{V}/\mathbf{E}\right)\left[\overline{Y}_2\left(\mathbf{E}\right) \subset \mathbf{E}\right] \\
=& \prod_{i=0}^{1}\left((d-i)\zeta + i\eta\right) \cdot \left[\overline{Y}_2(d-2)\right]\left(\eta + \frac{2}{d-2}\eta, \zeta + \frac{2}{d-2}\eta\right) \\
=& d(d-3)(d+2)\zeta\sigma_1^2 + d(d-3)(d^2-8)\zeta^2\sigma_1 - 4d(d-2)(d-3)\zeta^3.
\end{aligned}$$

As its pushforward along p we get

$$\begin{aligned}
\left[\overline{\mathcal{T}_{2,2} Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^n)\right] =& \frac{1}{2} p_!\left[I_{2;2} \subset \mathrm{Fl}_{1,2}\right] = \frac{1}{2} d(d-1)(d-2)(d-3)\sigma_1^2 + 2d(d-2)(d-3)\sigma_2 \\
=& \frac{1}{2} d(d-1)(d-2)(d-3)s_2 + \frac{1}{2} d(d-2)(d-3)(d+3)s_{1,1},
\end{aligned}$$

agreeing with (3).

2.8. Further enumerative problems

2.8.1. The universal hypersurface and Plücker numbers for linear systems. A more general construction considers all hypersurfaces Z_f at once. Consider the vector bundle

$$A_u := \operatorname{Hom}\big(L, \operatorname{Pol}^d(S)\big) \longrightarrow \mathbb{P}\big(\operatorname{Pol}^d(\mathbb{C}^n)\big) \times \operatorname{Gr}_2(\mathbb{C}^n),$$

where L and S are the tautological bundles over $\mathbb{P}\big(\operatorname{Pol}^d(\mathbb{C}^n)\big)$ and $\operatorname{Gr}_2(\mathbb{C}^n)$.

A_u has a section

$$\sigma([f], V)(f) : f \mapsto f|_V,$$

the *universal section*. Applying the construction in Section 2.3.1 to $A_u \cong \operatorname{Pol}^d(S) \otimes L^\vee$ and $\mathrm{GL}(2)$-invariant subsets $Y_\lambda(d) \subset \operatorname{Pol}^d(\mathbb{C}^2)$ we get subbundles $Y_\lambda(A_u)$. The universal section is transversal to the subvarieties $\overline{Y}_\lambda(A_u)$ (see Example A.2.3). The cohomology classes $\big[\sigma^{-1}(\overline{Y}_\lambda(A_u))\big]$ are the source of answers for new enumerative problems and can be calculated using Corollary 2.3.7 from the equivariant classes $\big[\overline{Y}_\lambda(d)\big]$ expressed in Chern classes c_1, c_2 (see Section 2.3.2) by substituting

$$c_1 \mapsto c_1 + \frac{2}{d}\xi, \;\; c_2 \mapsto c_2 + \frac{1}{d}\xi c_1 + \frac{1}{d^2}\xi^2,$$

where on the right hand side of these substitutions c_i and ξ denote the Chern classes of the duals of S and L respectively.

For example,

$$\big[\sigma^{-1}(\overline{Y}_2\,(A_u))\big] = d(d-1)c_1 + 2(d-1)\xi.$$

Therefore $2d-2$, the coefficient of ξ, is the number of degree d curves in a pencil tangent to a given line.

Similarly,

$$\begin{aligned}[\sigma^{-1}(\overline{Y}_3(A_u))] =& d(d-1)(d-2)c_1^2 - d(d-2)(d-4)c_2 + 3d(d-2)\xi c_1 + 3(d-2)\xi^2\\ =& d(d-2)(d-1)s_2 + 3d(d-2)s_{1,1} + 3d(d-2)\xi s_1 + 3(d-2)\xi^2.\end{aligned}$$

The coefficient of ξ^2—the degree of the variety $\mathbb{P}\big(\overline{Y}_3(d)\big)$ (See [**FNR05**, Cor. 6.4])—was already calculated by Hilbert (for all $\overline{Y}_\lambda(d)$). The only new information is the coefficient of ξs_1, which for $n = 3$ is the number of lines that go through a point and are flex lines to a member of a pencil of degree d curves. In other words, $3d(d-2)$ is the degree of the curve in the projective plane of lines that consists of those lines that are flexes to a member of a given generic pencil.

2.8.2. m-flex points of λ-lines. The flag manifold $\mathrm{Fl}_{1,2}(\mathbb{C}^n)$ possesses another fibration $q : \mathrm{Fl}_{1,2}(\mathbb{C}^n) \to \mathbb{P}(\mathbb{C}^n)$. We call the q-image of the incidence variety $I_{\lambda';m}$ of Remark 2.7.3 the *variety of m-flex points of λ-lines*. Since $q : I_{\lambda';m} \to q(I_{\lambda';m})$ is generically one-to-one, we can calculate its cohomology class by pushing forward $[I_{\lambda';m}]$ along q.

EXAMPLE 2.8.1. Let $\lambda=(3,2)$. Remark 2.7.3 with $m=3$ and $\lambda'=(2)$ gives that for the incidence variety $I_{2;3}$ of flex points and $(3,2)$-lines

$$\begin{aligned}[I_{2;3}\subset \mathrm{Fl}_{1,2}(\mathbb{C}^n)] =& e\left(\mathbf{V}/\mathbf{E}\right)\left[\overline{Y}_2(\mathbf{E})\subset \mathbf{E}\right]\\ =& d\zeta((d-1)\zeta+\eta)((d-2)\zeta+2\eta)\cdot\left[\overline{Y}_2(d-3)\right]\left(\eta+\frac{3}{d-3}\eta,\zeta+\frac{3}{d-3}\eta\right)\\ =& d\zeta((d-1)\zeta+\eta)((d-2)\zeta+2\eta)\cdot(d-3)(d-4)\left(\eta+\frac{3}{d-3}\eta+\zeta+\frac{3}{d-3}\eta\right).\end{aligned}$$

The fibration $q:\mathrm{Fl}_{1,2}(\mathbb{C}^n)\to\mathbb{P}(\mathbb{C}^n)$ is isomorphic to the projective bundle $\mathbb{P}(\mathbb{C}^n/\gamma)\to\mathbb{P}(\mathbb{C}^n)$, where $\gamma\to\mathbb{P}(\mathbb{C}^n)$ denotes the tautological bundle. Because of this description, we can use Proposition 2.7.1 to calculate the pushforward $q_!$:

$$q_!\left(\eta^a\zeta^b\right)=s_{a-n+2}\zeta^b,\ \text{ where } s=1-\zeta \tag{16}$$

is the Segre class of C^n/γ and $\zeta=c_1(\gamma^\vee\to\mathbb{P}(\mathbb{C}^n))$.

Notice that $\gamma^\vee\to\mathbb{P}(\mathbb{C}^n/\gamma)$ corresponds to $(S^2/S^1)^\vee\to\mathrm{Fl}_{1,2}(\mathbb{C}^n)$ hence the use of η in (16) is consistent with our earlier choice of generators of $H^*(\mathrm{Fl}_{1,2}(\mathbb{C}^n))$.

Contrary to the fibration $p:\mathrm{Fl}_{1,2}(\mathbb{C}^n)\to\mathrm{Gr}_2(\mathbb{C}^n)$, the relative codimension of q depends on n. For $n=4$, the nonzero pushfowards are $q_!(\eta^2)=1$ and $q_!(\eta^3)=-\zeta$, hence for a generic degree d surface, the class of the curve consisting of the 3-flex points of the $(3,2)$-lines is

$$q_!\left[I_{2;3}\subset\mathrm{Fl}_{1,2}(\mathbb{C}^4)\right]=\left[q(I_{2;3})\subset\mathbb{P}(\mathbb{C}^4)\right]=d(d-4)(3d^2+5d-24)\zeta^2.$$

Similar calculation shows that for a generic degree d surface the cohomology class of the curve consisting of the 2-tangent points of the $(3,2)$-lines is

$$\left[q\left(I_{3;2}\right)\subset\mathbb{P}(\mathbb{C}^4)\right]=d(d-2)(d-4)(d^2+2d+12)\zeta^2.$$

EXAMPLE 2.8.2. The degree of the curve of 4-flex points on a surface is calculated in [**EH16**, p. 399]. The calculation is essentially the same, so we don't repeat it here.

2.8.3. m-flex points of λ-lines for a linear system. The previous two constructions can be combined without difficulty.

Let $\lambda=(\lambda',m)$ with m not necessarily equal to λ_1, and consider the vector bundle $\mathbf{V}_u=\mathrm{Hom}(L,\mathbf{V})\to\mathbb{P}\big(\mathrm{Pol}^d(\mathbb{C}^n)\big)\times\mathrm{Fl}_{1,2}(\mathbb{C}^n)$ and its subbundle

$$\mathbf{E}_u=\mathrm{Hom}\left(L,\mathbf{E}\right)=\mathrm{Hom}\left(L,\mathrm{Pol}^{d'}(S^2)\otimes\mathrm{Pol}^m(S^2/S^1)\right)\to\mathbb{P}\left(\mathrm{Pol}^d(\mathbb{C}^n)\right)\times\mathrm{Fl}_{1,2}\left(\mathbb{C}^n\right),$$

where L and S^i are the tautological bundles over $\mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^n))$ and $\mathrm{Fl}_{1,2}(\mathbb{C}^n)$.

$\mathbf{V}_u$ has a section

$$\overline{\sigma}\left([f],(W,L)\right):f\mapsto f|_W.$$

Applying the construction in Section 2.3.1 to $\mathbf{E}_u\cong\mathrm{Pol}^{d'}(S^2)\otimes\mathrm{Pol}^m(S^2/S^1)\otimes L^\vee$ and $\mathrm{GL}(2)$-invariant subsets $Y_{\lambda'}(d')\subset\mathrm{Pol}^{d'}(\mathbb{C}^2)$, we get subbundles $Y_{\lambda'}(\mathbf{E}_u)$. The section $\overline{\sigma}$ is transversal to these subbundles, hence the class of their pullbacks $\overline{\sigma}^{-1}(\overline{Y}_{\lambda'}(\mathbf{E}_u))$ can be calculated, using Lemma 2.3.5 and Corollary 2.3.7, as

$$\begin{aligned}\left[\overline{\sigma}^{-1}\left(\overline{Y}_{\lambda'}\left(\mathbf{E}_u\right)\right)\subset\mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^n))\times\mathrm{Fl}_{1,2}(\mathbb{C}^n)\right]&=\left[\overline{Y}_{\lambda'}\left(\mathbf{E_u}\right)\subset\mathbf{V}_u\right]\\ &=e\left(\mathrm{Hom}\left(L,\mathbf{V}/\mathbf{E}\right)\right)\cdot\left[\overline{Y}_{\lambda'}(d')\right]\left(\eta+\frac{1}{d'}(\xi+m\eta),\zeta+\frac{1}{d'}(\xi+m\eta)\right),\end{aligned}$$

where we substitute into the equivariant class $\left[\overline{Y}_{\lambda'}(d')\right]$ expressed in Chern roots a,b, see Section 2.3.2.

The variety $\overline{\sigma}^{-1}\big(\overline{Y}_{\lambda'}(\mathbf{E}_u)\big)$ can be identified with the *universal incidence variety of m-flex points and λ-lines* for degree d hypersurfaces in $\mathbb{P}(\mathbb{C}^n)$.

Since the composition

$$\mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^n)) \times \mathrm{Fl}_{1,2}(\mathbb{C}^n) \xrightarrow{\pi_{\mathrm{Fl}}} \mathrm{Fl}_{1,2}(\mathbb{C}^n) \xrightarrow{q} \mathbb{P}(\mathbb{C}^n)$$

restricted to $\overline{\sigma}^{-1}\left(\overline{Y}_{\lambda'}(\mathbf{E}_u)\right)$ is generically one-to-one, the class of the image is $q_!\pi_{\mathrm{Fl}!}\left[\overline{\sigma}^{-1}\left(\overline{Y}_{\lambda'}(\mathbf{E}_u)\right)\right]$. This class provides solutions to further enumerative problems about m-tangent points of λ -lines in linear systems of hypersurfaces.

Example 2.8.3. To calculate the degree of the curve consisting of tangent points of bitangent lines in a pencil of degree d curves, set $\lambda = (2,2)$, $m = 2$, $\lambda' = (2)$ and $n = 3$. Then the class of the universal incidence variety is

$$\begin{aligned}\left[\overline{\sigma}^{-1}\left(\overline{Y}_2(\mathbf{E}_u)\right) \subset \mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^3)) \times \mathrm{Fl}_{1,2}(\mathbb{C}^3)\right] &= e\big(\mathrm{Hom}(L, \mathbf{V}/\mathbf{E})\big) \cdot \left[\overline{Y}_2(\mathbf{E}_u) \subset \mathbf{E}_u\right] \\ &= (d\zeta + \xi)(\eta + (d-1)\zeta + \xi) \cdot (d-3)\big((d-2)(\eta+\zeta) + 2\xi + 4\eta\big).\end{aligned} \tag{17}$$

Restricting our attention to a generic pencil of degree d plane curves amounts to multiplying (17) by ξ^{N-1} where $N = \dim(\mathrm{Pol}^d(\mathbb{C}^3))$, while the pushforward along π_{Fl} gives the coefficient of the volume form ξ^N. Therefore

$$\left[\pi_{\mathrm{Fl}}\left(\overline{\sigma}^{-1}\left(\overline{Y}_2(\mathbf{E}_u)\right)\right) \subset \mathrm{Fl}_{1,2}(\mathbb{C}^3)\right] = (d+2)(d-3)\eta^2 + 2(d-3)(d^2+3d-2)\eta\zeta + (d-3)(4d^2-7d+2)\zeta^2.$$

Finally, Proposition 2.7.1 calculates the pushforward $q_!$ as in Section 2.8.2, and we get that the degree of the curve of tangent points of bitangent lines is

$$(d-3)(2d^2+5d-6),$$

in particular, for quintics the degree is 46.

Example 2.8.4. The degree of the curve of flex points in a pencil of degree d curves is calculated e.g. in [**EH16**, p. 407].

CHAPTER 3

Chern-Schwarz-MacPherson classes of coincident root strata and varieties of λ-lines

3.1. A brief introduction to the (equivariant) Chern-Schwartz-MacPherson class

The existence of Chern-Schwartz-MacPherson classes (CSM) of constructible functions on complex algebraic varieties was conjectured by Eligne and Grothendieck and proved by MacPherson ([**Mac74**]). Following [**FR18**], we prefer its cohomology counterpart we get by applying the Poincaré duality. It provides a deformation of the cohomology fundamental class of an algebraic variety. As we will see, it encodes finer numerical invariants of the variety than its fundamental class. In what follows, we restrict our attention to embeddings of complex varieties into a smooth variety M.

Our definition of the CSM class is due to Aluffi ([**Alu06**]): it is a motivic class for singular cohomology H^*: By a *motivic class for a complex oriented cohomology theory* h^* we mean an additive morphism m mapping a constructible (finite union of locally closed subsets) $U \subset M$ to $\mathrm{m}(U)$ in a sense that it has the *motivic property*,

$$\mathrm{m}(U \cup V \subset M) = \mathrm{m}(U \subset M) + \mathrm{m}(V \subset M) - \mathrm{m}(U \cap V \subset M)$$

and a property we will call *homology property*,

$$f_! \,\mathrm{m}(U \subset M) = \mathrm{m}(f(U) \subset N)$$

for $f : M \to N$ a proper map that is an isomorphism when restricted to $U \subset M$. Sometimes, if it is clear from the context, we will omit the ambient space from the notation. In particular, $\mathrm{m}(M)$ will always denote $\mathrm{m}(M \subset M)$.

As discussed in [**FRW21**], setting $\mathrm{m}(M)$ determines the motivic class. For example, for a closed embedding $i : X \subset M$ of a smooth variety

$$\mathrm{m}(X \subset M) = i_! \,\mathrm{m}(X)$$

by the homology property. We extend this to embeddings of not necessarily closed smooth varieties $i : U \hookrightarrow M$ using a *proper normal crossing extension* $\bar{i}$ of i:

Definition 3.1.1. Suppose $f : U \to M$ is a map of smooth varieties. Then a proper normal crossing extension of f is a proper map $\bar{f} : Y \to M$ with an embedding $j : U \hookrightarrow Y$ satisfying $f = \bar{f} \circ j$ such that the variety Y is smooth and the complement $Y \setminus j(U) = \bigcup_{k=1}^{s} D_k$ is a simple normal crossing divisor.

Such proper normal crossing extension always exists ([**Web17**, § 5]).

If for $K \subset \underline{s} = \{1, 2, \dots, s\}$ we set $D_K = \bigcap_{k \in K} D_k$ and $\bar{i}_K = \bar{i}|_{D_K}$, in particular $\bar{i}_\emptyset = \bar{i}$, we have

$$\mathrm{m}(U \subset M) = \sum_{K \subset \underline{s}} (-1)^{|K|} \bar{i}_{K!} \,\mathrm{m}(D_K).$$

Independence of the chosen extension follows from a refined version of the Weak Factorization, as formulated in [**Wlo09**]. Finally, note that complex subvarieties admit stratifications with smooth strata.

We define the CSM class mapping by setting $c^{SM}(M) = c(M) = c(TM)$ for every algebraic manifold M. Then for every pair $X \subset M$

$$c^{SM}(X \subset M) = [X] + \cdots + \chi(X)[*],$$

which shows that we indeed got a deformation of the fundamental class. That the highest degree term of $c^{SM}(X \subset M)$ is $\chi(X)$ times the class of a point $[\mathrm{pt}] \in H^*(M)$ is an immediate consequence of the definition for closed embeddings of smooth varieties. In general, it can be shown using the fact that for complex algabraic varieties the Euler characteristics is motivic (see e.g. [**Ful95**]).

Choosing $c^{SM}(M) = c(TM)$ as the base case has the added benefit that the *multiplicative property* of the total Chern class yields the same property of the CSM class:

$$c^{SM}(X_1 \times X_2 \subset M_1 \times M_2) = c^{SM}(X_1 \subset M_1) \times c^{SM}(X_2 \subset M_2). \tag{18}$$

Let us add here a further feature of motivic classes, the *local property*: If $i : U \hookrightarrow M$ is open, then for any subvariety $X \subset M$

$$\mathrm{m}(U \cap X \subset U) = i^* \,\mathrm{m}(X \subset M). \tag{19}$$

This is an easy consequence of the definition, but it will turn out to be very helpful is computations.

3.1.1. Divisor trick. Multiplicativity also implies that the CSM class of the zero locus $Z = \sigma^{-1}(0)$ of a vector bundle $E \to M$ can be easily determined using the following fact:

PROPOSITION 3.1.2. *Let $\sigma : M \to E$ be a section of the vector bundle E, which is transversal to the zero section. For $Z := \sigma^{-1}(0)$ we have*

$$\nu_{Z \subset M} \cong E|_Z,$$

where $\nu_{Z \subset M}$ is the normal bundle of $Z \subset M$.

This combined with the short exact sequence defining the normal bundle results in the following formulas; we will refer to them as *divisor trick*:

$$c^{SM}(Z) = \frac{c(TM)|_Z}{c(\nu_{Z \subset M})} = \left.\frac{c(TM)}{c(E)}\right|_Z, \tag{20}$$

and if $i : Z \hookrightarrow M$ denotes the inclusion, then, by the adjunction property of the pushforward, we also get

$$c^{SM}(Z \subset M) = i_! c^{SM}(Z) = i_! i^* \frac{c(TM)}{c(E)} = \frac{c(TM)}{c(E)} i_! 1 = \frac{c(TM)}{c(E)}[Z \subset M] = \frac{c(TM)}{c(E)} e(E).$$

3.1.2. The Segre-Schwartz-MacPherson class. There is a variant of the CSM class, the Segre-Schwartz-MacPherson class (SSM), we get by dividing with the CSM class of the ambient space:

$$s^{SM}(X \subset M) = \frac{c^{SM}(X \subset M)}{c(M)}.$$

The SSM class may have non-zero components in infinitely many degrees, strictly speaking, it is an element of the completion of $H^*(M)$. We will not denote this completion.

For a closed embedding $i : X \subset M$ with X (and M) smooth $s^{SM}(X \subset M) = i_! c(TX)/c(TM) = i_! c(-\nu_{X \subset M})$, where $c(-\nu_{X \subset M})$ denotes the inverse of the Chern class.

As a consequence, this Segre variant behaves well with respect to some suitable type of transversal pullbacks: Let $f : N \to M$ be a map of smooth varieties and $X \subset M$ a Whitney stratified closed subvariety. Assume that f is transversal to the strata of X. Then, see [**Ohm16**, Prop. 3.8],

$$s^{SM}(f^{-1}(X) \subset N) = f^* s^{SM}(X \subset M). \tag{21}$$

3.1.3. Equivariant CSM and SSM classes. The equivariant version of Chern-Schwartz-MacPherson classes for reductive linear groups was developed by Ohmoto ([**Ohm04**]). Again, our preferred version is the cohomological equivariant CSM class of G-invariant subvarieties of complex smooth G-varieties. These can be defined similarly to the non-equivariant case, as the above construction can be carried out in the presence of a group action since, by Bierstone and Milman ([**BM95**]), equivariant resolutions exist and the Weak Factorization can be realized in an invariant manner.

Motivic, homology and local properties remain true in the equivariant setting. Multiplicativity also holds: for G_i-invariant subvarieties $U_i \subset M_i$,

$$c^{SM}_{G_1\times G_2}(X_1 \times X_2 \subset M_1 \times M_2) = c^{SM}_{G_1}(X_1 \subset M_1) \times c^{SM}_{G_2}(X_2 \subset M_2), \tag{22}$$

or, if $G_1 = G_2 = G$, using the diagonal map $G \to G \times G$,

$$c^{SM}_{G}(X_1 \times X_2 \subset M_1 \times M_2) = c^{SM}_{G}(X_1 \subset M_1) \times c^{SM}_{G}(X_2 \subset M_2). \tag{23}$$

The following lemma, combined with the latter equation, will prove to be essential in our calculations of CSM classes of affine varieties.

LEMMA 3.1.3 (*motivic calculus for CSM*). *Let the torus $\mathbb{T}$ act on $\mathbb{C}$ by the weight a. Then for the $\mathbb{T}$-equivariant CSM classes we have*

$$c^{SM}(\mathbb{C} \subset \mathbb{C}) = 1 + a, \qquad c^{SM}(\{0\} \subset \mathbb{C}) = e(\mathbb{C}) = a, \qquad c^{SM}(\mathbb{C} \setminus \{0\} \subset \mathbb{C}) = 1.$$

For example, for our $\mathbb{T}^2$-representation $\mathrm{Pol}^d(\mathbb{C}^2)$ and invariant subset $\langle x^s y^{d-s}\rangle \setminus \{0\}$ we get

$$c^{SM}\left(\langle x^s y^{d-s}\rangle \setminus \{0\} \subset \mathrm{Pol}^d(\mathbb{C}^2)\right) =$$

$$c^{SM}\left(\langle x^s y^{d-s}\rangle \setminus \{0\} \subset \langle x^s y^{d-s}\rangle\right) \cdot \prod_{\substack{t=0\\ t\neq s}}^{d} c^{SM}\left(\{0\} \subset \langle x^t y^{d-t}\rangle\right) =$$

$$1 \cdot \prod_{\substack{t=0\\ t\neq s}}^{d} (ta + (d-t)b) = \frac{e\left(\mathrm{Pol}^d(\mathbb{C}^2)\right)}{sa + (d-s)b}.$$

Equivariant SSM classes in representations

$$s^{SM}_G(X \subset V) := \frac{c^{SM}_G(X \subset V)}{c^{SM}_G(V)}$$

are especially important as they are universal for SSM classes of degeneracy loci ([**Ohm16**, Theorem 3.13]):

First, suppose that $P \to M$ is a principal G-bundle and denote by $\kappa : M \to BG$ its classifying map. Let A be a smooth G-variety. We can associate A to both principal G-bundles $P \to M$ and $\mathrm{E}\,G \to \mathrm{B}\,G$. Denote by $\hat{\kappa}_A : P \times_G A \to \mathrm{E}\,G \times_G A$ the lift of κ to the total spaces of these associated bundles, and let

$$\mathbf{a} := \hat{\kappa}_A^* : H^*_G(A) \to H^*(P \times_G A).$$

PROPOSITION 3.1.4. *Let $X \subset A$ be G-invariant subvariety. Then*

$$s^{SM}\left(P \times_G X \subset P \times_G A\right) = \mathbf{a}\left(s^{SM}_G(X \subset A)\right).$$

If $\sigma : M \to P \times_G A$ is a section of the associated bundle that is transversal to a Whitney stratification of the subbundle $P \times_G X$, then we can apply (21) to get

COROLLARY 3.1.5. *Let $X \subset A$ be a G-invariant closed subvariety of the smooth variety A. Suppose that $P \to M$ is a principal G-bundle, and $\sigma : M \to P \times_G A$ is a section of the associated bundle that is transversal to a Whitney stratification of the subbundle $P \times_G X$. Then*

$$s^{SM}(\sigma^{-1}(P \times_G X) \subset M) = \sigma^* \mathbf{a}\left(s_G^{SM}(X \subset A)\right).$$

If A is a vector space, $H_G^*(A) \cong H_G^*$, hence G-equivariant SSM classes of invariant subvarieties are G-characteristic classes. Similarly, $\sigma^* : H^*(P \times_G A) \to H^*(M)$ is an isomorphism. Under these identifications the composition $\sigma^* \mathbf{a}$ becomes κ^*, that is, the class $s^{SM}(\sigma^{-1}(P \times_G X) \subset M)$ can be obtained as $s_G^{SM}(X \subset A)$ evaluated at the bundle $P \to M$.

Moreover, note that a G-invariant Whitney stratification of X induces a Whitney stratification of $P \times_G X$.

3.1.4. Structure of the chapter. The main focus of this chapter is analogous to that of Chapter 2: to investigate CSM classes of CRS and see how $c^{SM}(Y_\lambda(d) \subset \mathrm{Pol}^d(\mathbb{C}^2))$ can be used to gather further information about varieties of λ-lines of generic hypersurfaces. In particular, we prove that the d-dependency of homogeneous $\{a, b\}$-degree f parts of $c^{SM}(Y_\lambda(d))$ is polynomial for large enough d's: $c^{SM}(Y_\lambda(d))_f \in \mathbb{Q}[a, b; d]^{S_2}$, the ring of polynomials that are symmetric in variables a and b.

To prove the above polynomiality, we need to show that the CSM class of the ambient vector space $\mathrm{Pol}^d(\mathbb{C}^2)$ also has this property. In Section 3.2 we investigate $c^{SM}(\mathrm{Pol}^d(\mathbb{C}^n))$ for all n's, not necessarily 2. This is motivated by the fact that certain coefficients of these classes provide solutions to a different kind of enumerative question: In Section 3.2.4, following [**Man99**], we use $c^{SM}(\mathrm{Pol}^d(\mathbb{C}^n))$ to examine degrees of varieties of hypersurfaces containing linear subspaces.

In Section 3.3 we take a slight detour to show how the characteristic polynomials and the CSM classes of hyperplane arrangements are related. This can serve as an ilustration of the motivic property, and will also be used to give a formula for $c^{SM}(Y_\emptyset(d))$.

Section 3.4 is the CSM analog of Section 2.3: Using the same branched covering, we arrive at a formula for $c^{SM}(Y_\lambda(d))$ that allows us to calculate its homogeneous degree f parts, $c^{SM}(Y_\lambda(d))_f$ recursively.

In Section 3.5 we describe a fibered resolution that, combined with results from Section 3.3 for some Weyl type arrangements of hyperplanes, provides a non-recursive formula for $c^{SM}(Y_\emptyset(d))$. This section is independent from the rest of the chapter.

Next, we return to our main objective. Section 3.6 is devoted to the proof of the aforementioned polynomial property of $c^{SM}(Y_\lambda(d))_f$. We list some our conjectures as well as unpublished results of Balázs Kőműves about the threshold from where this polynomial dependence holds.

Finally, in Section 3.7 we turn back to generic hypersurfaces and see how CSM classes of CRS can be used to calculate CSM classes of varieties of λ-lines. We introduce a non-degenerate pairing on $H^*(\mathrm{Gr}_2(\mathbb{C}^n))$, and use this to show that the class $c^{SM}(\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n))$ is equivalent to Euler characteristics of generic Schubert cell sections $\chi(\mathcal{T}_\lambda Z_f \cap \Omega_\mu)$. This then implies that these Euler characteristics also form a polynomial in the degree of the hypersurfaces.

3.2. Polynomial behaviour of coefficients in $c^{SM}(\mathrm{Pol}^d(\mathbb{C}^n))$

The standard $\mathrm{GL}(n)$ action on $\mathbb{C}^n$ induces a $\mathrm{GL}(n)$-representation on $\mathrm{Pol}^d(\mathbb{C}^n) \cong \mathrm{Sym}^d(\mathbb{C}^n)^\vee$. The weights of this representation are $d_1x_1 + \cdots + d_nx_n$ with $\sum_{i=1}^n d_i = d$, where $x_i = -a_i = -c_1\left(E\,\mathbb{T}^n \times_{\pi_i} \mathbb{C}\right)$, see also Section 2.3.3. Using motivic calculus, we immediately see that the

$\mathrm{GL}(n)$-equivariant Chern class is

$$c(\mathrm{Pol}^d(\mathbb{C}^n)) = \prod_{\substack{(d_1,\dots,d_n)\\ d_1+\dots+d_n=d}} (1 + d_1x_1 + \dots + d_nx_n)\,. \tag{24}$$

Throughout this section we will consider the parameter n as fixed. Then for every d the class $c(\mathrm{Pol}^d(\mathbb{C}^n))$ is a symmetric polynomial of degree $d+1$ in variables $x_1,\dots,x_n$, hence can be written in the Schur polynomial basis. Some of the coefficients in this Schur polynomial basis have geometric interpretations, see Section 3.2.4, hence we are mostly interested in the d-dependence of them.

However, d-dependence in the monomial symmetric basis seems to be easier to tackle, therefore we will start by looking at the coefficients of monomials $x^H = \prod_{i=1}^n x_i^{h_i}$, and then make the transition to the Schur polynomial basis. In particular, in Section 3.2.2 we will prove that

THEOREM 3.2.1. *For every multi-index $H = (h_1,\dots,h_n)$, the coefficients $a_H(d)$ of $x^H = \prod_{i=1}^n x_i^{h_i}$ in*

$$c(\mathrm{Pol}^d(\mathbb{C}^n)) = \prod_{\substack{(d_1,\dots,d_n)\\ d_1+\dots+d_n=d}} 1 + d_1x_1 + \dots + d_nx_n = \sum_H a_H(d)x^H$$

form a polynomial $a_H \in \mathbb{Q}[d]$ whose leading term is

$$\frac{1}{H!}\left(\frac{1}{n!}\right)^{|H|} d^{n|H|},$$

where $H! = h_1!\dots h_n!$.

We can subdivide $c(\mathrm{Pol}^d(\mathbb{C}^n))$ into $n-1$ products

$$\prod_{d_1=0}^{d}\cdots\prod_{d_i=0}^{d-d_1-\dots-d_{i-1}}\cdots\prod_{d_{n-1}=0}^{d-d_0-\dots-d_{n-2}} (1 + d_1x_1 + \dots + d_{n-1}x_{n-1} + (d - d_1 - \dots - d_{n-1})x_n)\,.$$

This has the benefit of reducing the indices in (24) to a single d_i. The downside is that at each product its terms contain the remaining "parameters" $d, d_1,\dots,d_{i-1}$. In the following Section 3.2.1, to be able to treat all these $n-1$ products in a universal way, we formulate a suitable generalization of our situation. Results there will provide the induction step in the proof of Theorem 3.2.1.

3.2.1. Polynomiality in certain one parameter products. Suppose that the formal power series

$$P(d_0,\dots,d_r,t,x) = \sum_J p_J(d_0,\dots,d_r,t)x^J \in \mathbb{Q}\left[d_0,\dots,d_r,t\right][[x]]$$

satisfies $P(d_0,\dots,d_r,t,0) = 1$, where $x = (x_1,x_2,\dots)$ and we use the multi-index notation $x^J = \prod_i x_i^{J_i}$. Let $K : \mathbb{N}^{r+1} \to \mathbb{N}$ be a function on the "parameter space" and define

$$\prod_{t=0}^{K(d_0,\dots,d_r)} P(d_0,\dots,d_r,t,x) = \sum_H a_H(d_0,\dots,d_r)x^H. \tag{25}$$

The goal of this section is to give a description of the coefficients $a_H(d_0,\dots,d_r)$, investigate their polynomiality and possibly their degree. To make our formulas more concise, we sometimes in this section abbreviate by d the sequence of "parameters" $d_0,\dots,d_r$.

Let us further introduce coefficients in the expansion

$$p_J(d,t) = \sum_{m\in I_P(J)} p_{J,m}(d)t^m,$$

where we assume that $p_{J,m}(d) \neq 0$ for all $m \in I_P(J)$. We are going to express $a_H(d)$ in terms of these $p_{J,m}(d)$'s and substitutes of monomial symmetric polynomials

$$m_\lambda(y_0, y_1, \dots, y_{K(d)})\big|_{y_i=i} = m_\lambda(y_1, \dots, y_{K(d)})\big|_{y_i=i}.$$

According to the following proposition such an expression can provide polynomiality:

PROPOSITION 3.2.2. *For each partition λ, there exists a polynomial $M_\lambda \in \mathbb{Q}[v]$ of degree $|\lambda| + l(\lambda)$ such that for every $v \geq 0$*

$$M_\lambda(v) = m_\lambda(y_1, \dots, y_v)|_{y_i=i}, \tag{26}$$

where, by definition, $m_\lambda(y_1, \dots, y_v) = 0$ if $l(\lambda) > v$.

For $\lambda = (1^{e_1}, \dots, k^{e_k})$ the leading term of M_λ is

$$\frac{1}{e_1! \dots e_k!} \prod_i \left(\frac{1}{\lambda_i + 1}\right) v^{|\lambda|+l(\lambda)} = \frac{1}{e_1! \dots e_k!} \prod_i \frac{v^{\lambda_i+1}}{\lambda_i + 1}.$$

E.g. for the partition $\lambda = (2, 1, 1)$

$$M_{(2,1,1)}(v) = \frac{1}{720} v\,(v-1)\,(v-2)\,(v+1)\left(30\,v^3 + 35\,v^2 - 11\,v - 12\right)$$

is divisible by $v(v-1)(v-2)$ corresponding to the cases when $v < l(\lambda)$.

PROOF. For every $\lambda = (1^{e_1}, \dots, k^{e_k})$ the monomial symmetric polynomial m_λ can be expressed as a polynomial of power sum symmetric polynomials p_k:

$$m_\lambda = \frac{1}{e_1! \dots e_k!} p_\lambda + \sum_{\mu\in\partial\lambda} c_\mu p_\mu$$

for some $c_\mu \in \mathbb{Q}$, where $\partial\lambda$ denotes the set of partitions coming from λ with at least two of its elements merged and $p_\eta = p_{\eta_1} \dots p_{\eta_l}$ for $\eta = (\eta_1, \dots, \eta_l)$. The $y_i = i$ substitute of the RHS is, e.g. by Faulhaber's formula, a polynomial in v whose highest, $|\lambda| + l(\lambda)$ degree part comes from the λ summand and has leading term

$$\frac{1}{e_1! \dots e_r!} \frac{1}{\prod_i(\lambda_i + 1)} v^{|\lambda|+l(\lambda)}.$$

$\square$

For any exponent $H \in \mathbb{N}^\infty$ denote by

$$\mathcal{P}(H) = \left\{ \underline{J} = \left(J_1, J_2, \dots, J_{l(\underline{J})}\right) \big|\, 0 \neq J_s \in \mathbb{N}^\infty, \sum J_s = H \text{ and } J_1 \geq J_2 \geq \dots \right\}$$

the set of partitions of H, where $\geq$ denotes some (e.g. lexicographical) ordering of $\mathbb{N}^\infty$. For example,

$$\begin{aligned}\mathcal{P}\left((2,0,1,0,\dots)\right) = \big\{ & \left((2,0,1,0,\dots)\right), \\ & \left((2,0,0,0,\dots),(0,0,1,0,\dots)\right), \\ & \left((1,0,1,0,\dots),(1,0,0,0,\dots)\right), \\ & \left((1,0,0,0,\dots),(1,0,0,0,\dots),(0,0,1,0,\dots)\right)\big\}.\end{aligned}$$

Vectors of size 1 will be important, let us denote by 1_i in which the 1 is at the i-th place.

Define

$$\mu(\underline{J}) = \prod e_i!$$

for e_i's the multiciplities of vectors in $\underline{J}$. For example,

$$\mu\left(((1,0,0,0,\dots),(1,0,0,0,\dots),(0,0,1,0,\dots))\right) = 2!1!\,.$$

Finally, let

$$(l \hookrightarrow [v]) = \{\, f : \{1,\dots,l\} \to \{0,1,\dots,v\} \,|\, f \text{ injective}\}\,.$$

Using the above notations, we can express $a_H(d)$ as

$$a_H(d) = \sum_{\underline{J}\in\mathcal{P}(H)} \underbrace{\frac{1}{\mu(\underline{J})} \sum_{c\in(l(\underline{J})\hookrightarrow[K(d)])} \prod_{s=1}^{l(\underline{J})} p_{J_s}(d,c(s))}_{\underline{J}\text{-contribution}}\,. \tag{27}$$

To shorten our future formulas, let us extend definition (26) from partitions to (unordered) tuples possibly containing zeros: For $\lambda = (0^{e_0}, 1^{e_1}, \dots, k^{e_k})$ let

$$M_\lambda(v) := \frac{1}{e_1!\cdots e_k!} \sum_{c\in(l(\lambda)\hookrightarrow[v])} \prod_{s=1}^{l(\lambda)} c(s)^{\lambda_s}, \tag{28}$$

where $l(\lambda) = \sum_{i=0}^k e_i$ denotes the length of the tuple. Note that

$$M_\lambda(v) = \left(v+1-l(\lambda^\times)\right)\dots\left(v+1-(l(\lambda)-1)\right) M_{\lambda^\times}(v)$$

holds for every $v \geq 0$, where $\lambda^\times$ stands for the nonzero, ordered part of λ, e.g. $(1,0,0,2,0,1)^\times = (2,1,1)$. This shows that (28) is indeed a generalization of (26) with analogous leading term:

$$\frac{1}{e_1!\dots e_k!} \prod_i \left(\frac{1}{\lambda_i+1}\right) v^{|\lambda|+l(\lambda)} = \frac{1}{e_1!\dots e_k!} \prod_i \frac{v^{\lambda_i+1}}{\lambda_i+1}. \tag{29}$$

Note, however, that contrary to what is the case for partitions in Proposition 3.2.2, for a general tuple λ we only have

$$M_\lambda(v) = 0 \text{ if } v < l(\lambda)-1. \tag{30}$$

Then for a fixed $\underline{J} \in \mathcal{P}(E)$ we can write the corresponding $\underline{J}$-contribution of (27) as

$$\begin{aligned}
&\frac{1}{\mu(\underline{J})} \sum_{c\in(l(\underline{J})\hookrightarrow[K(d)])} \prod_{s=1}^{l(\underline{J})} p_{J_s}(d,c(s)) = \frac{1}{\mu(\underline{J})} \sum_{c\in(l(\underline{J})\hookrightarrow[K(d)])} \prod_{s=1}^{l(\underline{J})} \sum_{m\in I_P(J_s)} p_{J_s,m}(d)c(s)^m = \\
&\frac{1}{\mu(\underline{J})} \sum_{c\in(l(\underline{J})\hookrightarrow[K(d)])} \sum_{\substack{\lambda=\left(\lambda_1,\dots,\lambda_{l(\underline{J})}\right)\\ \lambda_s\in I_P(J_s)}} \prod_{s=1}^{l(\underline{J})} p_{J_s,\lambda_s}(d)c(s)^{\lambda_s} = \\
&\frac{1}{\mu(\underline{J})} \sum_{\substack{\lambda=\left(\lambda_1,\dots,\lambda_{l(\underline{J})}\right)\\ \lambda_s\in I_P(J_s)}} \prod_{s=1}^{\underline{J}} (p_{J_s,\lambda_s}(d)) \sum_{c\in(l(\underline{J})\hookrightarrow[K(d)])} \prod_{s=1}^{l(\underline{J})} c(s)^{\lambda_s} = \\
&\frac{1}{\mu(\underline{J})} \sum_{\substack{\lambda=\left(\lambda_1,\dots,\lambda_{l(\underline{J})}\right)\\ =(0^{e_0},1^{e_1},\dots,k^{e_k})\\ \lambda_s\in I_P(J_s)}} \underbrace{e_1!\dots e_k! M_\lambda(K(d)) \prod_{s=1}^{l(\underline{J})} p_{J_s,\lambda_s}(d)}_{\lambda-\text{contribution}}\,.
\end{aligned} \tag{31}$$

If e.g. $K \in \mathbb{Q}[d_0,\dots,d_r]$, then the substitutes $M_\lambda(K(d_0,\dots,d_r))$ will also be polynomials, proving

PROPOSITION 3.2.3. *Suppose that the formal power series $P(d_0,\dots,d_r,t,x) \in \mathbb{Q}[d_0,\dots,d_r,t][[x]]$ satisfies $P(d_0,\dots,d_r,t,0)=1$, where $x=(x_1,x_2,\dots)$. Let*

$$\prod_{t=0}^{K(d_0,\dots,d_r)} P(d_0,\dots,d_r,t,x) = \sum_H a_H(d_0,\dots,d_r)x^H,$$

where $K(d_0,\dots,d_r) \in \mathbb{Q}[d_0,\dots,d_r]$. Then coefficients $a_H(d_0,\dots,d_r) \in \mathbb{Q}[d_0,\dots,d_r]$.

We can use (31) to determine the degree of $a_H(d)$ by comparing for all $\underline{J} = \big(J_1,\dots,J_{l(\underline{J})}\big) \in \mathcal{P}(H)$ and for all tuples $\lambda = (\lambda_1,\dots,\lambda_{l(\underline{J})})$, $\lambda_s \in I_P(J_s)$ the degrees of the corresponding λ-contributions,

$$\deg\left(M_\lambda(K(d))\prod_{s=1}^{l(\underline{J})} p_{J_s,\lambda_s}(d)\right) \overset{(29)}{=} \deg(K)(l(\underline{J})+|\lambda|) + \sum_{s=1}^{l(\underline{J})}\deg(p_{J_s,\lambda_s}) = \sum_{s=1}^{l(\underline{J})}\deg(K)(1+\lambda_s)+\deg(p_{J_s,\lambda_s}). \tag{32}$$

Note that by "degree" we mean the total degree. We use $\deg_z$ or z-degree for degree with respect to a specific variable z.

In the cases of our interest we will have a particularly simple relationship between degree of the coefficients $p_J(d,t)$ and $a_H(d)$:

PROPOSITION 3.2.4. *Suppose that the formal power series*

$$P(d_0,\dots,d_r,t,x) = \sum_J p_J(d_0,\dots,d_r,t)x^J \in \mathbb{Q}[d_0,\dots,d_r,t][[x]]$$

satisfies $P(d_0,\dots,d_r,t,0)=1$, where $x=(x_1,x_2,\dots)$. Let $L(d_0,\dots,d_r)$ be a linear combination of the d_i's and define

$$\prod_{t=0}^{L(d_0,\dots,d_r)} P(d_0,\dots,d_r,t,x) = \sum_H a_H(d_0,\dots,d_r)x^H.$$

If there is linear form $W:\mathbb{Z}^\infty \to \mathbb{Z}$ such that for every exponent $J \in \mathbb{N}^\infty$, $\deg(p_J) \le W(J)$, then

i) $\deg(a_H) \le W(H)+|H|$,

ii) the degree $W(H)+|H|$ part of a_H comes from the

$$\prod_{t=0}^{L(d_0,\dots,d_r)}\left(1+\sum_{i\ge 1} p_{1_i}(d_0,\dots,d_r,t)x_i\right)$$

summand of $\prod_t P(d_0,\dots,d_r,t,x)$.

PROOF. Let $p_J(d_0,\dots,d_r,t) = \sum_m p_{J,m}(d_0,\dots,d_r)t^m$ as usual. The condition $\deg(p_J) \le W(J)$ and the linearity of L imply by (32) that for every exponent $H=(h_1,\dots,h_n)$ and every $\underline{J} \in \mathcal{P}(H)$

$$\deg(\underline{J}\text{-contribution}) \le \max_{\lambda=(\lambda_1,\dots,\lambda_{l(\underline{J})})} \sum_{s=1}^{l(\underline{J})} 1+\lambda_s+\deg\left(p_{J_s,\lambda_s}\right) \le \max_{\lambda=(\lambda_1,\dots,\lambda_{l(\underline{J})})} \sum_{s=1}^{l(\underline{J})} 1+\lambda_s+W(J_s)-\lambda_s = l(\underline{J})+W(H) \le W(H)+|H|,$$

showing i). $\mathcal{P}(H)$ has a single partition

$$\underline{J}_{\max} = \left(1_1^{h_1}, \dots, 1_n^{h_n}\right)$$

of length $|H|$, and hence for which the degree of its contribution can reach $W(H) + |H|$. This proves ii). $\square$

3.2.2. Coefficients of $c(\mathrm{Pol}^d(\mathbb{C}^n))$ in the monomial symmetric basis. In this section we complete the proof of Theorem 3.2.1 about the d-dependence of coefficients of $c(\mathrm{Pol}^d(\mathbb{C}^n))$ in the monomial symmetric polynomial basis.

Recall that $c(\mathrm{Pol}^d(\mathbb{C}^n))$ can be written as

$$\prod_{d_1=0}^{d} \cdots \underbrace{\prod_{d_i=0}^{d-d_1-\dots-d_{i-1}} \cdots \prod_{d_{n-1}=0}^{d-d_0-\dots-d_{n-2}} 1 + d_1 x_1 + \dots + d_{n-1}x_{n-1} + (d - d_1 - \dots - d_{n-1})\, x_n}_{P_{n-i}(d,d_1,\dots,d_{i-1},x)} .$$

As mentioned above, the proof relies on the iterative application of the approach described in Section 3.2.1: Let

$$\begin{aligned} P_0(d, d_1, \dots, d_{n-1}, x) &= 1 + d_1 x_1 + \dots + d_{n-1}x_{n-1} + (d - d_1 - \dots - d_{n-1})\, x_n \\ &= \sum_H a_{0,H}(d, d_1, \dots, d_{n-1}) x^H. \end{aligned}$$

Then for each $i = 1, \dots, n-1$

$$\begin{aligned} P_i(d, d_1, \dots, d_{n-i-1}, x) &= \prod_{d_{n-i}=0}^{d-d_1-\dots-d_{n-i-1}} P_{i-1}(d, d_1, \dots, d_{n-i}, x) \\ &= \sum_H a_{i,H}(d, d_1, \dots, d_{n-i-1}) x^H, \end{aligned} \tag{33}$$

is of the form (25) with $t = d_{n-i}$, $P = P_{i-1}$, parameters $d, d_1, \dots, d_{n-i}$ and $K(d, d_1, \dots) = d - d_1 - \dots - d_{n-i-1}$, the upper bound of the product. Therefore, the inductive usage of Proposition 3.2.3 gives that coefficients $a_{i,H}(d, d_1, \dots, d_{n-i-1}) \in \mathbb{Q}[d, d_1, \dots, d_{n-i-1}]$.

Since $\deg(a_{0,J}) = |J|$, successive application of Proposition 3.2.4 also shows that

$$\deg(a_{i,H}(d)) \leq i|H| + |H| = (i+1)|H|.$$

The last, $i = n-1$ step proves that $a_H(d) = a_{n-1,H}(d) \in \mathbb{Q}[d]$ and that $\deg(a_H(d)) \leq n|H|$.

To prove that this degree estimate is sharp, we calculate, by a series of reductions, that for every exponent $H = (h_1, \dots, h_n)$

$$a_H(d) = \frac{1}{H!}\left(\frac{1}{n!}\right)^{|H|} d^{n|H|} + \text{ (lower degree terms)}.$$

Throughout our calculations we will use that for every $n \geq 1$

$$\sum_{d_1=0}^{d} \sum_{d_2=0}^{d-d_1} \cdots \sum_{d_n=0}^{d-d_1-\dots-d_{n-1}} 1 = \binom{d+n}{n} \tag{34}$$

and that for every $n \geq 0, k \geq 1$

$$\frac{1}{n!} \sum_{t=0}^{n} \binom{n}{t} (-1)^t \frac{1}{t+k} = \frac{(k-1)!}{(n+k)!}. \tag{35}$$

Both equations can be proved using induction on n.

According to ii) of Proposition 3.2.4 the degree $n|H|$ part of $a_H(d)$ comes from the

$$\prod_{d_1=0}^{d}\left(1+\sum_{i=1}^{n} a_{n-2,1_i}(d,d_1)x_i\right)$$

summand of $\prod_{d_1=0}^{d} P_{n-2}$. Because of symmetry reasons and then straightforward applications of (34) and (35), this is equal to

$$\prod_{d_1=0}^{d}\left(1+\sum_{d_2=0}^{d-d_1}\cdots\sum_{d_{n-1}=0}^{d-d_1-\cdots-d_{n-2}}(d_1)\,x_1+\sum_{d_2=0}^{d-d_1}\cdots\sum_{d_{n-1}=0}^{d-d_1-\cdots-d_{n-2}}(d_2)\,(x_2+\cdots+x_n)\right)\overset{(34)}{=}$$
$$\prod_{d_1=0}^{d}\left(1+\left(\binom{d-d_1+n-2}{n-2}d_1\right)x_1+\left(\sum_{d_2=0}^{d-d_1}\binom{d-d_1-d_2+n-3}{n-3}d_2\right)(x_2+\cdots+x_n)\right)\overset{(35)}{=}$$
$$\prod_{d_1=0}^{d}\Bigg(1+\left(\frac{(d-d_1)^{n-2}d_1}{(n-2)!}+\text{ (lower }\{d,d_1\}\text{-degree terms)}\right)x_1+$$
$$\left(\frac{(d-d_1)^{n-1}}{(n-1)!}+\text{ (lower }\{d,d_1\}\text{-degree terms)}\right)(x_2+\cdots+x_n)\Bigg).$$

From this we see that the leading terms of the $a_H(d)$'s and the leading term of the $a'_H(d)$'s in the much simpler

$$\prod_{d_1=0}^{d}\Bigg(\underbrace{1+\frac{(d-d_1)^{n-2}d_1}{(n-2)!}x_1+\frac{(d-d_1)^{n-1}}{(n-1)!}(x_2+\cdots+x_n)}_{P'(d,d_1,x)}\Bigg)=\sum_H a'_H(d)x^H$$

are the same.

Let us denote coefficients in the expansion of $P'(d,d_1,x)$ as usual: $P'(d,d_1,x)=\sum_J p'_J(d,d_1)x^J=\sum_J\left(\sum_m p'_{J,m}d_1^m\right)x^J$,

$$P_{J_s,m}(d)=\begin{cases}\frac{1}{(n-2)!}\binom{n-2}{m-1}(-1)^{m-1}d^{n-1-m} & \text{if } s=1,\dots,h_1\\ \frac{1}{(n-1)!}\binom{n-1}{m}(-1)^{m}d^{n-1-m} & \text{otherwise.}\end{cases}\qquad(36)$$

Fix an exponent $H = (h_1, \dots, h_n)$. Once again, we can use (27) to express $a'_H(d)$ as a single $\underline{J} = (1_1^{h_1}, \dots, 1_n^{h_n})$-contribution by substituting (29) and (36) into (31). We get that

$$a'_H(d) = \frac{1}{H!}\sum_{\lambda}\left(\prod_{s=1}^{l(\underline{J})}\left(\frac{d^{\lambda_s+1}}{\lambda_s+1}\right) + \text{ (lower degree terms)}\right)\prod_{s=1}^{l(\underline{J})} p_{J_s,\lambda_s}(d) =$$

$$\frac{1}{H!}\sum_{\lambda}\left(\left(\prod_{s=1}^{l(\underline{J})}\frac{d^{\lambda_s+1}}{\lambda_s+1}p_{J_s,\lambda_s}(d)\right) + \text{ (lower degree terms)}\right) =$$

$$\frac{1}{H!}\prod_{s=1}^{l(\underline{J})}\left(\sum_{m=0}^{n-1}\frac{d^{m+1}p_{J_s,m}(d)}{m+1}\right) + \text{ (lower degree terms)} \overset{(36)}{=}$$

$$\frac{1}{H!}\left(\frac{d^n}{(n-2)!}\sum_{m=1}^{n-1}\binom{n-2}{m-1}\frac{(-1)^{m-1}}{m+1}\right)^{h_1}\left(\frac{d^n}{(n-1)!}\sum_{m=0}^{n-1}\binom{n-1}{m}\frac{(-1)^m}{m+1}\right)^{h_2+\dots+h_n} + \text{(l. d. t.)} \overset{(35)}{=}$$

$$\frac{1}{H!}\left(\frac{1}{n!}\right)^{h_1+\dots+h_n} d^{n|H|} + \text{ (lower degree terms)}.$$

3.2.3. Coefficients of $c(\mathrm{Pol}^d(\mathbb{C}^n))$ in the Schur polynomial basis. In this section we want to convert our results to the Schur polynomial basis; namely, our goal is to deduce

THEOREM 3.2.5. *Let $b_\lambda(d)$ denote the coefficients in the Schur polynomial expansion*

$$c(\mathrm{Pol}^d(\mathbb{C}^n)) = \sum_{\lambda} b_\lambda(d)s_\lambda.$$

Then the leading term of $b_\lambda(d)$ is

$$\frac{\prod_{1\le i<j\le n}(\lambda_i-\lambda_j+j-i)}{(\lambda_1+n-1)!(\lambda_2+n-2)!\dots\lambda_n!}\left(\frac{1}{n!}\right)^{|\lambda|}d^{n|\lambda|}.$$

PROOF. Schur polynomials can be defined as

$$s_\lambda(x_1,\dots,x_n) = \frac{A^{\lambda+\delta}}{A^\delta},$$

where $\delta = (n-1,\dots,1,0)$ and for a partition $\mu = (\mu_1,\dots,\mu_n)$ A^μ denotes the determinant of the n-by-n matrix,

$$A^\mu = \left|\left(x_j^{\mu_i}\right)_{i,j}\right| = \sum_{\rho\in S_n}\mathrm{sgn}(\rho)x^{\rho(\mu)}.$$

Note that if the partition μ is strictly decreasing, then A^μ has a single term, corresponding to $\rho = 1$, where the exponent $(\rho(\mu_1),\dots,\rho(\mu_n))$ of x is decreasing.

Multiplying $c(\mathrm{Pol}^d(\mathbb{C}^n))$ by the Vandermond determinant A^δ, we get that

$$A^\delta c(\mathrm{Pol}^d(\mathbb{C}^n)) = \sum_{\lambda} b_\lambda(d)A^{\lambda+\delta} = \sum_{\lambda} b_\lambda(d)\sum_{\rho\in S_n}\mathrm{sgn}(\rho)x^{\rho(\lambda+\delta)} =$$

$$\sum_{\lambda} b_\lambda(d)x^{\lambda+\delta} + \sum_{\lambda}\sum_{1\neq\rho\in S_n}\mathrm{sgn}(\rho)b_\lambda(d)x^{\rho(\lambda+\delta)},$$

where only the first summand of the RHS contains terms with (strictly) decreasing exponents. We also have

$$A^\delta c(\mathrm{Pol}^d(\mathbb{C}^n)) = \sum_{H}\sum_{\rho\in S_n}\mathrm{sgn}(\rho)a_H(d)x^{H+\rho(\delta)}.$$

Collecting terms with strictly decreasing exponents, we deduce that

$$b_\lambda(d) = \sum_{\substack{(H,\rho)\\ H+\rho(\delta)=\lambda+\delta}} \mathrm{sgn}(\rho) a_H(d). \tag{37}$$

Combining Theorem (3.2.1) and (37), we see that

$$b_\lambda(d) = \underbrace{\sum_{\substack{(H,\rho)\\ H+\rho(\delta)=\lambda+\delta}} \left(\mathrm{sgn}(\rho)\frac{1}{H!}\right)}_{M(\lambda)} \left(\frac{1}{n!}\right)^{|\lambda|} d^{n|\lambda|} + (\text{lower degree terms}).$$

The following proposition provides an exact formula for the term $M(\lambda)$. This formula immediately implies that $M(\lambda) > 0$ for every partition λ, and hence that the leading term of $b_\lambda(d)$ is

$$M(\lambda)\left(\frac{1}{n!}\right)^{|\lambda|} d^{n|\lambda|}.$$

PROPOSITION 3.2.6. *Let $\delta = (n-1,\dots,1,0)$. Then for any partition $\lambda = (\lambda_1,\dots,\lambda_n)$*

$$M(\lambda) = \sum_{\substack{(H,\rho)\\ H+\rho(\delta)=\lambda+\delta}} \left(\mathrm{sgn}(\rho)\frac{1}{H!}\right) = \frac{\prod_{1\le i<j\le n}(\lambda_i-\lambda_j+j-i)}{(\lambda_1+n-1)!(\lambda_2+n-2)!\dots\lambda_n!}. \tag{38}$$

PROOF. For any vector $K = (k_1,\dots,k_n)$ write $K \geq 0$, if for all its coordinates $k_i \geq 0$. Then, as $\rho(\delta)_i = n - \rho(i)$,

$$\sum_{\substack{(H,\rho)\\ H+\rho(\delta)=\lambda+\delta}} \left(\mathrm{sgn}(\rho)\frac{1}{H!}\right) = \sum_{\substack{\rho\in S_n\\ \lambda+\delta-\rho(\delta)\ge 0}} \mathrm{sgn}(\rho)\frac{1}{(\lambda+\delta-\rho(\delta))!} =$$

$$\sum_{\substack{\rho\in S_n\\ \lambda+\delta-\rho(\delta)\ge 0}} \mathrm{sgn}(\rho)\prod_{i=1}^{n}\frac{1}{(\lambda_i+\rho(i)-i)!} = |\gamma(i,j)_{i,j}|,$$

a determinant $|\gamma(i,j)_{i,j}|$ of the n-by-n matrix with entries

$$\gamma(i,j) = \frac{1}{\Gamma(\lambda_i+j-i+1)} = \begin{cases} \frac{1}{(\lambda_i+j-i)!} & \text{if } \lambda_i+j-i\geq 0\\ 0 & \text{otherwise.}\end{cases}$$

Factoring out $\frac{1}{(\lambda_i+n-i)!}$ from the i-th row, we get that

$$|\gamma(i,j)| = \frac{1}{(\lambda_1+n-1)!(\lambda_2+n-2)!\dots\lambda_n!}\left|(\lambda_i+j-i+1)\dots(\lambda_i+n-i)_{i,j}\right|.$$

By expanding its entries and then using multilinearity, the latter determinant can be further written as

$$|(\lambda_i-i+j+1)\dots(\lambda_i-i+n)_{i,j}| = \left|\left(\sum_{t=0}^{n-j}(\lambda_i-i)^t\sigma_{n-j-t}(j+1,\dots,n)\right)_{i,j}\right| = \sum_{\substack{J=(J_1,\dots,J_n)\\ 0\le J_j\le n-j}} |W_J|,$$

where σ_k denotes the k-th elementary symmetric polynomial and $|W_J|$ stands for the determinant

$$|W_J| = \left|(\lambda_i - i)^{J_j}\sigma_{n-j-J_j}(j+1,\dots,n)_{i,j}\right| = \prod_{j=1}^{n}\left(\sigma_{n-j-J_j}(j+1,\dots,n)\right)\left|\left((\lambda_i-i)^{J_j}\right)_{i,j}\right| =$$

$$\begin{cases}\left|\left((\lambda_i-i)^{n-j}\right)_{i,j}\right| = \prod_{1\le i<j\le n}(\lambda_i - i)-(\lambda_j-j) & \text{if } J=(n-1,\dots,0)\\ 0 & \text{otherwise.}\end{cases}$$

Putting everything together, we get (38). $\square$

Since all the terms $\lambda_i - \lambda_j + j - i$ in (38) are positive, $M(\lambda) > 0$ for every partition λ. $\square$

3.2.4. Degree of varieties of hypersurfaces containing linear subspaces. Manivel in [**Man99**] showed that for rectangular partitions coefficients of corresponding Schur polynomials in $c(\mathrm{Pol}^d(\mathbb{C}^n))$ can be given a geometric interpretation as degrees of certain subvarieties $\Sigma(d,m,k) \subset \mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^{m+1}))$. In this section, following [**CZ20**], we define these subvarieties and explain how Manivel's formula for $\deg(\Sigma(d,m,k) \subset \mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^{m+1})))$ translates to our equivariant setting.

The set of hypersurfaces $Z_f \subset \mathbb{P}^m$ is parametrized by $\mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^{m+1}))$. For each k and $f \in \mathrm{Pol}^d(\mathbb{C}^{m+1})$ the *Fano variety of k-planes $F_k(Z_f)$ of the projective variety Z_f* is, by definition, the variety of k-planes that are contained in Z_f. As usual, we will treat this variety as a subset of $\mathrm{Gr}_{k+1}(\mathbb{C}^{m+1})$, and identify it as the zero locus of the section $\sigma_f(W) = f|_W$ of the rank $\binom{d+k}{k}$ vector bundle $\mathrm{Pol}^d(S) \to \mathrm{Gr}_{k+1}(\mathbb{C}^{m+1})$. For a generic $f \in \mathrm{Pol}^d(\mathbb{C}^{m+1})$, σ_f is transversal to the zero section, hence $F_k(Z_f)$ is a

$$\delta(d,m,k) = (k+1)(m-k) - \binom{d+k}{k}$$

dimensional subvariety.

This also shows that if $\delta(d,m,k) < 0$ and $f \in \mathrm{Pol}^d(\mathbb{C}^{m+1})$ is generic, then $F_k(Z_f) = \emptyset$. Denote by

$$\Sigma(d,m,k) \subset \mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^{m+1}))$$

the subvariety whose points correspond to degree $d \ge 3$ hypersurfaces that do contain a k-plane. Then $\Sigma(d,m,k)$ is an irreducible subvariety of codimension $-\delta(d,m,k)$ in $\mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^{m+1}))$, its generic point corresponds to a hypersurface that carries a unique k-plane and its degree is

$$\deg(\Sigma(d,m,k)) = \int_{\mathrm{Gr}_{k+1}(\mathbb{C}^{m+1})} c_{(k+1)(m-k)}(\mathrm{Sym}^d(S^\vee)), \tag{39}$$

where $S \to \mathrm{Gr}_{k+1}(\mathbb{C}^{m+1})$ denotes the tautologival bundle, see [**Man99**].

By definition, the $\mathrm{GL}(n)$-equivariant Chern class $c(\mathrm{Pol}^d(\mathbb{C}^n))$ is the Chern class of the Borel construction $\mathrm{B}_{\mathrm{GL}(n)}\mathrm{Pol}^d(\mathbb{C}^n) \to \mathrm{B}\,\mathrm{GL}(n) \sim \mathrm{Gr}_n(\mathbb{C}^\infty)$. Setting $n = k+1$, the vector bundle $\mathrm{Sym}^d(S^\vee) \cong \mathrm{Pol}^d(S) \to \mathrm{Gr}_{k+1}(\mathbb{C}^{m+1})$ is just the restriction of $\mathrm{B}_{\mathrm{GL}(k+1)}\mathrm{Pol}^d(\mathbb{C}^{k+1})$, therefore

$$c(\mathrm{Sym}^d(S^\vee)) = c(\mathrm{Pol}^d(\mathbb{C}^{k+1}))|_{c_i \mapsto c_i(S^\vee)},$$

where $c_i = \sigma_i(x_1,\dots,x_n)$ is the i-th Chern class of the dual of the tautological bundle over the infinite Grassmannian $\mathrm{Gr}_{k+1}(\mathbb{C}^\infty)$. As integration over $\mathrm{Gr}_{k+1}(\mathbb{C}^{m+1})$ returns the coefficient of the volume form $s_{(m-k)^{k+1}}$, we get that

$$\deg(\Sigma(d,m,k)) = \text{coefficient of } s_{(m-k)^{k+1}} \text{ in } c(\mathrm{Pol}^d(\mathbb{C}^{k+1})). \tag{40}$$

In [**Man99**] Manivel also claims without proof that fixing m and k, $\deg(\Sigma(d,m,k))$ depends on d as a degree $2(k+1)(m-k)$ polynomial. By looking at examples for k other than 2, one immediately sees that this is not the case. In fact, (40) combined with Theorem 3.2.5 implies that

PROPOSITION 3.2.7. *Let $\Sigma(d,m,k)$ denote the subvariety of $\mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^{m+1}))$ whose points correspond to degree $d \geq 3$ hypersurfaces in $\mathbb{P}^m$ that contain a k-plane. Fixing m and k, degrees $\deg(\Sigma(d,m,k))$ form a polynomomial in d , whose leading term is*

$$\frac{\prod_{1\leq i<j\leq k+1}(j-i)}{m!\dots(m-k)!}\left(\frac{1}{(k+1)!}\right)^{(k+1)(m-k)} d^{(k+1)^2(m-n)}.$$

EXAMPLE 3.2.8. For $m=3$ and $k=2$ Proposition 3.2.7 tells us that $\deg(\Sigma(d,3,2))$, the degree of the variety of degree d surfaces in $\mathbb{P}^3$ containing a projective plane, is a degree 9 polynomial in d. Then by e.g. interpolating its values for $d=\{3,\dots,13\}$, we get that

$$\deg(\Sigma(d,3,2)) = \frac{1}{1296} d\,(d+3)\,(d+2)\,(d+1)\left(d^2+2\right)\left(d^3+3\,d^2+2\,d+12\right).$$

REMARK 3.2.9. For degrees of varieties $\Sigma(d,m,m-1)$—such as the above example—we even have a closed formula, i.e. no interpolation is needed: The map

$$\varphi : \mathbb{P}(\mathrm{Pol}^1(\mathbb{C}^{m+1})) \times \mathbb{P}(\mathrm{Pol}^{d-1}(\mathbb{C}^{m+1})) \hookrightarrow \mathbb{P}(\mathrm{Pol}^d(\mathbb{C}^{m+1}))$$

induced by multiplication of polynomials provides a resolution of $\Sigma(d,m,m-1)$. Such a resolution can be used to calculate the degree as

$$\deg(\Sigma(d,m,m-1)) = \int_{\mathbb{P}(\mathrm{Pol}^1(\mathbb{C}^{m+1}))\times\mathbb{P}(\mathrm{Pol}^{d-1}(\mathbb{C}^{m+1}))} \varphi^* c_1^D,$$

where $D=\binom{d+m-1}{m}+m-1$ is the dimension of the domain and $c_1=c_1(\gamma^\vee)$ for γ the tautological line bundle. Let u and v denote the first Chern classes of the duals of tautological bundles over the first and second factor of the domain. As $\varphi^*\gamma^\vee$ is the tensor product of these dual bundles,

$$\varphi^*\left(c_1^D\right) = (u+v)^D = \sum_{t=0}^{D}\binom{D}{t}u^t v^{d-t}.$$

The above integration amounts to taking the coefficient of the volume form $u^m v^{D-m}$, hence substituting $D=\binom{d+m-1}{m}+m-1$, we get

$$\deg(\Sigma(d,m,m-1)) = \binom{\binom{d+m-1}{m}+m-1}{m}.$$

3.3. Scalar equivariant CSM classes and characteristic polynomials of hyperplane arrangements

Let $\mathrm{GL}(1)$ act on $\mathbb{C}$ and denote by $w:\mathrm{GL}(1)\to\mathbb{C}$ the weight of this representation. Every linear subspace of $\mathbb{C}^n$ is invariant under the direct sum of n copies of this representation. In particular, every (vector) hyperplane arrangement and its complement are $\mathrm{GL}(1)$-invariant. In this section we show that the characteristic polynomial of such a hyperplane arrangement $\mathcal{A}$ is essentially the same as the $\mathrm{GL}(1)$-equivariant CSM class $c^{SM}(\mathbb{C}^n\setminus\bigcup\mathcal{A})$.

Applying this result to Weyl arrangements will help us providing a formula for CSM classes of the open CRS, $c^{SM}(Y_\emptyset(d))$.

The characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^n$ is defined by

$$\chi(\mathcal{A},t)=\sum_{\substack{\mathcal{B}\subset\mathcal{A}\\ \cap\mathcal{B}\neq\emptyset}}(-1)^{|\mathcal{B}|}t^{\dim(\bigcap\mathcal{B})},$$

where we set $\bigcap\emptyset=\mathbb{C}^n$ ([**Sag99**]). Using the motivic and the multiplicative property of the CSM class, we will show that, similarly,

$$c^{SM}(\mathbb{C}^n\setminus\bigcup\mathcal{A})=\sum_{\substack{\mathcal{B}\subset\mathcal{A}\\ \cap\mathcal{B}\neq\emptyset}}(-1)^{|\mathcal{B}|}(1+w)^{\dim(\bigcap\mathcal{B})}w^{n-\dim(\bigcap\mathcal{B})}.$$

In other words,

Proposition 3.3.1. *Let $\mathcal{A}$ be a (vector) hyperplane arrangement in $\mathbb{C}^n$, denote by $\chi(\mathcal{A},t)$ its characteristic polynomial. Let $\mathrm{GL}(1)$ act on $\mathbb{C}$ with $w:\mathrm{GL}(1)\to\mathbb{C}$ the weight of this representation. Then $\mathbb{C}^n\setminus\bigcup\mathcal{A}$ is invariant under the induced action on $\mathbb{C}^n$ and its $\mathrm{GL}(1)$-equivariant CSM class can be computed as*

$$c^{SM}(\mathbb{C}^n\setminus\bigcup\mathcal{A})=\chi(\mathcal{A},t)|_{t^k\mapsto(1+w)^kw^{n-k}}\,.$$

Proof. First, using the motivic calculus of Lemma 3.1.3 we see that for any k-dimensional linear subspace

$$c^{SM}(\mathbb{C}^k\subset\mathbb{C}^n)=(1+w)^kw^{n-k}.$$

Denote by $\mathcal{I}$ the set of all possible nonempty proper intersections:

$$\mathcal{I}:=\left\{\bigcap\mathcal{B}\,\middle|\,\emptyset\neq\mathcal{B}\subset\mathcal{A},\bigcap\mathcal{B}\neq\emptyset\right\}.$$

For every $W\in\mathcal{I}$ count the number of subsets $\mathcal{B}\subset\mathcal{A}$ of size t whose intersection contains W:

$$s_t(W):=\left|\left\{\mathcal{B}\subset\mathcal{A}|\,|\mathcal{B}|=t,W\subset\bigcap\mathcal{B}\right\}\right|.$$

Then for any $W\in\mathcal{I}$, by the inclusion-exclusion principle,

$$1=s_1(W)-s_2(W)+\dots+(-1)^{|\mathcal{A}|+1}s_{|\mathcal{A}|}(W).$$

Finally, for any $W\in\mathcal{I}$ denote by $\mathring{W}$ the "interior" of W:

$$\mathring{W}:=W\setminus\bigcup\{W'\in\mathcal{I}|\,W'\subsetneq W\}\,.$$

As $\bigcup\mathcal{A}=\amalg_{W\in\mathcal{I}}\mathring{W}$, by the motivic property of the CSM class,

$$c^{SM}\left(\bigcup\mathcal{A}\right)=\sum_{W\in\mathcal{I}}c^{SM}(\mathring{W})=\sum_{W\in\mathcal{I}}(s_1(W)-s_2(W)+\dots)c^{SM}(\mathring{W})= \tag{41}$$

$$\sum_{t=1}^{|\mathcal{A}|}(-1)^{t+1}\sum_{W\in\mathcal{I}}s_t(W)c^{SM}(\mathring{W})=\sum_{t=1}^{|\mathcal{A}|}(-1)^{t+1}\sum_{\substack{\mathcal{B}\subset\mathcal{A}\\ \bigcap\mathcal{B}\neq\emptyset,|\mathcal{B}|=t}}c^{SM}\left(\bigcap\mathcal{B}\right)=$$

$$\sum_{t=1}^{|\mathcal{A}|}(-1)^{t+1}\sum_{\substack{\mathcal{B}\subset\mathcal{A}\\ \bigcap\mathcal{B}\neq\emptyset,|\mathcal{B}|=t}}(1+w)^{\dim(\bigcap\mathcal{B})}w^{n-\dim(\bigcap\mathcal{B})}=\sum_{\substack{\emptyset\neq\mathcal{B}\subset\mathcal{A}\\ \bigcap\mathcal{B}\neq\emptyset}}(-1)^{|\mathcal{B}|+1}(1+w)^{\dim(\bigcap\mathcal{B})}w^{n-\dim(\bigcap\mathcal{B})}$$

Subtracting (41) from $c^{SM}(\mathbb{C}^n)=c^{SM}(\bigcap\emptyset)=(-1)^0(1+w)^n$, we get our formula for the complement $c^{SM}(\mathbb{C}^n\setminus\bigcup\mathcal{A})$. $\square$

EXAMPLE 3.3.2. Let us apply Proposition 3.3.1 to calculate $c^{SM}(\mathbb{C}^n \setminus \mathcal{A}_n)$ for $n \geq 2$ and the *Weyl arrangement of type* A,

$$\mathcal{A}_n := \{x_i - x_j = 0 | \, 1 \leq i < j \leq n\} .$$

The characteristic polynomial of $\mathcal{A}_n$ is well-known, see for example [**Sag99**];

$$\chi(\mathcal{A}_n, t) = (t)_n = t(1-t)\dots(t-(n-1)), \tag{42}$$

we only need to understand the effect of the substitutions $t^k \mapsto (1+w)^k w^{n-k}$.

We claim that

$$t(t-1)\dots(t-(n-1))|_{t^k \mapsto (1+w)^k w^{n-k}} = (1+w)(1-w)\dots(1-(n-2)w) : \tag{43}$$

By the definition of the Stirling number of the first kind, the left-hand side can be written as

$$\sum_{l=1}^{n} (-1)^{n-l} \begin{bmatrix} n \\ l \end{bmatrix} \left(\sum_{s=0}^{l} \binom{l}{s} (1+w)^s w^{n-l} \right) = \sum_{u=0}^{n} \left(\sum_{t=0}^{u} (-1)^t \begin{bmatrix} n \\ n-t \end{bmatrix} \binom{n-t}{u-t} \right) w^u. \tag{44}$$

We use the identity [**GKPL89**, Table 251]

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{k=m}^{n} (-1)^{m-k} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m}$$

to further simplify the coefficient of w^u in (44) to

$$\sum_{s=n-u}^{n} (-1)^{n-s} \begin{bmatrix} n \\ s \end{bmatrix} \binom{s}{n-u} = \sum_{s=n-u}^{n} (-1)^{n-s} \begin{bmatrix} n \\ s \end{bmatrix} \left(\binom{s-1}{n-u-1} + \binom{s-1}{n-u} \right) =$$
$$(-1)^u \begin{bmatrix} n-1 \\ n-u-1 \end{bmatrix} + (-1)^{u-1} \begin{bmatrix} n-1 \\ n-u \end{bmatrix}.$$

This is clearly equal to the coefficient of w^u in the right-hand side of (43).

We got that

$$c^{SM}(\mathbb{C}^n \setminus \bigcup \mathcal{A}_n) = (1+w)(1-w)\dots(1-(n-2)w). \tag{45}$$

EXAMPLE 3.3.3. For future use, let us extend Example 3.3.2 to a hyperplane arrangement in $\mathbb{C}^1$, and define

$$\mathcal{A}_1 := \emptyset.$$

Although choosing $\mathcal{A} = \{x_1 = 0\}$ might seem to be the logical extension, the above choice has the benefit that formulas (42) and (45) describe its characteristic polyomial and the CSM class of its complement.

3.4. A recursive formula for the CSM class of CRS

In this section we give a formula for $c^{SM}(Y_\lambda(d))$ that provides a recursive method for calculating these classes. Its proof is based on the following fundamental property of the equivariant CSM class:

LEMMA 3.4.1. *Let $f : M \to N$ be a proper G-equivariant map of smooth varieties and $\tilde{Y} \subset M$ a constructible subset. Suppose that $f|_{\tilde{Y}}$ is k-to-1 to its image $Y \subset N$. Then*

$$c^{SM}(Y \subset N) = \frac{1}{k} f_! c^{SM}(\tilde{Y} \subset M).$$

This allows us to use the same proper GL(2)-equivariant map φ that we had in Chapter 2 for the fundamental class,

$$\begin{tikzcd} E = \mathrm{Pol}^{d-m}(\mathbb{C}^2) \otimes \mathrm{Pol}^{m}(\mathbb{C}^2/\gamma) \arrow[r, hook, "j"'] \arrow[rr, bend left, "\varphi"] \arrow[dr, "p"'] & \mathbb{P}(\mathbb{C}^2) \times \mathrm{Pol}^{d}(\mathbb{C}^2) \arrow[r, "\pi"'] \arrow[d] & \mathrm{Pol}^{d}(\mathbb{C}^2) \\ & \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2) & \end{tikzcd}, \tag{46}$$

to get a recursive formula also for the CSM classes of CRS. Only that this time, as the CSM class is inhomogeneous and detects bigger codimensional parts, generic property of the restriction $\varphi|_{Y_{\lambda'}(E)}$ is no longer enough, we have to be more specific.

Recall the construction from Section 2.3.1: Let $\lambda = (2^{e_2}, \dots, m^{e_m})$ be a nonempty partition without 1's and $d \geq |\lambda|$. Let λ' denote the partition $(2^{e_2}, \dots, m^{e_m-1})$, where $e_m = 1$ is allowed. We also use the notation $d' = d - m$. Then there is an injective map $j : E = \mathrm{Pol}^{d-m}(\mathbb{C}^2) \otimes \mathrm{Pol}^{m}(\mathbb{C}^2) \to \mathbb{P}^1 \times \mathrm{Pol}^{d}(\mathbb{C}^2)$ defined via multiplication of polynomials and a subvariety $Y_{\lambda'}(E) \subset E$ such that for $\varphi = \pi \circ j$ its restriction $\varphi|_{Y_{\lambda'}(E)}$ is generically e_m-to-1 to its image.

In the special case of $\lambda = (m)$, λ' is defined to be the empty partition $\emptyset$. Recall that, by Definition 1.1.1, this partition corresponds to the open strata:

$$Y_{\emptyset}(d') := \begin{cases} \left\{ f \in \mathrm{Pol}^{0}(\mathbb{C}^2) : f \neq 0 \right\} & \text{if } d' = 0 \\ \left\{ f \in \mathrm{Pol}^{d'}(\mathbb{C}^2) : f = \prod_{j=1}^{d'} f_j \right\} & \text{if } d' > 0, \end{cases}$$

where $0 \neq f_j : \mathbb{C}^2 \to \mathbb{C}$ are linear and no two of them are scalar multiples of each other.

Then for every λ

$$\varphi(Y_{\lambda'}(E)) = Y_{\lambda}(d) \amalg \coprod_{\mu \in \vee\lambda(d)} Y_{\mu}(d), \tag{47}$$

for partitions $\mu \in \vee\lambda(d)$ of size at most d we get by adding m to an element of λ':

$$\vee\lambda(d) := \begin{cases} \left\{ (\dots, i^{e_i-1}, \dots, m^{e_m-1}, i+m) \,|\, 2 \leq i \leq m-1 \right\} & \text{if } d = |\lambda| \\ \left\{ (\dots, i^{e_i-1}, \dots, m^{e_m-1}, i+m) \,|\, 2 \leq i \leq m-1 \right\} \cup \left\{ (\dots, m^{e_m-1}, 1+m) \right\} & \text{if } d \geq |\lambda|+1, \end{cases}$$

where we only denoted multiciplities different from those in λ and all of them should be nonnegative. Note that for any λ the set $\vee\lambda(d)$ is stable for $d \geq |\lambda| + 1$. Omitting d from the notation, we will refer to this stable set. Also note that for every $\mu \in \vee\lambda$

$$\operatorname{codim}(Y_{\mu}) = \operatorname{codim}(Y_{\lambda}) + 1 : \tag{48}$$

If $\mu \in \vee\lambda$, we have either $l(\mu) = l(\lambda) - 1$ or $|\mu| = |\lambda| + 1$. For example, when $\lambda = (5, 5, 5, 3, 2, 2)$,

$$\vee(5,5,5,3,2,2)(d) = \begin{cases} \{(10,5,3,2,2), (8,5,5,2,2), (7,5,5,3,2)\} & \text{if } d = |\lambda| = 21 \\ \{(10,5,3,2,2), (8,5,5,2,2), (7,5,5,3,2), (6,5,5,3,2,2)\} & \text{if } d \geq |\lambda| + 1 = 22. \end{cases}$$

In addition to (47), we have

$$\varphi^{-1}\left(Y_{\lambda}(d) \amalg \coprod_{\mu \in \vee\lambda(d)} Y_{\mu}(d) \right) = Y_{\lambda'}(E).$$

The restriction of $\varphi : E \to \mathrm{Pol}^d(\mathbb{C}^2)$ to $\varphi^{-1}(Y_\lambda(d))$ is an e_m-to-1 covering of $Y_\lambda(d)$, while φ restricted to $\varphi^{-1}(Y_\mu(d))$ is an isomorphism for every $\mu \in \vee\lambda(d)$. Combining Lemma 3.4.1 with the motivic property of the CSM class, we get that

$$\begin{aligned}(49)\quad \varphi_!\left(c^{SM}(Y_{\lambda'}(E) \subset E)\right) = \varphi_!\left(c^{SM}\left(\varphi^{-1}\left(Y_\lambda(d)\right)\right) + \sum_{\mu\in\vee\lambda(d)} c^{SM}\left(\varphi^{-1}\left(Y_\mu(d)\right)\right)\right) = \\ \varphi_!\left(c^{SM}\left(\varphi^{-1}(Y_\lambda(d))\right)\right) + \sum_{\mu\in\vee\lambda(d)} \varphi_!\left(c^{SM}\left(\varphi^{-1}(Y_\mu(d))\right)\right) = \\ e_m c^{SM}\left(Y_\lambda(d)\right) + \sum_{\mu\in\vee\lambda(d)} c^{SM}(Y_\mu(d)).\end{aligned}$$

The rest of this section contains the analysis of the left-hand side of (49). This then leads to

THEOREM 3.4.2. *Let $\lambda = (2^{e_2}, \dots, m^{e_m})$ be a nonempty partition without 1's and $d \geq |\lambda|$. Let λ' denote the partition $(2^{e_2}, \dots, m^{e_m - 1})$, where $e_m = 1$ is allowed. We also use the notation $d' = d - m$. Then*
(50)

$$c^{SM}(Y_\lambda(d)) = \frac{1}{e_m}\left(\partial\left((1+b-a)\prod_{i=0}^{m-1}\big(ia+(d-i)b\big)c^{SM}\left(Y_{\lambda'}(d')\right)_{m/d'}\right) - \sum_{\mu\in\vee\lambda(d)} c^{SM}(Y_\mu(d))\right),$$

where for a polynomial $\alpha \in \mathbb{Z}[a,b]$ and $q \in \mathbb{Q}$ we use the notation

$$\alpha_q(a,b) = \alpha(a+qa, b+qa)$$

and

$$\partial(\alpha)(a,b) = \frac{\alpha(a,b)-\alpha(b,a)}{b-a}$$

denotes the divided difference operation.

For any polynomial $p(a,b)$ in Chern roots a and b, let p_f denote its homogeneous degree f part, e.g. for any d

$$c^{SM}(Y_\lambda(d)) = \sum_{f=\mathrm{codim}(Y_\lambda)}^{d+1} c^{SM}(Y_\lambda(d))_f.$$

Throughout this section, homogeneous terms of negative degree will always be 0.

The reason we call (50) recursive is that for any given degree f and partition λ, to calculate $c^{SM}(Y_\lambda(d))_f$ the necessary ingredients from the right-hand side are either homogeneous CSM classes in smaller degrees or CSM classes corresponding to partitions with bigger codimensional CRS, see Section 3.6 for more details.

3.4.1. The twisted CSM class. Similarly to what we had in Section 2.3.3, the GL(2)-equivariant $c^{SM}(Y_{\lambda'}(E) \subset E)$ can be obtained from $c^{SM}(Y_{\lambda'}(d'))$:

$$(51)\quad c^{SM}(Y_{\lambda'}(E)\subset E) = c(T\mathbb{P}^1)\cdot c^{SM}(Y_{\lambda'}(d'))\Big(a+\frac{m}{d'}(a+b+c_1(\gamma)), b+\frac{m}{d'}(a+b+c_1(\gamma))\Big),$$

where we substitute into $c^{SM}(Y_{\lambda'}(d')) \in H^*_{\mathrm{GL}(2)} \cong \mathbb{Z}[a,b]^{S_2}$, see Section 2.3.2 for our use of the variables a and b. Note here that $H^*_{\mathrm{GL}(2)}(E) \cong H^*_{\mathrm{GL}(2)}(\mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)) \cong H^*_{\mathrm{GL}(2)}(\mathbb{P}^1)$ are naturally isomorphic. We will express terms as elements in $H^*_{\mathrm{GL}(2)}(\mathbb{P}^1)$ without indicating these isomorphism.

To show 51, we can use SSM class analogs of the statements in Section 2.3.3 since, by Theorem 3.1.4, $(\mathrm{GL}(2)\times\mathrm{GL}(1))$-equivariant SSM classes are universal. We get that

$$\frac{c^{SM}_{\delta}(Y_{\lambda'}(E)\subset E)}{c^{SM}_{\delta}(E)}=\frac{\mathbf{a}\left(c^{SM}_{\tilde\rho}(Y_{\lambda'}(d'))\right)}{\mathbf{a}\left(c^{SM}_{\tilde\rho}(\mathrm{Pol}^{d'}(\mathbb{C}^2))\right)}.$$

Here, to makes things easier to follow, we got back into including group actions in the notation of equivariant classes: the $\mathrm{GL}(2)$-action on $E\to\mathbb{P}^1$ is denoted by δ, and we used notation $\tilde\rho$ from Section 2.3.1 for the scalar extension of $\rho:\mathrm{GL}(2)\to\mathrm{GL}(\mathrm{Pol}^{d'}(\mathbb{C}^2))$, the usual representation on $\mathrm{Pol}^{d'}(\mathbb{C}^2)$. Connecting them is the map

$$\mathbf{a}:H^*_{\mathrm{GL}(2)\times\mathrm{GL}(1)}\to H^*_{\mathrm{GL}(2)}(\mathbb{P}^1)$$

induced by the classifying map of the principal $(\mathrm{GL}(2)\times\mathrm{GL}(1))$-bundle

$$\mathrm{B}_{\mathrm{GL}(2)}\left((\mathbb{P}^1\times\mathrm{GL}(2))\times_{\mathbb{P}^1}\mathrm{Pol}^m(\mathbb{C}^2/\gamma)^\times\right)\to\mathrm{B}_{\mathrm{GL}(2)}\,\mathbb{P}^1.$$

The term $c^{SM}_{\delta}(E)$ can be computed using the fact that for any (G-)vector bundle $p:E\to M$ the (G-equivariant) sequence

$$0\longrightarrow p^*E\longrightarrow TE\xrightarrow{Tp}p^*TM\longrightarrow 0$$

is exact, implying that

$$c^{SM}_{\delta}(E)=c_\delta(TE)=c_\delta(T\mathbb{P}^1)c_\delta(E),$$

where we used the same letter for the induced actions on tangent bundles.

Under the isomorphism $H^*_{\mathrm{GL}(2)\times\mathrm{GL}(1)}(\mathrm{Pol}^{d'}(\mathbb{C}^2))\cong H^*_{\mathrm{GL}(2)\times\mathrm{GL}(1)}$ the class $c^{SM}_{\tilde\rho}(\mathrm{Pol}^{d'}(\mathbb{C}^2))=c_{\tilde\rho}(T\,\mathrm{Pol}^{d'}(\mathbb{C}^2))$ corresponds to $c_{\tilde\rho}(\mathrm{Pol}^{d'}(\mathbb{C}^2))$, hence, by the definition of $\mathbf{a}$, we have

$$\mathbf{a}\left(c^{SM}_{\tilde\rho}(\mathrm{Pol}^{d'}(\mathbb{C}^2))\right)=c_\delta(E).$$

Finally, as the representation ρ contains the scalars, the SSM variant of Proposition 2.3.6 implies that

$$s^{SM}_{\tilde\rho}(Y_{\lambda'}(d'))=s^{SM}_{\rho}(Y_{\lambda'}(d'))\left(a+\frac{1}{d'}x,b+\frac{1}{d'}x\right).\tag{52}$$

This substitution corresponds to a map that relates $(\mathrm{GL}(2)\times\mathrm{GL}(1))$-equivariant cohomology to $\mathrm{GL}(2)$-equivariant cohomology, see the proof of Proposition 2.3.6. As this map assigns weights of $\tilde\rho$ to weights of ρ,

$$c^{SM}_{\tilde\rho}(\mathrm{Pol}^{d'}(\mathbb{C}^2))=c^{SM}_{\rho}(\mathrm{Pol}^{d'}(\mathbb{C}^2))\left(a+\frac{1}{d'}x,b+\frac{1}{d'}x\right),$$

and hence the CSM version of (52) also holds. The map $\mathbf{a}$ amounts to substituting $x\mapsto c_1^{\mathrm{GL}(2)}(\mathrm{Pol}^m(\mathbb{C}^2/\gamma)$, therefore

$$\mathbf{a}\left(c^{SM}_{\tilde\rho}(Y_{\lambda'}(d'))\right)=c^{SM}(Y_{\lambda'}(d'))\Big(a+\frac{m}{d'}(a+b+c_1(\gamma)),b+\frac{m}{d'}(a+b+c_1(\gamma))\Big).$$

Putting everything together, we arrive at (51).

3.4.2. GL(2)-equivariant Chern class of $T\mathbb{P}^1$. The only thing left to finish the proof of the recursive formula is to calculate the GL(2)-equivariant Chern class $c(T\mathbb{P}^1)$. Luckily, maps in the Euler exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(\gamma,\gamma) & \longrightarrow & \mathrm{Hom}(\gamma,\mathbb{P}^1\times\mathbb{C}^2) & \longrightarrow & \mathrm{Hom}(\gamma,\gamma^\perp) & \longrightarrow & 0 \\ &&&&&& \wr\Vert &&\\ &&&&&& T\mathbb{P}^1 && \end{array}$$

are all GL(2)-equivariant with respect to actions induced by the standard GL(2)-action on $\mathbb{C}^2$. Hence, identifying these actions, we get that

$$c_{\mathrm{GL}(2)}(T\mathbb{P}^1) = (1-a-c_1^{\mathrm{GL}(2)}(\gamma))(1-b-c_1^{\mathrm{GL}(2)}(\gamma)) \equiv 1-a-b-2c_1^{\mathrm{GL}(2)}(\gamma) \in H^*_{\mathrm{GL}(2)}(\mathbb{P}^1).$$

We calculate the pushforward $\varphi_! = \pi_! j_!$ the same way as we did in Section 2.3.4 : By Lemma 2.3.5, pushing forward along $j : E \to \mathbb{P}^1 \times \mathrm{Pol}^d(\mathbb{C}^2)$ amounts to multiplying by

$$e\left((\mathbb{P}^1\times\mathrm{Pol}^d(\mathbb{C}^2))/E\right) = \frac{\prod_{i=0}^d \big(ia+(d-i)b\big)}{\prod_{i=0}^{d'}\big(m(c_1(\gamma)+a+b)+ia+(d'-i)b\big)}$$

and the pushforward $\pi_!$ can be computed— using the ABBV integral formula—as the divided difference of the restriction to the $\langle e_2\rangle$ fixed point. This restriction can be obtained by substituting $c_1^{\mathrm{GL}(2)}(\gamma) \mapsto -b$, e.g.

$$c(T\mathbb{P}^1)\big|_{\langle e_2\rangle} = 1-a-b-2c_1(\gamma)|_{c_1(\gamma)\mapsto -b} = 1+b-a.$$

Together with (51), this gives

$$\varphi_! c^{SM}\left(Y_{\lambda'}(E)\subset E\right) = \partial\left((1+b-a)\prod_{i=0}^{m-1}\big(ia+(d-i)b\big)c^{SM}(Y_{\lambda'}(d'))_{m/d'}\right), \tag{53}$$

where we used the notation of Theorem 2.2.5. Substituting (53) into (49) finishes the proof of Theorem 3.4.2.

3.5. A non-recursive formula for $c^{SM}(Y_\emptyset(d))$

The above recursive method uses the motivic property of CSM class to calculate $c^{SM}(Y_\emptyset)_f$ as a difference

$$c^{SM}(Y_\emptyset)_f = c^{SM}(\mathrm{Pol}^d(\mathbb{C}^2)^\times)_f - \sum_{\lambda\in P_f(d)} c^{SM}(Y_\lambda(d))_f,$$

where $\mathrm{Pol}^d(\mathbb{C}^2)^\times = \mathrm{Pol}^d(\mathbb{C}^2)\setminus\{0\}$ and for each d and f we set

$$P_f(d) := \{\lambda : |\lambda| \le d \text{ and } \operatorname{codim}(Y_\lambda)\le f\},$$

the set of partitions of size at most d such that CSM classes of the corresponding CRS in $\mathrm{Pol}^d(\mathbb{C}^2)$ can contain nonzero degree f terms. In this part we take a detour and describe a $d!$-fold covering of this open CRS that, combined with results from Secion 3.2.4 about the CSM classes of hyperplane arrangements, provides a direct, non-recursive formula for $c^{SM}(Y_\emptyset(d))$:

PROPOSITION 3.5.1. *For every $d \geq 1$ the CSM class of the open CRS is*

$$c^{SM}(Y_\emptyset(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)) =$$
$$\frac{e(\mathrm{Pol}^d(\mathbb{C}^2))}{d!w^d}\Bigg(\frac{(-1)^0}{db}(1+w)\prod_{j=1}^{d-2}(1-jw) + \frac{(-1)^d}{da}(1-w)\prod_{j=1}^{d-2}(1+jw) +$$
$$(1-w^2)\sum_{s=1}^{d-1}\binom{d}{s}\frac{(-1)^{d-s}}{(d-s)a+sb}\prod_{j=1}^{s-2}(1-jw)\prod_{j=1}^{d-s-2}(1+jw)\Bigg).$$

PROOF. Consider the following GL(2)-equivariant diagram

$$\begin{tikzcd}
L = \otimes_{t=1}^d (\mathbb{C}^2/\gamma_t)^\vee \setminus \{0\} \arrow[r, hook, "j"'] \arrow[rd] \arrow[rr, bend left, "\varphi"] & \bigtimes_{t=1}^d (\mathbb{P}(\mathbb{C}^2)_t) \times \mathrm{Pol}^d(\mathbb{C}^2) \arrow[r, "\pi"'] \arrow[d] & \mathrm{Pol}^d(\mathbb{C}^2) \\
& M = \bigtimes_{t=1}^d \mathbb{P}(\mathbb{C}^2)_t &
\end{tikzcd},$$

where j is induced by multiplication of polynomials and π is the projection to the second factor. Let us denote by $i : X = X_\emptyset(d) \subset \bigtimes_{t=1}^d \mathbb{P}(\mathbb{C}^2)_t$ the complement of the fat diagonal

$$\Delta := \left\{ (v_1, \ldots, v_n) \in \times_{t=1}^d \mathbb{P}(\mathbb{C}^2)_t \middle| \, v_t = v_s \text{ for some } t \neq s \right\}.$$

Then the restriction of φ to $\mathring{L} = L|_X$ is a GL(2)-equivariant $d!$-fold covering of $Y_\emptyset(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)$, so we can use Lemma 3.4.1 to get

$$c^{SM}(Y_\emptyset(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)) = \frac{1}{d!}\varphi_!\left(c^{SM}(\mathring{L} \subset L)\right) = \frac{1}{d!}\pi_!\left(c^{SM}\left(j(\mathring{L}) \subset M \times \mathrm{Pol}^d(\mathbb{C}^2)\right)\right).$$

Restricting the above GL(2)-actions to a maximal complex torus $\mathbb{T}^2 \subset \mathrm{GL}(2)$, we can use the ABBV integral formula to write the pushforward as

$$\pi_!\left(c^{SM}\left(j(\mathring{L}) \subset M \times \mathrm{Pol}^d(\mathbb{C}^2)\right)\right) = \sum_{f \in \mathcal{F}(M)} \frac{c^{SM}\left(j(\mathring{L}) \subset M \times \mathrm{Pol}^d(\mathbb{C}^2)\right)\Big|_f}{e(T_f M)},$$

where $\mathcal{F}(M)$ is the fixed point set of M consisting of tuples $f = (f_1, \ldots, f_n) \in M$ with coordinates $f_t = \langle e_1 \rangle$ or $f_t = \langle e_2 \rangle$. Let us call such a fixed point with s of its coordinates equal to $\langle e_1 \rangle$ a *fixed point of type* $\langle e_1 \rangle^s \langle e_2 \rangle^{d-s}$. For each $s = 0, \ldots, d$ there are $\binom{d}{s}$ of them and their corresponding Euler class is

$$e(T_f M) = \prod_{t=1}^{s} e(T_{\langle e_1 \rangle}\mathbb{P}^1) \prod_{t=1}^{d-s} e(T_{\langle e_2 \rangle}\mathbb{P}^1) = (a-b)^s(b-a)^{d-s} = (-1)^{d-s}(a-b)^d.$$

We can calculate the numerators by first restricting to $\mathring{L}|_{U_f}$, where $T_f M \overset{\psi_f}{\cong} U_f \subset M$ is an torus-invariant affine trivializing neighbourhood of the fixed point f given by e.g. the standard charts of $\mathbb{P}^1$. Motivic classes are local, so this restriction shows, using the multiplicative property, that

$$c^{SM}\left(j(\mathring{L}) \subset M \times \mathrm{Pol}^d(\mathbb{C}^2)\right)\Big|_f = c^{SM}\left(\psi_f^{-1}(U_f \cap X) \subset T_f M\right) c^{SM}\left(j(L_f) \subset \mathrm{Pol}^d(\mathbb{C}^2)\right).$$

For a fixed point f of type $\langle e_1\rangle^s\langle e_2\rangle^{d-s}$ motivic calculus shows that

$$c^{SM}\left(j(L_f)\subset \mathrm{Pol}^d(\mathbb{C}^2)\right)=\frac{e(\mathrm{Pol}^d(\mathbb{C}^2))}{e(\langle x^{d-s}y^s\rangle)}=\frac{\prod_{t=0}^d (ta+(d-t)b)}{(d-s)a+sb}.$$

The first term of the numerator we will compute by relating it to CSM classes of hyperplane arrangements. For an $\langle e_1\rangle^s\langle e_1\rangle^{d-s}$ type fixed point $T_fM\cong \mathbb{C}_w^s\oplus\mathbb{C}_{-w}^{d-s}$, where by $\mathbb{C}_w$ we mean the one-dimensional GL(2)-representation with weight $w=a-b:\mathbb{T}^2\to\mathbb{C}$. Under these isomorphisms

$$\begin{array}{ccc} T_fM & \cong & \mathbb{C}_w^d \\ \cup & & \cup \\ \psi_f^{-1}(U_f\cap X) & \cong & \mathbb{C}_w^d\setminus\mathcal{A}_d \end{array} \qquad \text{for } f=\langle e_1\rangle^d \text{ and } d\geq 1,$$

$$\begin{array}{ccccc} T_fM & \cong & \mathbb{C}_w^s & \oplus & \mathbb{C}_{-w}^{d-s} \\ \cup & & \cup & & \cup \\ \psi_f^{-1}(U_f\cap X) & \cong & \mathbb{C}_w^s\setminus\bigcup\mathcal{A}_s & \times & \mathbb{C}_{-w}^{d-s}\setminus\bigcup\mathcal{A}_{d-s} \end{array} \qquad \text{for } f \text{ of type } \langle e_1\rangle^s\langle e_2\rangle^{d-s} \text{ with } s\geq 1 \text{ and } d\geq 2.$$

In Examples 3.3.2 and 3.3.3 we calculated CSM classes of complements of such hyperplane arrangements. From (45) we deduce that

$$c^{SM}\left(\psi_f^{-1}(U_f\cap X)\subset T_fM\right)=$$
$$\begin{cases}(1+w)\prod_{j=1}^{d-2}(1-jw) & \text{if } f=\langle e_1\rangle^d \text{ and } d\geq 2\\ (1-w^2)\prod_{j=1}^{s-2}(1-jw)\prod_{j=1}^{d-s-2}(1+jw) & \text{if } f=\langle e_1\rangle^s\langle e_2\rangle^{d-s} \text{ and } d\geq 2, s\geq 1.\end{cases}$$

Putting everything together finishes the proof. □

3.6. Polynomial d-dependence of $c^{SM}(Y_\lambda(d))$

This section is the CSM analog of Section 2.4: Generalizing results there, we show that the d-dependece of coefficients of $c^{SM}(Y_\lambda(d))$ is also polynomial; in other words, that for any $f\geq \mathrm{codim}(Y_\lambda)$ the degree f homogeneous parts of the $c^{SM}(Y_\lambda(d))$'s form a polynomial in $\mathbb{Q}[a,b;d]^{S_2}$ for large enough d's.

THEOREM 3.6.1. *For any partition $\lambda=(2^{e_2},\dots,m^{e_m})$ and degree f there is a polynomial $p_{\lambda,f}\in\mathbb{Q}[a,b;d]^{S_2}$ such that*

$$c^{SM}(Y_\lambda(d))_f=p_{\lambda,f}(d)$$

for every $d\geq|\lambda|+2(f-\mathrm{codim}(Y_\lambda))$.

We will refer to these polynomials $p_{\lambda,f}\in\mathbb{Q}[a,b;d]^{S_2}$ as *stable homogeneous parts of CSM classes* of CRS, and often denote them the same way, as $c^{SM}(Y_\lambda(d))_f$, see also Remark 3.6.2.

PROOF. Our inductive proof is built on the recursive formula (50). More precisely, we will use its homogeneous counterpart,

$$c^{SM}(Y_\lambda(d))_f=\frac{1}{e_m}\left(\partial\left(\prod_{i=0}^{m-1}\left(ia+(d-i)b\right)\left(c^{SM}\left(Y_{\lambda'}(d')\right)_{f-m+1}\right)_{m/d'}\right)+\right.$$
$$\left.\partial\left((b-a)\prod_{i=0}^{m-1}\left(ia+(d-i)b\right)\left(c^{SM}\left(Y_{\lambda'}(d')\right)_{f-m}\right)_{m/d'}\right)-\sum_{\mu\in\vee\lambda_f(d)}c^{SM}(Y_\mu(d))_f\right), \tag{54}$$

where, for the sake of uniformity, we introduced

$$\vee\lambda_f(d) := \{\mu \in \vee\lambda(d) |\operatorname{codim}(Y_\mu) \leq f\} = \begin{cases} \emptyset & \text{if } f = \operatorname{codim}(Y_\lambda) \\ \vee\lambda(d) & \text{if } f > \operatorname{codim}(Y_\lambda). \end{cases}$$

The set $\vee\lambda_f(d)$ is stable for $d \geq |\lambda|$ if $f = \operatorname{codim}(Y_\lambda)$ and for $d \geq |\lambda| + 1$ if $f \geq \operatorname{codim}(Y_\lambda)$. Let $\vee\lambda_f$ denote these stable sets. As substitution $()_{m/d'}$ does not change the $\{a, b\}$-degree, (54) follows immediately from (50).

A critical part of the proof is to keep track of the boundary from where polynomial property holds. For this reason, we set

$$k(\lambda, f) := |\lambda| + 2(f - \operatorname{codim}(Y_\lambda)).$$

To describe the induction scheme, let us recall that for each d and f we defined

$$P_f(d) = \{\lambda : |\lambda| \leq d \text{ and } \operatorname{codim}(Y_\lambda) \leq f\},$$

the set of partitions of size at most d such that CSM classes of the corresponding CRS in $\operatorname{Pol}^d(\mathbb{C}^2)$ can contain nonzero degree f terms. Note that $P_f(d)$ doesn't increase after $d = 2f$. We will denote the stable set by P_f. On each P_f fix a linear extension of the partial order corresponding to the codimension of its members. For example,

$$P_3 = \{\emptyset \leq (2) \leq (3) \leq (2, 2) \leq (4) \leq (3, 2) \leq (2, 2, 2)\}.$$

The proof goes by induction on the degree f: Starting from $f = 0$, for each P_f we prove in reverse order that for every $\lambda \in P_f$ for $d \geq k(\lambda, f)$ homogeneous parts $c^{SM}(Y_\lambda(d))_f$ form a polynomial $p_{\lambda,f} \in \mathbb{Q}[a, b; d]^{S_2}$.

The theorem holds in the $f = 0$ case: $P_0 = \{\emptyset\}$ and the $c^{SM}(Y_\emptyset(d))_0 = 1$'s for $d \geq k(\emptyset, 0) = 0$ form the constant polynomial $p_{\emptyset,0} = 1 \in \mathbb{Q}[a, b; d]^{S_2}$.

For $f > 0$ and $\emptyset \neq \lambda \in P_f$ the induction step is provided by (54). We first show that the divided differences of (54) form a polynomial for $d \geq k(\lambda, f)$: $m \geq 2$ implies $f - m < f - m + 1 < f$, therefore, by the induction hypothesis, there exists polynomials $p_{\lambda',f-m}, p_{\lambda',f-m+1} \in \mathbb{Q}[a, b; d']^{S_2}$ such that

$$c^{SM}(Y_{\lambda'}(d'))_{f-m} = p_{\lambda',f-m}(d') \text{ and } c^{SM}(Y_{\lambda'}(d'))_{f-m+1} = p_{\lambda',f-m+1}(d')$$

hold for every $d' \geq k(\lambda', f - m + 1) > k(\lambda', f - m)$.

Substitutions $d' = d - m$ and $()_{m/d'}$ result in rational functions $\hat{p}_{\lambda',f-m}$ and $\hat{p}_{\lambda',f-m+1}$. Analogously to what we had in Section 2.4, coefficients of $a^i b^j$ in both of them have integer values for $d >> 0$. Hence, by Lemma 2.4.2, $\hat{p}_{\lambda',f-m}, \hat{p}_{\lambda',f-m+1} \in \mathbb{Q}[a, b; d]^{S_2}$ such that

$$\hat{p}_{\lambda',f-m}(d) = \left(c^{SM}(Y_{\lambda'}(d - m))_{f-m}\right)_{m/(d-m)} \text{ and } \hat{p}_{\lambda',f-m+1}(d) = \left(c^{SM}(Y_{\lambda'}(d - m))_{f-m+1}\right)_{m/(d-m)}$$

hold for every $d \geq k(\lambda', f - m + 1) + m = k(\lambda, f)$. The terms $(b - a) \prod_{i=0}^{m-1} \big(ia + (d - i)b\big)$ and $\prod_{i=0}^{m-1} \big(ia + (d - i)b\big)$ are clearly polynomials. Multiplying them and taking divided difference preserve polynomiality.

Next, we prove that sums $\sum_{\mu\in\vee\lambda_f(d)} c^{SM}(Y_\mu(d))_f$ in (54) form a polynomial for $d \geq k(\lambda, f)$. By (48), we can apply the induction hypotesis which says that for every $\mu \in \vee\lambda_f(d)$ there exists $p_{\mu,f} \in \mathbb{Q}[a, b; d]^{S_2}$ such that

$$c^{SM}(Y_\mu(d))_f = p_{\mu,f}(d) \text{ for every } d \geq k(\mu, f).$$

As $k(\mu, f) > k(\lambda, f)$ for all $\mu \in \vee\lambda_f$, and $d \geq k(\lambda, f)$ implies $\vee\lambda_f(d) = \vee\lambda_f$, we get for every $d \geq k(\lambda, f)$ that

$$\sum_{\mu \in \vee\lambda_f(d)} c^{SM}(Y_\mu(d)) = \left(\sum_{\mu \in \vee\lambda_f} p_{\mu,f}\right)(d)$$

finishing the induction step for the $Y_\emptyset \neq Y_\lambda$ case.

Lastly, for $f > 0$ and $\emptyset \in P_f$ we can use the motivic property of the CSM class to prove the polynomiality of $c^{SM}(Y_\emptyset(d))_f$. As $P_f(d) = P_f$ for every $d \geq 2f = k(\emptyset, 0)$,

$$Y_\emptyset(d) = \mathrm{Pol}^d(\mathbb{C}^2)^\times \setminus \coprod_{\lambda \in P_f \setminus \{\emptyset\}} Y_\lambda(d).$$

Then we deduce from Theorem 3.2.1 that for every degree f there is a polynomial $p_f^\times \in \mathbb{Q}[a, b; d]^{S_2}$ such that

$$c^{SM}\left(\mathrm{Pol}^d(\mathbb{C}^2)^\times\right)_f = p_f^\times(d) \text{ for every } d \geq f. \tag{55}$$

Also, as $|\lambda| \geq 2l(\lambda)$ for every partition λ,

$$\max_{\emptyset \neq \lambda \in P_f} k(\lambda, f) = \max_{\emptyset \neq \lambda \in P_f} \left(|\lambda| + 2\left(f - (|\lambda| - l(\lambda))\right)\right) = 2f = k\left(2^f, f\right),$$

so we can use the induction hypothesis, and get that for every $d \geq k(\emptyset, f)$

$$c^{SM}(Y_\emptyset(d))_f = \left(p_f^\times - \sum_{\emptyset \neq \lambda \in P_f} p_{\lambda,f}\right)(d).$$

□

REMARK 3.6.2. The same induction scheme shows that starting from $f = 0$ and for all f going through partitions in P_f in reverse order we can, using the homogeneous recursion formula (54), compute these stable homogeneous parts of CSM classes

$$c^{SM}(Y_\lambda(d))_f \in \mathbb{Q}[a, b; d]^{S_2}.$$

The only input we need is stable homogeneous parts of $c^{SM}\left(\mathrm{Pol}^d(\mathbb{C}^2)^\times\right)_f \in \mathbb{Q}[a, b; d]^{S_2}$, which, by Theorem 3.2.1, are homogeneous polynomials of degree $2f$, and can be interpolated using values $c^{SM}\left(\mathrm{Pol}^f(\mathbb{C}^2)^\times\right)_f, \ldots, c^{SM}\left(\mathrm{Pol}^{3f}(\mathbb{C}^2)^\times\right)_f$.

The recursion in easy to implement for example in `Maple`. The vast number of examples we got this way led us to the assumption that the boundary $|\lambda| + 2(f - \mathrm{codim}(Y_\lambda))$ in Theorem 3.6.1 can be strengthened.

3.6.1. Remarks about the threshold for the polynomiality property. In this section we collect conjectures and remarks about possible ways to strenghten our result about the threshold $k(\lambda, f) = |\lambda| + 2(f - \mathrm{codim}(Y_\lambda))$ for the polynomiality property of $c^{SM}(Y_\lambda(d))_f$. These often imply divisibility of $c^{SM}(Y_\lambda(d))_f$ by certain linear factors, which we will also describe.

CONJECTURE 3.6.3. *For any partition $\lambda = (2^{e_2}, \ldots, m^{e_m})$ and degree f there is a polynomial $p_{\lambda,f} \in \mathbb{Q}[a, b; d]^{S_2}$ such that*

$$p_{\lambda,f}(d) = \begin{cases} 0 & \text{if } f \leq d < |\lambda| \\ c^{SM}(Y_\lambda(d))_f & \text{if } |\lambda| \leq d. \end{cases}$$

The above distinction reflects the fact that $Y_\lambda(d)$ is only defined for $d \geq |\lambda|$. As an easy consequence of Conjecture 3.6.3 we obtain

CONJECTURE 3.6.4. *Let* $\lambda = (2^{e_2}, \dots, m^{e_m})$ *be a partition. Then for any degree* $f \in \{\operatorname{codim}(Y_\lambda), \dots, |\lambda|-1\}$ *the stable homogeneous part* $c^{SM}(Y_\lambda(d))_f \in \mathbb{Q}[a,b;d]^{S_2}$ *is divisible by*

$$(d-f)\dots(d-(|\lambda|-1)).$$

For example,

$$\begin{aligned} c^{SM}(Y_{3,2,2}(d))_4 = {} & \frac{1}{2}\, d(d-1)(d-2)(d-3)(d-4)(d-5)(d-6)\left(a^4+b^4\right) \\ & + \frac{1}{2}\, d(d-4)(d-5)(d-6)(3\,d^3+d^2-14\,d+24)\left(a^3b+ab^3\right) \\ & + d(d-4)(d-5)(d-6)(2\,d^3+5\,d^2+3\,d-18)a^2b^2 \end{aligned}$$

$$\begin{aligned} c^{SM}(Y_{3,2,2}(d)_5 = {} & -\frac{1}{4}\, d(d-1)(d-2)(d-3)(d-4)^2(d-5)(d-6)(d-7)\left(a^5+b^5\right) \\ & -\frac{1}{4}\, d(d-5)(d-6)(d-7)(4\,d^5-29\,d^4+17\,d^3+230\,d^2-672\,d+288)\left(a^4b+ab^4\right) \\ & -\frac{1}{4}\, d(d-5)(d-6)(d-7)(7\,d^5-25\,d^4-92\,d^3-40\,d^2+448\,d-192)\left(a^3b^2+a^2b^3\right) \end{aligned}$$

$$\begin{aligned} c^{SM}(Y_{3,2,2}(d))_6 = {} & \frac{1}{48}\, d(d-1)(d-2)(d-3)(d-4)^2(d-5)^2(d-6)^2(3d-25)\left(a^6+b^6\right) \\ & + \frac{1}{48}\, d(d-6)(15\,d^9-466\,d^8+5726\,d^7-\dots)\left(a^5b+ab^5\right) \\ & + \frac{1}{48}\, d(d-6)(33\,d^9-869\,d^8+8170\,d^7-\dots)\left(a^4b^2+a^2b^4\right) \\ & + \frac{1}{24} d(d-6)(21\,d^9-518\,d^8+4326\,d^7-\dots)a^3b^3 \end{aligned}$$

REMARK 3.6.5. Balázs Kőműves has results regarding the boundary in Theorem 3.6.1 that are, in most cases, even stronger than Conjecture 3.6.3. He claims that the polynomial property holds for

$$d \geq \begin{cases} |\lambda|+3 & \text{if } l(\lambda)=0 \\ |\lambda|+2 & \text{if } l(\lambda)=1 \\ |\lambda|+1 & \text{if } l(\lambda)=2 \\ |\lambda| & \text{if } l(\lambda)\geq 3. \end{cases}$$

In other words, substituting a specific $d \geq |\lambda|$ in the stable homogeneous part $c^{SM}(Y_\lambda(d))_f \in \mathbb{Q}[a,b;d]^{S_2}$ gives the degree f homogeneous part of the CSM class of $Y_\lambda(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)$ except for partitions λ of length 0,1 or 2, where for $f \geq |\lambda|+1$ substituting $|\lambda|, \dots, |\lambda|+2-l(\lambda)$ may not give the correct value.

The top degree parts of equivariant CSM classes of affine cones are the equivariant Euler classes of the ambient, see [**FR18**, p. 7]. This implies that $c^{SM}(Y_\lambda(d))_{d+1} = 0$ for any partition λ. Combined with the above remark, we get that $c^{SM}(Y_\lambda(d))_f$ is divisible by

$$\begin{array}{lll} (d-(|\lambda|+3))\dots(d-(f-1)) & \text{for } f=|\lambda|+4,\dots & \text{if } l(\lambda)=0, \\ (d-(|\lambda|+2))\dots(d-(f-1)) & \text{for } f=|\lambda|+3,\dots & \text{if } l(\lambda)=1, \\ (d-(|\lambda|+1))\dots(d-(f-1)) & \text{for } f=|\lambda|+2,\dots & \text{if } l(\lambda)=2, \\ (d-|\lambda|)\dots(d-(f-1)) & \text{for } f=|\lambda|+1,\dots & \text{if } l(\lambda)\geq 3. \end{array} \tag{56}$$

Based on examples we calculated, an even stronger assumption can be made:

Conjecture 3.6.6. *Let $\lambda = (2^{e_2}, \dots, m^{e_m})$ be a partition. Then the stable homogeneous part $c^{SM}(Y_\lambda(d))_f \in \mathbb{Q}[a,b;d]^{S_2}$ is divisible by*

$$\begin{aligned}
&(d-(|\lambda|+3))\dots(d-f) && \text{for } f = |\lambda|+3,\dots && \text{if } l(\lambda)=0,\\
&(d-(|\lambda|+2))\dots(d-f) && \text{for } f = |\lambda|+2,\dots && \text{if } l(\lambda)=1,\\
&(d-(|\lambda|+1))\dots(d-f) && \text{for } f = |\lambda|+1,\dots && \text{if } l(\lambda)=2,\\
&(d-|\lambda|)\dots(d-f) && \text{for } f = |\lambda|,\dots && \text{if } l(\lambda)\geq 3.
\end{aligned} \tag{57}$$

3.7. Invariants of the variety of tangent lines of type λ

In this section we turn back to hypersurfaces, and see what can we can deduce about them using CSM classes of CRS.

Let $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ be a nonzero homogeneous polynomial of degree d and denote by $Z_f \subset \mathbb{P}(\mathbb{C}^n)$ the hypersurface it defines. Recall that in Section 1.2 we identified the variety of λ-lines $\mathcal{T}_\lambda Z_f$ with the degeneracy locus $\sigma_f^{-1}\left(Y_\lambda\left(\mathrm{Pol}^d(S)\right)\right)$ in

$$\begin{tikzcd}
Y_\lambda\left(\mathrm{Pol}^d(S)\right) \arrow[r, phantom, "\subset"] & \mathrm{Pol}^d(S) \arrow[d, shift left] \\
\sigma_f^{-1}\left(Y_\lambda\left(\mathrm{Pol}^d(S)\right)\right) \arrow[r, phantom, "\subset"] & \mathrm{Gr}_2(\mathbb{C}^n). \arrow[u, bend left, "\sigma_f"]
\end{tikzcd}$$

Equivariant SSM classes are universal, evaluating them at the defining bundle, we can calculate the SSM class of such a degeneracy locus: As detailed in Remark 2.2.2, $\mathrm{Pol}^d(S)$ admits a Whitney stratification adapted to our coincident root stratification. For a generic homogeneous polynomial $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ the section $\sigma_f : V \mapsto f|_V$ is transversal to all strata of $\overline{Y}_\lambda\left(\mathrm{Pol}^d(S)\right)$ in this Whitney stratification, see Proposition A.1.1. This means that we can combine the motivic property of the CSM class with Corollary 3.1.5 for $P \to \mathrm{Gr}_2(\mathbb{C}^n)$, the frame bundle of the tautological bundle $S \to \mathrm{Gr}_2(\mathbb{C}^n)$, to calculate the SSM class of $\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n)$. With the right choice of generators, see our conventions in Section 2.3.2, this gives

Proposition 3.7.1. *For a generic homogeneous polynomial $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ the class $s^{SM}(\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n))$ is obtained from the equivariant class $s^{SM}(Y_\lambda(d)) \in H^*_{\mathrm{GL}(2)} \cong \mathbb{Z}[c_1,c_2]$ by substituting $c_i(S^\vee)$ into c_i for $i=1,2$.*

In other words,

$$c^{SM}(\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n)) = c\left(T\,\mathrm{Gr}_2(\mathbb{C}^n)\right) \left.\frac{c^{SM}(Y_\lambda(d))}{c^{SM}(\mathrm{Pol}^d(\mathbb{C}^2))}\right|_{c_i \mapsto c_i(S^\vee)}. \tag{58}$$

Since $\mathrm{Gr}_2(\mathbb{C}^n)$ has nonzero cohomology only in degrees at most its dimension $2(n-2)$ and the smallest degree part of the term $c^{SM}(Y_\lambda(d))$ is the equivariant fundamental class in degree $\mathrm{codim}(Y_\lambda)$, to calculate $c^{SM}(\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n))$, we only need homogeneous parts of $c^{SM}(Y_\lambda(d))$ in degrees at most $2(n-2)$ and homogeneous parts of $c(T\,\mathrm{Gr}_2(\mathbb{C}^n))$ and $1/c^{SM}(\mathrm{Pol}^d(\mathbb{C}^2))$ in degrees at most $2(n-2) - \mathrm{codim}(Y_\lambda)$.

To obtain the multiplicative inverse of $c^{SM}(\mathrm{Pol}^d(\mathbb{C}^2))$, we apply Wronski's formula, see e.g. [**Hen74**, Thm. 1.3], to get

Proposition 3.7.2. *Let $1 + \sum_{i=1}^\infty b_i t^i$ the formal multiplicative inverse of the power series $1 + \sum_{i=1}^\infty a_i t^i$. Then*

$$b_i = (-1)^i \Delta_{1^i}(a_1, \dots, a_i), \tag{59}$$

where $\Delta_{1^i}(a_1,\dots,a_i)$ is the determinant of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_i \\ 1 & a_1 & a_2 & \dots & a_{i-1} \\ 0 & 1 & a_1 & \dots & a_{i-2} \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & 0 & 1 & a_1 \end{pmatrix}.$$

This proposition also shows that homogeneous parts of $c^{SM}(\mathrm{Pol}^d(\mathbb{C}^2))$ needed in (58) are also of degree at most $2(n-2)-\operatorname{codim}(Y_\lambda)$.

In Section 3.6 and Section 3.2 we described approaches to calculate stable homogeneous parts of $c^{SM}(Y_\lambda(d))$ and $c^{SM}(\mathrm{Pol}^d(\mathbb{C}^2))$. The remaining term of the right-hand side can be computed as follows.

3.7.1. Total Chern classes of Grassmannians. Let Q denote the quotient bundle $(\mathrm{Gr}_k\,\mathbb{C}^n\times\mathbb{C}^n)/S$. Then $T(\mathrm{Gr}_k(\mathbb{C}^n))\cong\mathrm{Hom}(S,Q)$, therefore applying the functor $\mathrm{Hom}(S,-)$ to the split exact sequence

$$0\longrightarrow S\longrightarrow \mathrm{Gr}_k(\mathbb{C}^n)\times\mathbb{C}^n\longrightarrow Q\longrightarrow 0,$$

we see that

$$c(T\,\mathrm{Gr}_k(\mathbb{C}^n))=\frac{c\left(\mathrm{Hom}(S,\mathrm{Gr}_k(\mathbb{C}^n)\times\mathbb{C}^n\right)}{c\left(\mathrm{Hom}(S,S)\right)}=\frac{\prod_{i=1}^k(1+x_i)^n}{\prod_{i=1}^k\prod_{j=1}^k(1+x_i-x_j)},\tag{60}$$

where the x_i's are Chern roots of $S^\vee$.

Example 3.7.3. Let us use (58) to calculate the Euler characteristics of the dual of a generic (non-singular) plane curve. The dual of such a degree d plane curve $Z_f\subset\mathbb{P}(\mathbb{C}^3)$ corresponds to $\overline{T_2Z_f}\subset\mathrm{Gr}_2(\mathbb{C}^2)$. As $\sigma_f^{-1}\big(\overline{Y}_\lambda(\mathrm{Pol}^d(S))\big)=\overline{\mathcal{T}_\lambda Z_f}$, see Remark 2.2.2, its class can be calculated as

$$c^{SM}\left(\overline{\mathcal{T}_2Z_f}\subset\mathrm{Gr}_2(\mathbb{C}^3)\right)=c\left(T\,\mathrm{Gr}_2(\mathbb{C}^3)\right)\left.\frac{c^{SM}\left(\overline{Y_2(d)}\right)}{c^{SM}(\mathrm{Pol}^d(\mathbb{C}^2))}\right|_{c_i\mapsto c_i(S^\vee)}.$$

Here we only need degree 1 and 2 homogeneous parts of $c^{SM}\left(\overline{Y_2(d)}\right)$:

$$c^{SM}\left(\overline{Y_2(d)}\right)_1=[Y_2(d)]=d(d-1)c_1,$$
$$c^{SM}\left(\overline{Y_2(d)}\right)_2=c^{SM}\left(Y_2(d)\right)_2+[Y_3(d)]+[Y_3(d)]=-\frac{1}{2}d(d-4)(d-1)^2c_1^2-d(d-2)(d-4)c_2,$$

as all other strata in the closure are of codimension bigger than 2, and the lowest degree term of $c^{SM}(Z\subset M)$ is $[Z\subset M]$ for any smooth variety M and (closed) subvariety Z.

For the other two terms in the right-hand side, (60) applied either directly to $\mathrm{Gr}_2(\mathbb{C}^3)$ or to $\mathbb{P}\left(\mathbb{C}^{3\vee}\right)\cong\mathrm{Gr}_2(\mathbb{C}^3)$ implies that

$$c\left(T\,\mathrm{Gr}_2(\mathbb{C}^3)\right)=1+3c_1+3c_1^2$$

in $H^*(\mathrm{Gr}_2(\mathbb{C}^3))\cong\mathbb{Z}[c_1,c_2]/(c_1^2-c_2,c_1^3-2c_1c_2)$, while e.g. interpolation for the degree 2 coefficient of c_1 in $c(\mathrm{Pol}^d(\mathbb{C}^2))$ together with (59) gives

$$\frac{1}{c^{SM}(\mathrm{Pol}^d(\mathbb{C}^2))}=1-\frac{1}{2}d(d+1)c_1+\text{ (higher degree terms)}.$$

Putting everything together, we get

$$c^{SM}\left(\overline{\mathcal{T}_2Z_f}\subset\mathrm{Gr}_2(\mathbb{C}^3)\right)=d(d-1)c_1-\frac{1}{2}d(d-3)(d^2+d-4)c_1^2.$$

As the highest degree part of $c^{SM}\left(\overline{\mathcal{T}_2 Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^3)\right)$ encodes the Euler characteristics, we obtain

$$\chi\left(\overline{\mathcal{T}_2 Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^3)\right) = \int_{\mathrm{Gr}_2(\mathbb{C}^3)} c^{SM}\left(\overline{\mathcal{T}_2 Z_f} \subset \mathrm{Gr}_2(\mathbb{C}^3)\right) = -\frac{1}{2}d(d-3)(d^2+d-4).$$

EXAMPLE 3.7.4. Similarly, we can use (58) to calculate CSM classes of (locally closed) varieties of lines tangent to degree d surfaces.

Here, again, $c(T\,\mathrm{Gr}_2(\mathbb{C}^4))$ can be calculated in two ways, either by (60) or by the divisor trick, (20), using the fact that the Plücker embedding embeds $\mathrm{Gr}_2(\mathbb{C}^4)$ as a quadratic hypersurface in $\mathbb{P}(\mathbb{C}^5)$:

$$c^{SM}(\mathrm{Gr}_2(\mathbb{C}^4)) = \frac{(1+c_1)^6}{1+2c_1} = 1 + 4c_1 + 7c_1^2 + 6c_1^3 + 3c_1^4.$$

For e.g. $\lambda = (3)$ we get that

$$\begin{aligned} (61)\quad & c^{SM}(\mathcal{T}_3 Z_f \subset \mathrm{Gr}_2(\mathbb{C}^4)) = d(d-1)(d-2)c_1^2 - d(d-2)(d-4)c_2 - \\ & d(d^4 - d^3 - 19d^2 + 42d - 4)c_1c_2 + \frac{1}{2}d(d^6 - 2d^5 - 31d^4 + 100d^3 + 194d^2 - 1032d + 864)c_1^2c_2. \end{aligned}$$

We know that the lowest and the highest degree terms correspond to the fundamental class and the Euler characteristics. In the following section we will see how the rest of such a CSM class can be endowed with geometric interpretation.

3.7.2. A pairing and the Aluffi transformation for the cohomology of Grassmannians. Aluffi showed in [**Alu13**] that the CSM class of a projective subvariety can be expressed from the Euler characteristics of generic linear sections. We generalize this idea to Grassmannian manifolds.

For any compact smooth variety M we can define a bilinear map on $H^*(M)$:

$$\langle \alpha, \beta \rangle := \int_M \frac{\alpha\beta}{c(M)}. \tag{62}$$

This is non-degenerate because the cup product pairing is non-degenerate. If M has nonzero cohomology only in even degrees, e.g. $M = \mathrm{Gr}_k(\mathbb{C}^n)$, (62) is also symmetric. The key feature of this pairing, see e.g. [**Sch17**], is that

PROPOSITION 3.7.5. *If the closed subvarieties $X, Y \subset M$ are* Whitney transversal*, i.e. both are endowed with a Whitney stratification such that all strata of X and all strata of Y are transversal, then*

$$\langle c^{SM}(X), c^{SM}(Y) \rangle = \chi(X \cap Y).$$

From now on, let $M = \mathrm{Gr}_k(\mathbb{C}^n)$. In this case, CSM classes of the Schubert cells $\Omega_\lambda \subset \mathrm{Gr}_k(\mathbb{C}^n)$ form a basis of $H^*(\mathrm{Gr}_k(\mathbb{C}^n))$. Denote by $\{f_\lambda\}$ the basis dual to $\{c^{SM}(\Omega_\lambda)\}$ with respect to this non-degenerate pairing. Then for any closed subvariety $Y \subset M$ and $c^{SM}(Y \subset M) = \sum_\mu a_\mu f_\mu$, as the Schubert stratification of the Grassmannian is Whitney (see [**Nic12**]), Proposition 3.7.5 implies that

$$\chi(Y \cap \Omega_\lambda) = \langle c^{SM}(Y), c^{SM}(\Omega_\lambda) \rangle = \langle \sum_\mu a_\mu f_\mu, c^{SM}(\Omega_\lambda) \rangle = a_\lambda \tag{63}$$

if Ω_λ is chosen such that the Schubert variety $\overline{\Omega}_\lambda$ is Whitney-transversal to (the canonical stratification of) Y. This shows that Euler characteristics of generic Schubert cell sections determine the CSM class of $Y \subset M$.

The dual basis was calculated in [**AMSS17**]: $f_\lambda = c^{SM}(\Omega_{\bar{\lambda}})$, where $\bar{\lambda}$ denotes the dual partition of λ. This is equivalent to the fact that Richardson "cells" $\Omega_\lambda \cap \overline{\Omega}_\mu$ —intersections of Schubert cells with opposite Schubert cells—have zero Euler characteristics unless $\mu = \bar{\lambda}$, in which case the Richardson cell is a point, and the Euler characteristics is 1.

To see this, let us equip $\mathrm{Gr}_k(\mathbb{C}^n)$ with the standard $\mathrm{GL}(n)$-action. Let $\mathbb{T} \subset \mathrm{GL}(n)$ denote a maximal torus and consider the restriction of this $\mathbb{T}$-action to the $\mathbb{T}$-invariant $\Omega_\lambda \cap \overline{\Omega}_\mu$. If $\mu \neq \bar{\lambda}$, $\Omega_\lambda \cap \overline{\Omega}_\mu$ contains no torus fixed point, $\mathcal{F}(\Omega_\lambda \cap \overline{\Omega}_\mu) = \emptyset$, therefore, by the ABBV integral formula,

$$\chi(\Omega_\lambda \cap \overline{\Omega}_\mu) = \int_{\Omega_\lambda \cap \overline{\Omega}_\mu} c^{SM}(\Omega_\lambda \cap \overline{\Omega}_\mu) = \int^{(\mathbb{T})}_{\Omega_\lambda \cap \overline{\Omega}_\mu} c^{SM}_{\mathbb{T}}(\Omega_\lambda \cap \overline{\Omega}_\mu) = \sum_{f \in \mathcal{F}(\Omega_\lambda \cap \overline{\Omega}_\mu)} \frac{c^{SM}_{\mathbb{T}}(\Omega_\lambda \cap \overline{\Omega}_\mu)\big|_f}{e\left(T_f \Omega_\lambda \cap \overline{\Omega}_\mu\right)} = 0.$$

The same could be extended to (complex) partial flag manifolds. This generalization we will not need in what follows.

3.7.3. Euler characteristics of generic Schubert cell sections of varieties of tangent lines. Applying the above change of basis to varieties of lines of type λ, we get

COROLLARY 3.7.6. *Let λ denote a partition without 1's and μ a partition inside the 2-by-$(n-2)$ rectangle. For each d let*

$$X_{\lambda,\mu,d} = \chi(\mathcal{T}_\lambda Z_{f_d} \cap \Omega_\mu)$$

for a generic polynomial $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ and a generic Schubert cell $\Omega_\mu \subset \mathrm{Gr}_2(\mathbb{C}^n)$. Then the $X_{\lambda,\mu,d}$'s form a polynomial in $\mathbb{Q}[d]$ for large enough d's.

PROOF. By (58) and (59), for a generic polynomial $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ every homogeneous part of $c^{SM}(\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n))$ can be written as a polynomial in homogeneous parts of $c^{SM}(\mathrm{Gr}_2(\mathbb{C}^n))$, $c^{SM}(Y_\lambda(d))$ and $c^{SM}(\mathrm{Pol}^d(\mathbb{C}^2))$. By Theorem 3.6.1 and 3.2.1, the latter two form polynomials in $\mathbb{Q}[a,b;d]^{S_2}$ for large enough d's. Therefore, coefficients of e.g. any Schur polynomial s_μ in $c^{SM}(\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n))$ form a polynomial in $\mathbb{Q}[d]$ for large enough d's.

By (63), Euler characteristics $\chi(\mathcal{T}_\lambda Z_f \cap \Omega_\mu)$ for a sufficiently transversal Ω_μ is the coefficient of $c^{SM}(\Omega_{\bar{\mu}} \subset \mathrm{Gr}_2(\mathbb{C}^n))$ in $c^{SM}(\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n))$, and changing from the $\{s_\mu\}$ basis to the $\{c^{SM}(\Omega_\mu \subset \mathrm{Gr}_2(\mathbb{C}^n))\}$ basis of $H^*(\mathrm{Gr}_2(\mathbb{C}^n))$ does not change this polynomiality property. □

EXAMPLE 3.7.7. To illustrate the above corollary, let us calculate Euler characteristics of Schubert cell sections of the variety of 3-flex lines for a generic degree d surface in $\mathbb{P}^3$. All we have to do is to write (61) in the $\{c^{SM}(\Omega_\mu \subset \mathrm{Gr}_2(\mathbb{C}^4))\}$ basis.

Understanding the simple geometry of these Schubert cells gives an "elementary" way to calculate these classes: Closures of $\Omega_\mu \subset \mathrm{Gr}_2(\mathbb{C}^4)$ for $\mu \neq (1)$ are all isomorphic to projective spaces, while the closure of $\Omega_1 \subset \mathrm{Gr}_2(\mathbb{C}^4)$ can be described as a cone over a smooth degree 2 surface, hence its CSM class can be computed using [**Feh21**, Prop. 3.1] and the fact that it is symmetric in s_2 and $s_{1,1}$. We get that

$$\begin{aligned}
\mathrm{c}^{\mathrm{SM}}(\Omega_{2,2}) &= s_{2,2},\\
\mathrm{c}^{\mathrm{SM}}(\Omega_{2,1}) &= s_{2,1} + s_{2,2},\\
\mathrm{c}^{\mathrm{SM}}(\Omega_{2}) &= s_2 + 2\,s_{2,1} + s_{2,2},\\
\mathrm{c}^{\mathrm{SM}}(\Omega_{1,1}) &= s_{1,1} + 2\,s_{2,1} + s_{2,2},\\
\mathrm{c}^{\mathrm{SM}}(\Omega_{1}) &= s_1 + 2\,s_{1,1} + 2\,s_2 + 3\,s_{2,1} + s_{2,2},\\
\mathrm{c}^{\mathrm{SM}}(\Omega_{\emptyset}) &= 1 + 3s_1 + 4\,s_{1,1} + 4\,s_2 + 4\,s_{2,1} + s_{2,2}.
\end{aligned}$$

After the change of basis, (61) becomes

$$c^{SM}(\mathcal{T}_3 Z_f \subset \mathrm{Gr}_2(\mathbb{C}^4)) = 3d(d-2)\omega_{1,1} + d(d-1)(d-2)\omega_2 - \\ d(d^4 - d^3 - 17d^2 + 42d - 12)\omega_{2,1} + \\ \frac{1}{2}d(d^6 - 2d^5 - 29d^4 + 98d^3 + 158d^2 - 948d + 848)\omega_{2.2},$$

where $\omega_\mu = c^{SM}(\Omega_\mu \subset \mathrm{Gr}_2(\mathbb{C}^4))$. By (63), for a generic $f \in \mathrm{Pol}^d(\mathbb{C}^4)$ and generic Ω_μ's the coefficient of e.g. $\omega_{2,1}$ is equal to $\chi(\mathcal{T}_3 Z_f \cap \Omega_1)$, i.e. the Euler characteristics of the variety of 3-flexes of Z_f that intersect a generic puncture line in $\mathbb{P}^3$. The coefficient of $\omega_{1,1}$ is the Euler characteristics that counts the the number of 3-flexes of Z_f contained in a generic $\mathbb{P}^2 \setminus \mathbb{P}^1$. This is equal to the number of 3-flexes of a generic degree d plane curve.

There are general formulas computing CSM classes of Schubert cells of an arbitrary Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ in e.g. [**FR18**]:

$$c^{SM}(\Omega_\mu \subset \mathrm{Gr}_k(\mathbb{C}^n)) = \mathcal{S}\left(\prod_{j=1}^{k} z_j^{\mu_j} \prod_{j=1}^{k}(1+z_j)^{n-i_{k+1-j}} \frac{1}{\prod_{1\le i<j\le k}(1+z_i-z_j)}\right),$$

where $i_t = \mu_{k+1-t} + t$ and the operation $\mathcal{S} = \mathcal{S}_{z_1,\dots,z_k}$ is defined as $\mathcal{S}(z_1^{\lambda_1}\dots z_k^{\lambda_k}) = s_{\lambda_1,\dots,\lambda_k}$, possibly combined with the straightening laws, for a monomial $z_1^{\lambda_1}\dots z_k^{\lambda_k}$, and extended linearly for a polynomial.

This makes it possible to use our approach to calculate Euler characteristics of generic Schubert cell sections of varieties of λ-lines for hypersurfaces of arbitrary dimensions.

CHAPTER 4

Motivic Chern classes of coincident strata and varieties of λ-lines

4.1. A brief introduction to (equivariant) motivic Chern classes

In this last chapter we focus on calculating equivariant motivic Chern classes of coincident root strata and deducing motivic Chern classes of varieties of λ-lines. The motivic Chern class can be regarded as a K-theory analog of the Chern-Schwartz-MacPherson class.

4.1.1. An overview of K-theory. K-theory of a complex projective variety X comes in different flavors: In algebraic geometry we can consider the Grothendieck group of locally free sheaves $K^0(X)$ and the Grothendieck group of coherent sheaves $K_0(X)$. For a smooth X the existence of finite locally free resolutions of coherent sheaves implies that $K^0(X)$ and $K_0(X)$ are isomorphic.

We can also look at the Grothendieck group of complex vector bundles $K^0_{top}(X)$ for the analytic topology on X. Using Bott periodicity, the functor $K^0_{top}(X)$ can be extended to a $\mathbb{Z}_2$-graded complex-oriented cohomology theory $K^*_{top}(X)$ with, among many others, K-theoretic Euler and Chern classes and pushforwards $f_!$ along proper maps defined. This extension we won't need, all our calculations will be inside $K^0_{top}(X)$. In fact, if all cells of a finite CW complex X have even dimension, then $K^1_{top}(X) = 0$.

Connecting the algebraic and the topological versions is the natural forgetful morphism $K^0(X) \to K^0_{top}(X)$. This, by the Riemann-Roch theorem, commutes with the respective push-forwards, but is an isomorphism only in some rare cases, for example when X admits a decomposition into algebraic cells. Luckily, in this thesis we are only interested in the K-groups of $X = \mathrm{Gr}_k(\mathbb{C}^n)$, where all three variants coincide. This we will denote by $K(X)$.

Likewise, G-equivariant K-theory has algebraic K^G_{alg} and topological K^G_{top} versions. For a brief summary and comparison of them, see [**FRW21**]. [**FRW21**, Thm. 9.1] implies that the natural map $K^{\mathrm{GL}(2)}_{alg}(\mathrm{Pol}^d(\mathbb{C}^2)) \to K^{U(2)}_{top}(\mathrm{Pol}^d(\mathbb{C}^2))$ is an isomorphism for our GL(2)-representation $\mathrm{Pol}^d(\mathbb{C}^2)$. In such cases, the isomorphic groups we will denote by $K_G(X)$. Then e.g. isomorphisms

$$K^{\mathrm{GL}(2)}_{alg}(\mathrm{Pol}^d(\mathbb{C}^2)) \cong K^{\mathrm{GL}(2)}_{alg}(\mathrm{pt}) \cong \mathrm{R}(\mathrm{GL}(2)) \cong \mathrm{R}(\mathbb{T})^W \tag{64}$$

—where $\mathbb{T} \subset \mathrm{GL}(2)$ is a maximal complex torus, $W \cong S_2$ is the Weyl group of GL(2) and R refers to the representation rings—allow us to restrict our GL(2)-action to $\mathbb{T}$ without losing information.

As we have mentioned, our main objective for this chapter is to compute $\mathbb{T}$-equivariant mC classes of CRS. These are elements of

$$K_{\mathbb{T}}(\mathrm{Pol}^d(\mathbb{C}^2)) \cong K_{\mathbb{T}}(\mathrm{pt}) \cong R(\mathbb{T}) \cong \mathbb{Z}\left[X^{\pm 1}, Y^{\pm 1}\right],$$

where we abbreviated projections $\pi_i \in R(\mathbb{T})$ to the i-th factor as $X := \pi_1$ and $Y := \pi_2$. We will also use generators $\alpha := X^{-1}$ and $\beta := Y^{-1}$ to avoid negative exponents in the (multiplicative) characters $\alpha^i\beta^{d-i}$ of the representation $\mathrm{Pol}^d(\mathbb{C}^2) \cong \mathrm{Sym}^d\left(\mathbb{C}^{2\vee}\right)$.

In case of a torus action, the Lefschetz-Riemann-Roch theorem—a K-theory counterpart of the ABBV integral formula—can help us to determine K-theory pushforwards. The following special case ([**FRW21**, Prop. 7.5]) will be particularly useful for calculating torus-equivariant motivic Chern classes of CRS:

PROPOSITION 4.1.1. *Suppose that the torus $\mathbb{T}$ acts on the vector space V with no zero weight and on the smooth and compact M with finitely many fixed points. Let $\pi : M \times V \to V$ denote the projection onto V. Then for all $\omega \in K_{\mathbb{T}}(M \times V)$ we have*

$$\pi_! \omega = \sum_{f \in \mathcal{F}(M)} \frac{\omega|_f}{e_K(T_f M)},$$

where $\mathcal{F}(M)$ denotes the fixed point set of M and e_K refers to the equivariant K-theoretic Euler class.

Note that inclusions $0 \hookrightarrow V$ and $M \hookrightarrow M \times V$ induce isomorphism on K-theory, and that we identified the fixed points $f \in M$ with $(f, 0) \in M \times V$ accordingly.

4.1.2. Definition and some properties of the motivic Chern class. The motivic Chern (mC) class can be defined similarly to the Chern-Schwartz-MacPherson class. More precisely —restricting again our attention to constructible subsets X of a smooth algebraic variety M— it is a motivic class taking $X \subset M$ to $\mathrm{mC}(X \subset M) \in K^0_{alg}(M)[y]$, where y is a formal variable. The normalizing condition is given by

$$\mathrm{mC}(M) = \lambda_y(T^\vee M),$$

where for a vector bundle E we define $\lambda_y(E) = \sum_{i=0}^{\operatorname{rank} E} [\Lambda^i E]\, y^i$.

The motivic Chern class analog of the local property (19) is again an easy consequence of the definition, and the fact that λ_y is multiplicative implies the analog of the multiplicative property (18). The divisor trick expressing mC classes of zero loci, see Proposition 3.1.2, takes the following form

$$\mathrm{mC}(Z) = \frac{\lambda_y(T^\vee M)|_Z}{\lambda_y(\nu^\vee_{Z \subset M})} = \left.\frac{\mathrm{mC}(M)}{\lambda_y(E^\vee)}\right|_Z \text{ and } \mathrm{mC}(Z \subset M) = \frac{\mathrm{mC}(M)}{\lambda_y(E^\vee)} e_K(E), \tag{65}$$

where $e_K(E)$ stands for the K-theoretic Euler class of E.

The motivic Chern class also has a Segre variant,

$$\mathrm{mS}(X \subset M) = \frac{\mathrm{mC}(X \subset M)}{\mathrm{mC}(M)}.$$

This motivic Segre class behaves well for transversal pullbacks for a fine notion of transversality ([**FRW18**, § 8.1]):

DEFINITION 4.1.2. Let N be a smooth variety. Then $g : N \to M$ is *motivically transversal* to a map of smooth varieties $f : U \to M$ if there is a proper normal crossing extension $\bar{f} : Y \to M$ of f such that g is transversal to all the $\bar{f}_K$'s.

Here we used notations of Definition 3.1.1. Motivically transversal pullbacks of proper normal crossing extensions are proper normal crossing extensions. Therefore, if $g : N \to M$ is motivically transversal to a smooth variety $U \subset M$, we have ([**FRW18**, Thm. 8.5])

$$\mathrm{mS}(g^{-1}(U) \subset N) = g^* \,\mathrm{mS}(U \subset M). \tag{66}$$

4.1.3. The Todd genus and the χ_y-genus of Hirzebruch. Compared to cohomology, K-theory has a new feature ([**Feh21**, § 2.7]): Let X be a smooth, closed projective variety. Then the *Todd genus of* X, the K-theoretic pushforward of $1 \in K(X)$ along the collapse map $\mathrm{co}_X : X \to \mathrm{pt}$,

$$\mathrm{Td}(X) := \int_X 1 := \mathrm{co}_{X!}1 \in K(\mathrm{pt}) \cong \mathbb{Z}$$

is a non-trivial invariant.

The following is a key result in K-theory, it will be used throughout this chapter.

THEOREM 4.1.3 ([**AH62**]). *The Todd genus of the projective space,*

$$\mathrm{Td}(\mathbb{P}^n) = 1.$$

A straightforward consequence is that for any linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$ the pushforward of its K-class,

$$\int_{\mathbb{P}^n} \left[\mathbb{P}^k \subset \mathbb{P}^n\right] = 1.$$

These classes generate the K-theory of $\mathbb{P}^n$: $K(\mathbb{P}^n) \cong \mathbb{Z}[H]/(H^{n+1})$ for $H = [\mathbb{P}^{n-1} \subset \mathbb{P}^n] = 1 - [\gamma]$, where $\gamma \to \mathbb{P}^n$ denotes the tautological bundle, see Section 4.4.1. This means that we can calculate the integral for any $\omega \in K(\mathbb{P}^n)$.

A straighforward extension of the Todd genus is the χ_y-genus of Hirzebruch:

$$\chi_y(X) = \int_X \lambda_y(T^\vee X) = \int_X \mathrm{mC}(X),$$

for X a smooth, closed projective variety ([**Feh21**, § 2.9]).

4.1.4. Equivariant mC and mS classes. The equivariant version of the motivic Chern class for a linear algebraic group G acting on a smooth G-variety was developed in [**FRW21**]. The definition is parallel to what we have sketched for the non-equivariant setting: For example, a G-action on M lifts to make TM a G-vector bundle, the exterior powers $\Lambda^i T^\vee M$ are also G-vector bundles, so $\mathrm{mC}_{\mathbb{T}}(M) := \lambda_y(T^\vee M)$ can be considered as an element in $K^G_{alg}(M)[y]$.

Analogs of the multiplicative property (22) and (23) hold for the equivariant mC class. These together with the following lemma form the backbone of our calculations of mC classes of affine varieties.

Let α denote a torus action $\mathbb{T} \cong (\mathbb{C}^\times)^r$ on $\mathbb{C}$. The inclusion $0 \hookrightarrow \mathbb{C}$ induces an isomorphism $K^{\mathbb{T}}_{alg}(\mathbb{C}) \cong K^{\mathbb{T}}_{alg}(\mathrm{pt})$ which we won't denote in our formulas. Under this isomorphism $[T\mathbb{C}] \in K^{\mathbb{T}}_{alg}(\mathbb{C})$, the class represented by the natural lift of α to $T\mathbb{C}$, corresponds to the $\mathbb{T}$-equivariant line bundle over the point with α the $\mathbb{T}$-action on its total space. Corresponding to $K^{\mathbb{T}}_{alg}(\mathrm{pt}) \cong R(\mathbb{T})$, we use the same letter α for this $\mathbb{T}$-equivariant line bundle. Under these identifications, by definition, we have:

LEMMA 4.1.4 (*motivic calculus for mC*). *Let α denote a torus action on $\mathbb{C}$. Then for the torus-equivariant motivic Chern classes we have*

$$\mathrm{mC}(\mathbb{C} \subset \mathbb{C}) = 1 + y/\alpha, \qquad \mathrm{mC}(\{0\} \subset \mathbb{C}) = 1 - 1/\alpha, \qquad \mathrm{mC}(\mathbb{C} \setminus \{0\} \subset \mathbb{C}) = (1+y)/\alpha.$$

As shown in [**FRW18**, § 8.2], G-equivariant motivic Segre class

$$\mathrm{mS}_G(X \subset M) = \frac{\mathrm{mC}_G(X \subset M)}{\mathrm{mC}_G(M)}$$

is a universal formula for motivic Chern classes of degeneracy loci. This statement can be dissected into two levels.

First, suppose that $P \to M$ is a principal G-bundle over the smooth M and A is a smooth G-variety. Then we can define a map

$$\mathbf{a} : K_G(A) \to K(P \times_G A) \tag{67}$$

by association: For any G-vector bundle E over A the associated bundle $P \times_G E$ is a vector bundle over $P \times_G A$. Then by [**FRW18**, Prop. 8.7] we have

Proposition 4.1.5. *Let $Y \subset A$ be G-invariant. Then*

$$\mathrm{mS}(P \times_G Y \subset P \times_G A) = \mathbf{a}\big(\mathrm{mS}_G(Y \subset A)\big).$$

The second part ([**FRW18**, Cor. 8.8]) can be deduced by applying (66):

Corollary 4.1.6. *Suppose that $\sigma : M \to P \times_G A$ is a section motivically transversal to $P \times_G Y$. Then the mS class of the Y-locus of the section σ is*

$$\mathrm{mS}(\sigma^{-1}(P \times_G Y) \subset M) = \sigma^* \mathbf{a}\big(\mathrm{mS}_G(Y \subset A)\big). \tag{68}$$

If A is a vector space then $K_G(A) \cong K_G(\mathrm{pt})$, σ^* can be identified with the identity map $K(P \times_G A) = K(M)$, and under these identifications $\mathbf{a} : K_G(\mathrm{pt}) \to K(M)$ maps the class of a G-representation $[V] \in K_G(\mathrm{pt})$ to $[P \times_G V] \in K(M)$.

4.1.5. Structure of the chapter. In Section 4.2 we calculate motivic Chern classes of plane curves in terms of their degree, sum of Milnor numbers and sum of delta invariants. Later we apply the resulting formula to duals of generic plane curves.

Section 4.3 starts with an example showing that the motivic Chern class doesn't behave well with respect to pushforwards along branched coverings. As a consequence, we describe a resolution for CRS, and we use this to deduce a recursive formula for equivariant mC classes of CRS. We also conjecture that in an appropriate basis d-dependence of the coefficients of $\mathrm{mC}(Y_\lambda(d))$ is polynomial.

We conclude with Section 4.4, where we show for generic polynomials how the motivic Chern class of the corresponding variety of λ-lines can be calculated from $\mathrm{mC}(Y_\lambda(d))$. We then combine Section 4.2 with this approach for ordinary tangent lines to prove polynomial property for some of the coefficients in $\mathrm{mC}(\overline{Y}_2(d))$.

4.2. Motivic Chern class of plane curves

To illustrate some characteristic features of K-theory and the motivic Chern class, let us calculate the mC classes of projective plane curves. Applying the resulting formula to the dual of a generic plane curve connects this and Section 4.4.3. Let us start with the smooth case.

4.2.1. Motivic Chern classes of smooth plane curves. A smooth degree d hypersurface $Z_d \subset \mathbb{P}^n$ is the zero locus of a section of the line bundle $(\gamma^\vee)^d$. Using the divisor trick (65), we get

$$\mathrm{mC}(Z_d \subset \mathbb{P}^n) = \frac{(1+yt)^{n+1}}{1+y} \cdot \frac{1-t^d}{1+yt^d}, \tag{69}$$

where t denotes the class of the tautological bundle $[\gamma]$. This is an elegant formula, but we will also need the χ_y-genus. It can be deduced from (69) by first taking the n-th Taylor polynomial in $H = [\mathbb{P}^{n-1} \subset \mathbb{P}^n] = 1 - t$ then substituting $H = 1$:

To find the multiplicative inverse of $1 + yt^d$, we can apply 3.7.2. Luckily for $n = 2$, as $K(\mathbb{P}^2) \cong \mathbb{Z}[H]/(H^3)$ and $1 - t^d = 1 - (1-H)^d = dH - \binom{d}{2}H^2 + \dots$, we only need its linear term:

$$\frac{1}{1+y-ydH} = \frac{1}{(1+y)\left(1 - \frac{dy}{1+y}H\right)} = \frac{1}{1+y}\left(1 + \frac{yd}{1+y}H\right) = \frac{1}{(1+y)^2}\left(1+y+ydH\right).$$

Then expanding terms in (69), we get

$$\mathrm{mC}(Z_d \subset \mathbb{P}^2) \equiv \frac{1}{(1+y)^3} \cdot ((1+y)^3 - 3y(1+y)^2 H)(dH - \binom{d}{2} H^2)(1+y+dyH),$$

which is congruent to

$$\mathrm{mC}(Z_d \subset \mathbb{P}^2) = (1+y)dH + \left(-\binom{d}{2}\right) H^2 + \left(\frac{d^2-5d}{2}\right) yH^2.$$

Substituting $H = 1$, we get the χ_y-genus of the degree d smooth plane curve:

$$\chi_y(Z_d) = \left(\binom{d-1}{2} - 1\right)(y-1). \tag{70}$$

4.2.2. Motivic Chern classes of singular plane curves.

PROPOSITION 4.2.1. *The motivic Chern class of a degree d plane curve $C \subset \mathbb{P}^2$ with M the sum of Milnor numbers and Δ the sum of delta invariants is*

$$\mathrm{mC}(C \subset \mathbb{P}^2) = (1+y)dH + \left(-\frac{d^2-d}{2} + M - \Delta\right) H^2 + \left(\frac{d^2-5d}{2} - \Delta\right) yH^2. \tag{71}$$

To prove this proposition, let $\varphi : \tilde{Z} \to \mathbb{P}^2$ be a normalization of $Z \subset \mathbb{P}^2$. We will see that the pushforward $\varphi_!$ provides a comparison between $\chi_y(\tilde{Z})$ and $\chi_y(Z)$. From $\chi_y(Z)$, using the Aluffi transformation, we can calculate $\mathrm{mC}(Z \subset \mathbb{P}^2)$.

4.2.2.1. *Aluffi transformation for $\mathbb{P}^2$.* Motivic Chern classes of subvarieties of partial flag manifolds can be calculated from the χ_y-genera of their intersections with Schubert varieties. The simplest use case for this is the computation of $\mathrm{mC}(Z \subset \mathbb{P}^2)$, which we do next. Let us write

$$\mathrm{mC}(Z \subset \mathbb{P}^2) = aH + byH + cH^2 + dyH^2,$$

and let $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ be motivically transversal to Z. Denoting by $\mathbb{P}^1$ its image, we have

$$\mathrm{mS}(Z \cap \mathbb{P}^1 \subset \mathbb{P}^1) = i^* \,\mathrm{mS}(Z \subset \mathbb{P}^2).$$

As $\mathrm{mC}(Z \cap \mathbb{P}^1 \subset \mathbb{P}^1) = aH$, this implies that $a = b$, meaning that if

$$\chi_y(Z \cap \mathbb{P}^1) = z_{10} \text{ and } \chi_y(Z) = z_{20} + z_{21}y,$$

then

$$\mathrm{mC}(Z \subset \mathbb{P}^2) = z_{10}(1+y)H + (z_{20} - z_{10})H^2 + (z_{21} - z_{10})yH^2. \tag{72}$$

4.2.2.2. *χ_y-genus of a plane curve from its normalization.* The deviation of the geometric genus, the genus of the normalization $\tilde{Z} = \Sigma_g$, from the smooth case can be localized to delta invariants of the singular points:

$$g = \binom{d-1}{2} - \Delta.$$

Generalizing (70), we have

$$\begin{aligned}\chi_y(\Sigma_g) = h^{0,0} - h^{0,1} + (h^{1,0} - h^{1,1})y = 1 - g + (g-1)y = (1-g)(1-y) = \\ \left(1 - \binom{d-1}{2} + \Delta\right) + y\left(\binom{d-1}{2} - \Delta - 1\right)\end{aligned} \tag{73}$$

The Milnor-Jung formula

$$\mu = 2\delta - r + 1,$$

where μ is the Milnor number, δ is the delta invariant and r is the branching number of the singularity, implies that the number of "extra" preimages of the normalization is $2\Delta - M$. This combined with the homology and the motivic property of the motivic Chern class gives that

$$\varphi_! \,\mathrm{mC}(\tilde{Z}) - (2\Delta - M)\varphi_! \,\mathrm{mC}(\mathrm{pt} \subset \tilde{Z}) = \mathrm{mC}(Z \subset \mathbb{P}^2).$$

Integrating the left-hand side over $\tilde{Z}$, the right-hand side over $\mathbb{P}^2$, we obtain

$$\chi_y(Z) = \chi_y(\tilde{Z}) - (2\Delta - M) = \left(1 - \binom{d-1}{2} + M - \Delta\right) + y\left(\binom{d-1}{2} - \Delta - 1\right).$$

Substituting this and $\chi_y(Z \cap \mathbb{P}^1) = z_{10} = \deg(Z) = d$ into (72), we complete the proof.

4.3. A recursive formula for motivic Chern classes of CRS

In this section we describe a recursive formula for GL(2)-equivariant motivic Chern classes of CRS. As explained in Section 4.1.2, the motivic Chern class can be regarded as a K-theory counterpart of the Chern-Schwartz-MacPherson class. Accordingly, they share lots of common features. There is, however, a major difference: as the next example shows, the motivic Chern class doesn't behave well with respect to branched coverings.

EXAMPLE 4.3.1. Consider the composition $\varphi = i_2 f_d : \mathbb{P}^1 \to \mathbb{P}^2$, where $f_d : \mathbb{P}^1 \to \mathbb{P}^1$ has degree d and $i_t : \mathbb{P}^1 \hookrightarrow \mathbb{P}^t$ denotes a linear embedding. Then φ is a d-fold branched covering of $\mathbb{P}^1 \subset \mathbb{P}^2$ with exceptional points $A, B \in \mathbb{P}^1 \subset \mathbb{P}^2$ having unique preimages. We will show that, contrary to what a motivic Chern class analog of Lemma 3.4.1 would imply,

$$\varphi_!(\mathrm{mC}(\mathbb{P}^1)) \neq d\,\mathrm{mC}(\mathbb{P}^1 \setminus \{A, B\} \subset \mathbb{P}^2) + \mathrm{mC}(\{A, B\} \subset \mathbb{P}^2).$$

Let $Z_d \subset \mathbb{P}^d$ denote the degree d rational normal curve , the image of the Veronese embedding $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$. Then its K-class is

$$\nu_{d!}1 = \left[Z_d \subset \mathbb{P}^d\right] = dH^{d-1} - (d-1)H^d,$$

see [**Feh21**], while $\nu_{d!}H = H^d$ trivially holds. As the composition $i_d f_d$ is homotopic to ν_d, $i_{d!}1 = H^{d-1}$ and $i_{d!}H = H^d$ imply that $f_{d!}1 = d - (d-1)H$ and $f_{d!}H = H$, hence

$$\varphi_! 1 = dH - (d-1)H^2 \text{ and } \varphi_! H = H^2.$$

As λ_y is multiplicative, we can use the Euler exact sequence to show that

$$\mathrm{mC}(\mathbb{P}^n) = \sum_{i=1}^{n} \binom{n+1}{i} (-y)^i (1+y)^{n-i} H^i.$$

In particular, $\mathrm{mC}(\mathbb{P}^1) = 1 + y - 2yH$ and we can conclude that

$$\begin{aligned}\varphi_!(\mathrm{mC}(\mathbb{P}^1)) = \varphi_!(1 + y - 2yH) = d(1+y)H - ((d-1) + (d+1)y)\,H^2 \neq \\ d\,\mathrm{mC}(\mathbb{P}^1 \setminus \{A, B\} \subset \mathbb{P}^2) + \mathrm{mC}(\{A, B\} \subset \mathbb{P}^2) = d(1+y)H - (2(d-1) + 2dy)H^2.\end{aligned}$$

This means that we can no longer apply the branched covering (46) we used to calculate the CSM classes to compute motivic Chern classes of CRS. In the next section we remedy this situation by providing proper maps whose suitable restrictions are isomorphisms.

4.3.1. A resolution of CRS. Let us first remark that the method we will describe in this section was the starting point for our investigation of GL(2)-equivariant invariants of CRS. After discovering the following method, we realized that if the invariant behaves well with respect to branched coverings, it can be modified to better suit our needs. In particular, this modification helped us to prove the polynomial "d-dependence" of the CSM classes of CRS, see Theorem 3.6.1. Correspondingly, the subsequent recursive formula can be regarded as somewhat outdated: Many of the results we have for singular cohomology invariants can possibly be transferred to the K-theory case.

The biggest difference is that for motivic Chern classes we will stop short of proving results about "d-dependence" (but see Conjecture 4.3.3). Correspondingly, in this chapter partitions can contain 1's— to distinguish these partitions we will denote them by $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_r)$—, and by the corresponding CRS we will always mean a subset of $\mathrm{Pol}^{|\hat{\lambda}|}(\mathbb{C}^2)$:

$$Y_{\hat{\lambda}} = Y_\lambda(|\hat{\lambda}|),$$

where the right-hand side is as in Definition 1.1.1 and λ denotes the partition we get by removing all the 1's from $\hat{\lambda}$.

The resolution we describe next will provide the main step in a recursive algorithm computing GL(2)-equivariant classes $\mathrm{mC}\left(Y_{\hat{\lambda}}\right) = \mathrm{mC}\left(Y_{\hat{\lambda}} \subset \mathrm{Pol}^{|\hat{\lambda}|}(\mathbb{C}^2)\right)$: Given a partition $\hat{\lambda}$, however, the procedure calculating the mC class of the corresponding CRS invokes the procedure itself for a slightly bigger class of subsets, *shifted CRS*,

$$x^k y^l Y_{\hat{\mu}} = \left\{ x^k y^l p \,\middle|\, p \in Y_{\hat{\mu}} \right\} \subset \mathrm{Pol}^{k+l+|\hat{\mu}|}(\mathbb{C}^2).$$

Note that this subset is invariant only for the restriction of our usual GL(2)-action to its maximal torus $\mathbb{T}^2$, and it is its $\mathbb{T}^2$-equivariant $\mathrm{mC}(x^k y^l Y_{\hat{\mu}} \subset \mathrm{Pol}^{k+l+|\hat{\mu}|}(\mathbb{C}^2))$ we will compute next.

Let $\hat{\lambda} = (1^{e_1}, \dots, m^{e_m})$ be a partition, where $m = \max(\hat{\lambda}) = 1$ is allowed. Let us write $\hat{\lambda}''$ for the partition $(1^{e_1}, \dots, (m-1)^{e_{m-1}})$ of $|\hat{\lambda}| - me_m$ we get by removing all the maximal elements from $\hat{\lambda}$. Recall that in the $m = 1$ case, $\hat{\lambda}'' = \emptyset$ with $Y_{\hat{\emptyset}} = \mathrm{Pol}^0(\mathbb{C}^2) \setminus \{0\}$ the corresponding CRS. Consider the equivariant diagram for $x^k y^l Y_{\hat{\lambda}}$:

(74)

$$\begin{tikzcd}
E = \gamma^{\otimes m} \otimes \mathrm{Pol}^{k+l+|\hat{\lambda}|-me_m}(\mathbb{C}^2) \arrow[r, hook, "j"'] \arrow[dr] \arrow[rr, bend left, "\varphi"] & \mathbb{P}(\mathrm{Pol}^{e_m}(\mathbb{C}^2)) \times \mathrm{Pol}^{k+l+|\hat{\lambda}|}(\mathbb{C}^2) \arrow[r, "\pi"'] \arrow[d, "p"] & \mathrm{Pol}^{k+l+|\hat{\lambda}|}(\mathbb{C}^2) \\
 & \mathbb{P}^{e_m} = \mathbb{P}(\mathrm{Pol}^{e_m}(\mathbb{C}^2)) &
\end{tikzcd},$$

where $\gamma \to \mathbb{P}^{e_m}$ denotes the tautological line bundle, and the embedding j is induced by multiplication of polynomials. The $\mathbb{T}^2$-invariant subset $x^k y^l Y_{\hat{\lambda}''}$ determines a subbundle

$$x^k y^l Y_{\hat{\lambda}''}(E) \subset E,$$

see Section 2.3.1, and φ restricted to this subbundle is a $\mathbb{T}^2$-equivariant isomorphism to its image.

This image $\varphi(x^k y^l Y_{\hat{\lambda}''}(E))$ consists of partitions we get by first adding up some elements of (m^{e_m}), then merging some of those to elements of $\hat{\lambda}''$. More precisely, let τ be a partition of e_m, then for each subset $\upsilon \subset \tau$ denote by P_υ the set of partitions (of size $|\hat{\lambda}|$) we get by merging

υ^m into partitions we obtain by adding all the elements of $(\tau \setminus \upsilon)^m$ to elements of $\hat{\lambda}''$, where the exponentiation is meant coordinate-wise. Let

$$\mathbb{V}\hat{\lambda} = \bigcup_{\tau \vdash e_m} \bigcup_{\upsilon \subset \tau} P_\upsilon.$$

Then $\varphi(x^k y^l Y_{\hat{\lambda}''}(E)) = \bigcup_{\hat{\mu} \in \mathbb{V}\hat{\lambda}} x^k y^l Y_{\hat{\mu}}$. Here, choices $\tau = 1^{e_m}$ and $\upsilon = \tau$ correspond to the partition $\hat{\lambda}$. Since for all the other partitions in $\mathbb{V}\hat{\lambda}$ their length is strictly smaller, and

$$\operatorname{codim}(x^k y^l Y_{\hat{\mu}} \subset \operatorname{Pol}^{k+l+|\hat{\mu}|}(\mathbb{C}^2)) = k + l + |\hat{\mu}| - l(\hat{\mu}),$$

we see that $x^k y^l Y_{\hat{\lambda}}$ is the smallest codimensional stratum of the image. For e.g. $\hat{\lambda} = (5,5,3,3,3,1,1)$ partitions of $\mathbb{V}\hat{\lambda}$ are

$$(5,5,3,3,3,1,1), (8,5,3,3,1,1), (6,5,3,3,3,1), (8,8,3,1,1), (8,6,3,3,1), (6,6,3,3,3),$$
$$(10,3,3,3,1,1), (13,3,3,1,1), (11,3,3,3,1).$$

4.3.2. Deducing the recursive formula from the resolution. The preceding section, combined with the motivic property of the mC class, implies that

$$\mathrm{mC}(x^k y^l Y_{\hat{\lambda}} \subset \operatorname{Pol}^{k+l+|\hat{\lambda}|}(\mathbb{C}^2)) = \varphi_! \, \mathrm{mC}\left(x^k y^l Y_{\hat{\lambda}''}(E) \subset E\right) - \sum_{\hat{\lambda} \neq \hat{\mu} \in \mathbb{V}\hat{\lambda}} \mathrm{mC}(x^k y^l Y_{\hat{\mu}} \subset \operatorname{Pol}^{k+l+|\hat{\lambda}|}(\mathbb{C}^2)).$$

The pushforward can be calculated using Proposition 4.1.1:

$$\varphi_! \, \mathrm{mC}(x^k y^l Y_{\hat{\lambda}''}(E) \subset E) = \pi_! \, \mathrm{mC}\left(j\left(x^k y^l Y_{\hat{\lambda}''}(E)\right) \subset \mathbb{P}^{e_m} \times \operatorname{Pol}^{k+l+|\hat{\lambda}|}(\mathbb{C}^2)\right) = \sum_{f \in \mathcal{F}(\mathbb{P}^{e_m})} \frac{\mathrm{mC}\left(j\left(x^k y^l Y_{\hat{\lambda}''}(E)\right) \subset \mathbb{P}^{e_m} \times \operatorname{Pol}^{k+l+|\hat{\lambda}|}(\mathbb{C}^2)\right)\Big|_f}{e\left(T_f \mathbb{P}^{e_m}\right)}, \tag{75}$$

where the summation is over the $\mathbb{T}^2$-fixed points of $\mathbb{P}(\operatorname{Pol}^{e_m}(\mathbb{C}^2))$. In the rest of this section we show how terms in this localization formula can be calculated.

For the fixed point $f = \langle x^i y^{e_m - i} \rangle \in \mathbb{P}(\operatorname{Pol}^{e_m}(\mathbb{C}^2))$ the denominator of (75) is

$$e\left(T_f \mathbb{P}^{e_m}\right) = \prod_{\substack{j=0 \\ j \neq i}}^{e_m} \left(1 - \frac{\alpha^i \beta^{e_m - i}}{\alpha^j \beta^{e_m - j}}\right), \tag{76}$$

as we can apply motivic calculus to the representation

$$T_f \mathbb{P}^{e_m} \cong \bigoplus_{\substack{j=0 \\ j \neq i}} \langle x^i y^{e_m - i} \rangle^\vee \otimes \langle x^j y^{e_m - j} \rangle$$

with (multiplicative) characters $(\alpha^j \beta^{e_m - j})/(\alpha^i \beta^{e_m - i})$ for $i \neq j = 0, \ldots, e_m$.

To calculate the numerators, let $\omega = \mathrm{mC}\left(j\left(x^k y^l Y_{\hat{\lambda}''}(E)\right) \subset \mathbb{P}^{e_m} \times \operatorname{Pol}^{k+l+|\hat{\lambda}|}(\mathbb{C}^2)\right)$. To compute the local contribution $\omega|_f$, we first restrict ω to $p^{-1}(U_f)$, where $f \in U_f \subset \mathbb{P}^{e_m}$ is a trivializing neighbourhood of E. Then the isomorphisms

$$j\left(x^k y^l Y_{\hat{\lambda}''}(E)\right) \cap p^{-1}(U_f) \cong U_f \times x^k y^l Y_{\hat{\lambda}''}(E)_f \cong U_f \times x^k y^l (x^i y^{e_m - i})^m Y_{\hat{\lambda}''},$$

combined with the fact that motivic classes are local and multiplicative, gives that

$$\omega|_f = \omega|_{U_f}\Big|_f = \mathrm{mC}\left(U_f \times x^k y^l (x^i y^{e_m - i})^m Y_{\hat{\lambda}''} \subset U_f \times \operatorname{Pol}^{k+l+|\hat{\lambda}|}(\mathbb{C}^2)\right)\Big|_f = \mathrm{mC}(\mathbb{P}^{e_m})\big|_f \, \mathrm{mC}\left(x^{k+mi} y^{l+m(e_m - i)} Y_{\hat{\lambda}''} \subset \operatorname{Pol}^{k+l+|\hat{\lambda}|}(\mathbb{C}^2)\right). \tag{77}$$

The first term of (77) is

$$\mathrm{mC}(\mathbb{P}^{e_m})|_f = \frac{1}{1+y}\left(\prod_{j=0}^{e_m}\left(1+\frac{y}{\alpha^j\beta^{e_m-j}}\right)-\prod_{j=0}^{e_m}\left(1-\frac{1}{\alpha^j\beta^{e_m-j}}\right)\right)\Bigg|_{\substack{\alpha\mapsto\alpha^{1-i/e_m}\beta^{i/e_m-1}\\ \beta\mapsto\alpha^{-i/e_m}\beta^{i/e_m}}}. \tag{78}$$

This can be shown in two steps. First, following [**Feh21**], we show that

$$\mathrm{mC}\left(\mathbb{P}^{e_m}\right)=\frac{1}{1+y}\,\mathrm{mC}(\mathrm{Pol}^{e_m}(\mathbb{C}^2)\setminus\{0\}\subset\mathrm{Pol}^{e_m}(\mathbb{C}^2))\big|_{\substack{\alpha\mapsto\alpha t^{-1/e_m}\\ \beta\mapsto\beta t^{-1/e_m}}},$$

where $t\in K_{\mathbb{T}^2}\left(\mathbb{P}^{e_m}\right)$ denotes the class of the tautological bundle with the natural $\mathbb{T}^2$-action:

For the K-theoretic Kirwan map κ

$$\kappa\left(\mathrm{mS}_{\mathbb{T}^2\times\mathrm{GL}(1)}\left(\mathrm{Pol}^{e_m}(\mathbb{C}^2)\setminus\{0\}\subset\mathrm{Pol}^{e_m}(\mathbb{C}^2)\right)\right)=\mathrm{mS}_{\mathbb{T}^2}\left(\mathbb{P}(\mathrm{Pol}^{e_m}(\mathbb{C}^2))\right),$$

where $\mathrm{GL}(1)$ acts on $\mathrm{Pol}^{e_m}(\mathbb{C}^2)$ via scalar multiplication. As the $\mathbb{T}^2$-representation $\mathrm{Pol}^{e_m}(\mathbb{C}^2)$ contains the scalars, see Section 2.3.3, the former class can be obtained from $\mathrm{mS}_{\mathbb{T}^2}\left(\mathrm{Pol}^{e_m}(\mathbb{C}^2)\setminus\{0\}\subset\mathrm{Pol}^{e_m}(\mathbb{C}^2)\right)$. We can get back to the mC class correspondence by multiplying with

$$\frac{\mathrm{mC}_{\mathbb{T}^2}(\mathbb{P}^{e_m})}{\kappa(\mathrm{mC}_{\mathbb{T}^2\times\mathrm{GL}(1)}(\mathrm{Pol}^{e_m}(\mathbb{C}^2)))}=\frac{\lambda_y\left(T^\vee\mathbb{P}^{e_m}\right)}{\kappa\left(\lambda_y(T^\vee\,\mathrm{Pol}^{e_m}(\mathbb{C}^2))\right)}=\frac{1}{1+y},$$

as e.g. exterior powers of the Euler exact sequence show.

The second step is the restriction to the fixed point $f=\langle x^iy^{e_m-i}\rangle$. This amounts to further substituting $t\mapsto\alpha^i\beta^{e_m-i}$. Combining these substitutions, we get (78).

To calculate the second term of (77),

$$\mathrm{mC}\left(x^{k+mi}y^{l+m(e_m-i)}Y_{\hat\lambda''}\subset\mathrm{Pol}^{k+l+|\hat\lambda|}(\mathbb{C}^2)\right) \tag{79}$$

we have to solve a problem like the original. We do this the same way, using an $x^{k+mi}y^{l+m(e_m-i)}Y_{\hat\lambda''}$ analog of the diagram (74). In case $\hat\lambda=m^{e_m}$, $\hat\lambda''=\emptyset$, and (79) becomes

$$\mathrm{mC}(x^{k+mi}y^{l+m(e_m-i)}\setminus\{0\}\subset\mathrm{Pol}^{k+l+me_m}(\mathbb{C}^2))=\frac{1+y}{\alpha^{k+mi}\beta^{l+m(e_m-i)}}\prod_{\substack{j=0\\ j\neq k+mi}}^{k+l+me_m}\left(1-\frac{1}{\alpha^j\beta^{k+l+me_m-j}}\right).$$

The above recursion works, since the classes $\mathrm{mC}(x^sy^tY_{\hat\mu}\subset\mathrm{Pol}^{k+l+|\hat\lambda|}(\mathbb{C}^2))$ needed to calculate $\mathrm{mC}(x^ky^lY_{\hat\lambda}\subset\mathrm{Pol}^{k+l+|\hat\lambda|}(\mathbb{C}^2))$ are for $\hat\mu$'s that are either smaller or their corresponding shifted CRS is of bigger codimension.

For example, the diagram below shows some of the steps the recursion has to go through when calculating $\mathrm{mC}(Y_{5,5,1,1,1}\subset\mathrm{Pol}^{13}(\mathbb{C}^2))$. More precisely, each column lists the shifted CRS whose mC class we have to compute to determine the underlined class to its left.

- $\underline{Y_{5,5,1,1,1}}$
  - $\underline{Y_{6,5,1,1}}$
    - $\underline{Y_{11,1,1}}$
      - $\underline{Y_{12,1}}$
        - $\underline{Y_{13}}$
          - $x^{13}Y_\emptyset$
          - $y^{13}Y_\emptyset$
        - $x^{12}Y_1$
        - $y^{12}Y_1$
      - $x^{11}Y_{1,1}$
      - $y^{11}Y_{1,1}$
    - $Y_{7,5,1}$
    - $x^6Y_{5,1,1}$
    - $y^6Y_{5,1,1}$
  - $Y_{6,6,1}$
  - $Y_{10,1,1,1}$
  - $Y_{11,1,1}$
  - $x^{10}Y_{1,1,1}$
  - $x^5y^5Y_{1,1,1}$
  - $y^{10}Y_{1,1,1}$

4.3.3. Bases of $K_{GL(2)}(\mathrm{pt})$ and a conjecture on polynomial d-dependence of $\mathrm{mC}(Y_\lambda(d)$. The algorithm gives the motivic Chern classes of CRS in the α,β-monomial base. CRS are GL(2)-invariant, so these classes are symmetric in α,β, and the isomorphisms (64) correspond to writing them as polynomials in

$$\alpha+\beta=[\mathbb{C}^2_{\mathrm{Id}^\vee_{\mathrm{GL}(2)}}] \text{ and } \alpha\beta=[\Lambda^2\mathbb{C}^2_{\mathrm{Id}^\vee_{\mathrm{GL}(2)}}]\in K_{\mathrm{GL}(2)}(\mathrm{pt}),$$

where $[\mathbb{C}^2_{\mathrm{Id}^\vee_{\mathrm{GL}(2)}}]$ is the class represented by the dual of the standard representation of GL(2). This basis we will call the *representation theoretic basis* of $K_{\mathrm{GL}(2)}(\mathrm{pt})$.

Analyzing the recursive formula (75), it is easy to see that in $\mathrm{mC}(Y_{\hat\lambda})$ the total degrees of monomials with nonzero coefficients are divisible by $|\hat\lambda|$. For example, after changing to the variables $X=\alpha^{-1}, Y=\beta^{-1}$, we have

$$\begin{aligned}\mathrm{mC}(Y_2)=\left(-X^3Y^3+XY\right)y^2+\left(-X^3Y-2\,X^2Y^2-XY^3+X^2+2\,XY+Y^2\right)y+\\ X^3Y^3-X^3Y-2\,X^2Y^2-XY^3+X^2+XY+Y^2,\end{aligned}$$

$$\begin{aligned}\mathrm{mC}(Y_{2,1})=\left(-X^6Y^6-X^5Y^4-X^4Y^5+X^4Y^2+X^3Y^3+X^2Y^4\right)y^3+\\ \left(-2\,X^6Y^3-4\,X^5Y^4-4\,X^4Y^5-2\,X^3Y^6+X^5Y+3\,X^4Y^2+4\,X^3Y^3+3\,X^2Y^4+XY^5\right)y^2+\\ (\dots)y-X^5Y^4-X^4Y^5+X^2Y+XY^2.\end{aligned}$$

This shows that we cannot expect d-dependence of the coefficients in the representation theoretic basis to be polynomial.

However, based on the examples we calculated, we conjecture that stability holds for the coefficients in the topological basis of $K_{\mathrm{GL}(2)}(\mathrm{pt})$: The Koszul complex definition of the Euler class works in the equivariant setting, meaning that (equivariant) Chern roots of $\mathbb{C}^2_{\mathrm{Id}^\vee_{\mathrm{GL}(2)}}$ are

$$U:=1-\alpha^{-1}=1-X \text{ and } V:=1-\beta^{-1}=1-Y,$$

while its (equivariant) Chern classes can be defined, using the splitting principle, as elementary symmetric polynomials in U and V.

The *topological basis*, i.e. monomials of e.g. U,V, seems to have several advantages over the representation theoretic basis. For example, it seems to respect codimension:

CONJECTURE 4.3.2. *Written in the topological basis, degree of the lowest degree term of the motivic Chern class of a subvariety (closed or not) is the codimension.*

Also, polynomial d-dependence seems to be true:

CONJECTURE 4.3.3. *Let λ denote a partition without 1's and fix exponents j,k and l. Then, for large enough d's, coefficients of the monomial $y^jU^kV^l$ in $\mathrm{mC}(Y_\lambda(d)\subset \mathrm{Pol}^d(\mathbb{C}^2))$ form a polynomial in $\mathbb{Q}[d]$.*

For example, coefficients of y^0U^2V in $\mathrm{mC}(Y_2(d))$ seems to form a polynomial

$$\frac{3}{2}d^6-\frac{21}{2}d^5+\frac{295}{12}d^4-\frac{3}{2}d^3-\frac{643}{12}d^2+\frac{39}{2}d.$$

We yet to give a complete proof for both conjectures. Relating $\mathrm{mC}(\overline{Y}_2(d))$ to the motivic Chern class of the dual of a generic plane curve, see Section 4.4.3, proves the polynomiality assumption for a small number of coefficients in $\mathrm{mC}(\overline{Y}_2(d))$.

4.4. Motivic Chern classes of varieties of λ-lines

In this part, for the last time, we make use of the fact that varieties of λ-lines of generic hypersurfaces are coincident root loci: We see what further information about them can be deduced from motivic Chern classes of CRS.

In Section 1.2 we identified $\mathcal{T}_\lambda Z_f$ with the locus $\sigma_f^{-1}(Y_\lambda(d))$. This means that we can use Corollary 4.1.6 to show that for a polynomial $f\in \mathrm{Pol}^d(\mathbb{C}^n)$ such that $\sigma_f:\mathrm{Gr}_2(\mathbb{C}^n)\to \mathrm{Pol}^d(S)$ is motivically transversal to $Y_\lambda(\mathrm{Pol}^d(S))$

$$\mathrm{mS}(\mathcal{T}_\lambda Z_f\subset \mathrm{Gr}_2(\mathbb{C}^n))=\mathbf{a}\left(\mathrm{mS}_{\mathrm{GL}(2)}(Y_\lambda(d))\right). \tag{80}$$

Corollary A.1.3 shows that this transversality condition holds for a generic polynomial in $\mathrm{Pol}^d(\mathbb{C}^n)$. First, let us describe the K-theory of Grassmannians.

4.4.1. K-theory and motivic Chern classes of complex Grassmannians. As we have mentioned, (topological) K-theory is a complex oriented cohomology theory, so it shares many common properties with singular cohomology. For example, K-theoretic Euler class

$$e_K(L)=1-[L^\vee]\ \text{ for a line bundle } L$$

and K-theoretic Chern classes

$$c_i^K(E):=\sigma_i(e_K(L_1),\dots,e_K(L_r))\text{ for a vector bundle } E=\oplus_{t=1}^r L_t$$

(in K^0_{top}) are defined, and the corresponding version of the projective bundle formula holds. This implies for the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ (or similarly for partial flag manifolds) that the map $c_i^H(S^\vee)\mapsto c_i^K(S^\vee)$ induces a ring isomorphism $H^*(\mathrm{Gr})\to K(\mathrm{Gr})$, where S denotes the tautological bundle.

From now on, let us restrict our attention to the $k=2$ case. Monomials in $c_1(S^\vee),c_2(S^\vee)$ (or monomials in Chern roots of $S^\vee$: $U=1-[L_1]=1-X$ and $V=1-[L_2]=1-Y$ for some $S=L_1\oplus L_2$) we will call the *topological basis* of $K(\mathrm{Gr}_2(\mathbb{C}^n))$.

The above also implies that monomials in $[S]=X+Y$ and $[\Lambda^2S]=XY$ also generate $K(\mathrm{Gr}_2(\mathbb{C}^n))$. An independent subset of these we will call the *representation theoretic basis* of $K(\mathrm{Gr}_2(\mathbb{C}^n))$ for reasons that will become apparent in the next section. Relations in the K-theory of the Grassmannians in this basis are complicated.

There is a third, well-known basis of $K(\mathrm{Gr})$ consisting of Grothendieck polynomials G_λ, the K-classes of Schubert varieties. Intergration in this basis is easy: $\int_{\mathrm{Gr}}G_\lambda=1$.

The mC class of the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ can be expressed (in the representation theoretic basis) analogously to the CSM case (60) as

$$\lambda_y(T^\vee\mathrm{Gr}_2(\mathbb{C}^n))=\frac{\lambda_y\left(\mathrm{Hom}\left(S,\mathrm{Gr}_2(\mathbb{C}^n)\times\mathbb{C}^n\right)^\vee\right)}{\lambda_y\left(\mathrm{Hom}(S,S)^\vee\right)}=\frac{(1+yX)^n(1+yY)^n}{(1+y)^2(1+yXY^{-1})(1+yX^{-1}Y)}.$$

4.4.2. Map by association to the frame bundle of the tautological. For the frame bundle $P \to \mathrm{Gr}_2(\mathbb{C}^n)$ of the tautological bundle

$$\mathbf{a} : K_{\mathrm{GL}(2)}(\mathrm{pt}) \cong \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]^{S_2} \to K(\mathrm{Gr}_2(\mathbb{C}^n)),$$

maps $X+Y = [\mathbb{C}^2_{\mathrm{Id}_{\mathrm{GL}(2)}}]$ to $[P \times_{\mathrm{Id}_{\mathrm{GL}(2)}} \mathbb{C}^2] = [S]$ and $XY = [\Lambda^2\mathbb{C}^2_{\mathrm{Id}_{\mathrm{GL}(2)}}]$ to $[\Lambda^2 S]$, that is, $\mathbf{a}$ maps basis elements of $K_{\mathrm{GL}(2)}(\mathrm{pt})$ to respective basis elements of $K(\mathrm{Gr}_2(\mathbb{C}^n))$, e.g. $X+Y \in K_{\mathrm{GL}(2)}(\mathrm{pt})$ to $X+Y \in K(\mathrm{Gr}_2(\mathbb{C}^n))$.

As a consequence, applying $\mathbf{a} : K_{\mathrm{GL}(2)}(\mathrm{pt}) \to K(\mathrm{Gr}_2(\mathbb{C}^n))$ to elements written in the representation theoretic or in the topological basis, e.g. to $\mathrm{mC}_{\mathrm{GL}(2)}\left(\mathrm{Pol}^d(\mathbb{C}^2)\right) = \prod_{t=0}^d \left(1 + yX^tY^{d-t}\right)$ amounts to factoring out by relations of $K(\mathrm{Gr}_2(\mathbb{C}^n))$.

Equation (80) can be written as

$$\mathrm{mC}(\mathcal{T}_\lambda Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n)) = \mathrm{mC}(\mathrm{Gr}_2(\mathbb{C}^n)) \frac{\mathbf{a}(\mathrm{mC}_{\mathrm{GL}(2)}(Y_{\hat{\lambda}}))}{\mathbf{a}(\mathrm{mC}_{\mathrm{GL}(2)}(\mathrm{Pol}^{|\hat{\lambda}|}(\mathbb{C}^2)))}. \tag{81}$$

These show that for any partition $\hat{\lambda}$ and generic $f \in \mathrm{Pol}^{|\hat{\lambda}|}(\mathbb{C}^2)$ we have a way of calculating $\mathrm{mC}(\mathcal{T}_{\hat{\lambda}} Z_f \subset \mathrm{Gr}_2(\mathbb{C}^n))$.

REMARK 4.4.1. Let us remark that, by introducing a pairing for the K-theory of partial flag manifolds, a K-theoretic analog of the analysis in Section 3.7.2 can be carried out. This shows that the motivic Chern class of a constructible subset of a partial flag manifold is encoded in the χ_y-genera of its sufficiently transversal Schubert cell sections.

4.4.3. Motivic Chern classes of dual curves. We can calculate the mC classes of duals of generic plane curves in two ways: The first one is to substitute Plücker formulas into the formula (71) describing mC classes of plane curves. The second is by applying Section 4.4.2 to deduce it from the equivariant $\mathrm{mC}(\overline{Y}_2(d) \subset \mathrm{Pol}^d(\mathbb{C}^2))$. Comparing the two, we show polynomial d-dependence of some of the coefficients of $\mathrm{mC}(\overline{Y}_2(d) \subset \mathrm{Pol}^d(\mathbb{C}^2))$:

Recall that the Milnor number of the cusp is 2, of the node is 1. The delta invariant is 1 for both. Therefore previous calculations of the degree $d^\vee$, the number of ordinary double points $\delta^\vee$ and the number of cusps $\kappa^\vee$ of the dual curve $C^\vee$ of a generic degree d plane curve C give that

$$d^\vee = d(d-1),\ M^\vee = 2\kappa^\vee + \delta^\vee = \frac{1}{2}d(d-2)(d^2+3),\ \Delta^\vee = \kappa^\vee + \delta^\vee = \frac{1}{2}d(d-2)(d^2-3),$$

where $M^\vee$ and $\Delta^\vee$ denotes the sum of Milnor numbers and the sum of delta invariants of $C^\vee$. Therefore

(82)
$$\begin{aligned}\mathrm{mC}(C^\vee \subset \mathbb{P}^2) &= (1+y)d^\vee H + \left(-\frac{d^\vee(d^\vee-1)}{2} + M^\vee - \Delta^\vee\right) H^2 + \left(\frac{(d^\vee)^2 - 5d^\vee}{2} - \Delta^\vee\right) yH^2 \\ &= (1+y)d(d-1)H - \frac{1}{2}d(d^3 - 2d^2 - 6d + 13)H^2 - \frac{1}{2}d(d+1)yH^2.\end{aligned}$$

On the other hand, combining formula (81) for $\overline{Y}_2, \overline{Y}_{2,1}, \dots$ with the isomorphism $K(\mathbb{P}(\mathbb{C}^3)^\vee) \to K(\mathrm{Gr}_2(C^3))$ induced by the Plücker embedding $\mathrm{Gr}_2(\mathbb{C}^3) \to \mathbb{P}(\mathbb{C}^3)^\vee$ also gives mC classes for duals of generic degree d curves for each $d \geq 2$.

Let us introduce coefficients for the topological basis,

$$\mathrm{mC}\left(\overline{Y}_2(d) \subset \mathrm{Pol}^d(\mathbb{C}^2)\right) = \sum_{j \geq 0} y^j \sum_{k,l \geq 0} p_{j,k,l}(d) U^k V^l.$$

Comparing the two approaches, we get that

$$p_{0,1,0}(d) = d(d-1) \quad p_{1,1,0}(d) = d^2(d-1) \quad p_{2,1,0}(d) = \frac{1}{2}d^2(d-1)^2 \quad \dots$$

$$p_{0,1,1}(d) - p_{0,2,0}(d) = -\frac{1}{2}d(d^3-2d^2-4d+11) \quad p_{1,1,1}(d) - p_{1,2,0}(d) = -\frac{1}{2}d(d^4-2d^3-2d^2+11d-6) \quad \dots .$$

Appendix A. Transversality

A.1. Transversality of a generic section

PROPOSITION A.1.1. *(**Bertini for globally generated bundles**: [**FNR12**, Prop. 6.4]) Let $E \to X$ be an algebraic vector bundle, B a vector space and $\varphi : B \to \Gamma(E)$ is a linear family of algebraic sections. Suppose that E is generated by the sections $\varphi(b), b \in B$, i.e. $\Phi(b, x) := \varphi(b)(x) : B \times X \to E$ is surjective, and Y is a closed subvariety of the total space E. Then there is an open subset U of B such that for all $b \in U$ the section $\varphi(b)$ is transversal to Y.*

EXAMPLE A.1.2. For our family of algebraic sections

$$\varphi : \mathrm{Pol}^d(\mathbb{C}^n) \to \Gamma(\mathrm{Pol}^d(S)),\ f \mapsto \sigma_f$$

$\mathrm{Pol}^d(S)$ is clearly generated by the sections σ_f, hence for a generic $f \in \mathrm{Pol}^d(\mathbb{C}^n)$ σ_f is transversal to any subvariety $\overline{Y}_\lambda(\mathrm{Pol}^d(S))$.

Since the definition of proper normal crossing extension involves finitely many maps, the above proposition implies the following motivic version:

COROLLARY A.1.3. *Let $E \to X$, $\varphi : B \to \Gamma(E)$ and $Y \subset E$ be as above. Then there is an open subset U of B such that for all $b \in U$ the section $\varphi(b)$ is motivically transversal to Y.*

A.2. Transversality of the universal section

The following is a modification of the idea of [**FRW18**, Prop. 8.11]. We show that, with the proper definitions, universal sections are transversal.

DEFINITION A.2.1. Let V be a G-vector space and assume that $j : P \hookrightarrow V$ is an open G-invariant subset such that $\pi : P \to P/G$ is a principal G-bundle over the smooth $M := P/G$. Let W be another G-vector space and let $\vartheta : V \to W$ be a G-equivariant linear map. Then $\vartheta \circ j : P \to W$ is G-equivariant, therefore

$$\sigma_\vartheta : M \to P \times_G W,\ [p] \mapsto [p, \vartheta(p)]$$

determines a section of the associated bundle $P \times_G W$. We call σ_ϑ the *universal section of* ϑ.

PROPOSITION A.2.2. *If $\vartheta : V \to W$ is surjective then the universal section σ_ϑ is transversal to $P \times_G Z$ for any G-invariant constructible subset $Z \subset W$.*

PROOF. The question is local, so let $\varphi : U \to P$ be a local slice to P. Then $\pi\varphi : U \to M$ is a chart of M. In this local trivialization σ_ϑ has a particularly simple form: If $m := \pi\varphi(u)$ for some $u \in U$, then $\sigma_\vartheta(m) = (m, \vartheta(\varphi(u)))$. By definition, a local slice is transversal to every G-orbit of P. As $\vartheta^{-1}(Z) \subset V$ is G-invariant, for every G-orbit of P, $\vartheta^{-1}(Z)$ either contains the whole orbit or they are disjoint. Therefore σ_{Id} is transversal to $P \times_G \sigma^{-1}(Z)$. If ϑ is a surjective G-equivariant linear map, this implies that σ_ϑ is transversal to $P \times_G Z$. □

EXAMPLE A.2.3. In Section 2.8.1 we introduced the vector bundle

$$A_u := \mathrm{Hom}\big(L, \mathrm{Pol}^d(S)\big) \quad \to \quad \mathbb{P}\big(\mathrm{Pol}^d(\mathbb{C}^n)\big) \times \mathrm{Gr}_2(\mathbb{C}^n),$$

and its section

$$\sigma([f], A)(f) := f|_A,$$

where L and S are the tautological bundles over $\mathbb{P}\big(\operatorname{Pol}^d(\mathbb{C}^n)\big)$ and $\operatorname{Gr}_2(\mathbb{C}^n)$.

This is a special case of the previous construction with the choices $V = \operatorname{Pol}^d(\mathbb{C}^n) \oplus \operatorname{Hom}(\mathbb{C}^2, \mathbb{C}^n)$, $W = \operatorname{Pol}^d(\mathbb{C}^2)$, $P = \big(\operatorname{Pol}^d(\mathbb{C}^n) \setminus \{0\}\big) \times \Sigma^0(\mathbb{C}^2, \mathbb{C}^n)$, $G = \mathrm{GL}(1) \times \mathrm{GL}(2)$ and

$$\vartheta(f, \beta) = f \circ \beta.$$

The surjectivity of ϑ is clear, so the universal section is transversal to the $\overline{Y}_\lambda(A_u)$'s, therefore our calculations for the cohomology class of $[\sigma^{-1}(\overline{Y}_\lambda(A_u))]$ is valid.

Appendix B. Degrees of Plücker numbers

This chapter is dedicated for the proof of Theorem 2.4.8. The proof is a straightforward application of the recursive formula in Theorem 2.2.5 and is very technical. To make it more concise, let us in this chapter use the shorthand $\rho \vdash k$ for partitions ρ of k with length at most 2. We also introduce an ordering on $\{\rho | \, \rho \vdash k\}$ according to the usual ordering of $\pi_2(\rho)$, the projection to the second coordinate:

$$(k,0) \leq (k-1,1) \leq \cdots \leq \left(\left\lceil \frac{k}{2} \right\rceil, \left\lfloor \frac{k}{2} \right\rfloor \right).$$

The proof can arranged into two theorems, the first one directly reflecting our recursive formula:

THEOREM B.2.4. *Let $\lambda = (2^{e_2}, \ldots, r^{e_r})$ be a partition without 1's. The class of the corresponding coincident root stratum can be expressed in Schur polynomials*

$$\left[\overline{Y}_\lambda(d)\right] = \sum_{\rho \vdash c} r_\rho s_\rho \quad \left(c = |\tilde{\lambda}| = \operatorname{codim}\left(Y_\lambda \subset \operatorname{Pol}^d\left(\mathbb{C}^2\right)\right)\right),$$

where $r_\rho \in \mathbb{Q}[d]$.

Let m be any member of λ ($e_m \neq 0$) and denote by $\lambda' = (2^{e_2}, \ldots, m^{e_m - 1}, \ldots, r^{e_r})$ the partition λ minus m.

i) Then for any $\rho \vdash c$ the coefficients of s_ρ in

$$\left[\overline{Y}_\lambda(d)\right] \text{ and } \frac{1}{e_m}\left[\overline{Y}_m(d)\right]\left[\overline{Y}_{\lambda'}(d)\right]$$

have the same leading term.

ii) If m is such that $m-2 \leq c' := |\tilde{\lambda'}| = \operatorname{codim}(Y_{\lambda'} \subset \operatorname{Pol}^d(\mathbb{C}^2))$, (e.g. $m = \min(\lambda)$), then for any $\rho \vdash c$ the coefficients of s_ρ in

$$\left[\overline{Y}_\lambda(d)\right] \text{ and } \frac{1}{e_m} p_{(m-1,0)} s_{(m-1,0)} \left[\overline{Y}_{\lambda'}(d)\right]$$

have the same leading term, where $\left[\overline{Y}_m(d)\right] = \sum_{\mu \vdash m-1} p_\mu s_\mu$.

Note that according to Theorem 2.2.5, strictly speaking, we should compare e.g. $\left[\overline{Y}_\lambda(d+m)\right]$ with $1/e_m \left[\overline{Y}_m(d+m)\right]\left[\overline{Y}_{\lambda'}(d)\right]$. However, the "$+m$" translation doesn't change the leading coefficient, hence its omission from the above theorem.

The second is Theorem 2.4.8, slightly reformulated:

THEOREM B.2.5. *Let λ_1 be the largest number in the partition λ. Denote by $c = |\tilde{\lambda}|$ the codimension of the corresponding coincident root stata $Y_\lambda \subset \operatorname{Pol}^d(\mathbb{C}^2)$. Then for the coefficients r_ρ in its class $\left[\overline{Y}_\lambda(d)\right] = \sum_{\rho \vdash c} r_\rho s_\rho$*

$$\deg_d(r_\rho) = \begin{cases} |\lambda| & \text{if } \rho \leq (\lambda_1 - 1, c - \lambda_1 + 1) \\ |\lambda| - (\pi_2(\rho) - (c - \lambda_1 + 1)) & \text{if } \rho > (\lambda_1 - 1, c - \lambda_1 + 1). \end{cases}$$

Plainly speaking, r_ρ has degree $|\lambda|$ for $\rho=(c,0),\dots,(\lambda_1-1,c-\lambda_1+1)$, then by increasing ρ by one, the degree drops by one.

Throughout their proof we will use the following:

THEOREM B.2.6. *For any partition λ and index $\rho \vdash c := |\tilde{\lambda}|$ the leading coefficient of r_ρ in $\left[\overline{Y}_\lambda(d)\right] = \sum_{\rho\vdash c} r_\rho s_\rho$ is positive.*

PROOF. Theorem B.2.6 has a geometric proof: for big enough d's the values of $r_\rho \in \mathbb{Q}[d]$ are solutions to enumerative problems. Those solutions are necessarily nonnegative, therefore the leading coefficients of the r_ρ's must be nonnegative as well. We will prove the remaining two theorems at once using induction on the length of the partition λ.

Induction starts with $\lambda=(m)$, where by Theorem 2.5.1, $p_{(m-1-l,l)}$ in

$$\left[\overline{Y}_m(d)\right] = \sum_{\mu\vdash m'} p_\mu(d) s_\mu \qquad \left(m' := m-1 = \operatorname{codim}\left(Y_m \subset \operatorname{Pol}^d\left(\mathbb{C}^2\right)\right)\right)$$

has the expected degree, $\deg_d p_{(m-1-j,j)} = m-1-j$.

Let m be a member of λ, and denote by $\lambda' = (2^{e_2},\dots,m^{e_m-1},\dots,r^{e_r})$ the partition λ minus one instance of m. λ' has length 1 less than λ so we can assume Theorem B.2.4 and B.2.5 hold for the coefficients q_ν in

$$\left[\overline{Y}_{\lambda'}(d)\right] = \sum_{\nu\vdash c'} q_u s_\nu \qquad \left(c' := |\tilde{\lambda'}| = \operatorname{codim}\left(Y_{\lambda'} \subset \operatorname{Pol}^d\left(\mathbb{C}^2\right)\right)\right).$$

Then by Theorem 2.2.5

$$\begin{aligned}\left[\overline{Y}_\lambda(d+m)\right] &= \frac{1}{e_m}\partial\left(\left[\overline{Y}_{\lambda'}(d)\right]\Big|_{\substack{a=a+(m/d)a\\ b=b+(m/d)a}} \cdot \prod_{i=0}^{m-1}(ia+(d+m-i)b)\right)\\ &= \frac{1}{e_m}\partial\left(\left[\overline{Y}_{\lambda'}(d)\right]\Big|_{\substack{a=a+x\\ b=b+x}}\Bigg|_{x=(m/d)a} \cdot \prod_{i=0}^{m-1}(ia+(d+m-i)b)\right).\end{aligned} \tag{83}$$

Let us keep the variable x for a moment and introduce $B_t \in \mathbb{Q}\left[a,b;d\right]^{S_2}$, $t=0,\dots,c'$—symmetric of degree $c'-t$ in a,b —and its Schur polynomial coefficients $q_\mu \in \mathbb{Q}[d]$, $\mu \vdash c'-t$ as in

$$\left[\overline{Y}_{\lambda'}(d)\right]\Big|_{\substack{a=a+x\\ b=b+x}} = \sum_{t=0}^{c'} B_t x^t = \sum_{t=0}^{c'}\sum_{\nu\vdash c'-t}\left(q_\nu s_\nu\right)x^t.$$

Note that $B_0 = \left[\overline{Y}_{\lambda'}(d)\right]$. Then we can expand (83) as

$$\begin{aligned}\left[\overline{Y}_\lambda(d+m)\right] &= \frac{1}{e_m}\partial\left(\sum_{t=0}^{c'}\left(B_t\left(\frac{m}{d}a\right)^t\right)\cdot\prod_{i=0}^{m-1}(ia+(d+m-i)b)\right)\\ &= \frac{1}{e_m}\sum_{t=0}^{c'} B_t\left(\frac{m}{d}\right)^t\cdot\partial\left(a^t\prod_{i=0}^{m-1}(ia+(d+m-i)b)\right).\end{aligned} \tag{84}$$

Let us further introduce $A_t \in \mathbb{Q}\left[a,b;d\right]^{S_2}$, $t=0,\dots,c'$—symmetric of degree $m'+t$ in a,b—and its Schur polynomial coefficients $p_\mu \in \mathbb{Q}[d]$, $\mu\vdash m'+t$ as in

$$A_t = \partial\left(a^t\prod_{i=0}^{m-1}(ia+(d+m-i)b)\right) = \sum_{\mu\vdash m'+t} p_\mu s_\mu.$$

Note that $A_0 = [Y_m(d+m)]$. Then we can continue (84) as

$$\begin{aligned}\left[\overline{Y}_\lambda(d+m)\right] &= \frac{1}{e_m}\sum_{t=0}^{c'}\left(\frac{m}{d}\right)^t A_tB_t \\ &= \frac{1}{e_m}\sum_{t=0}^{c'}\left(\frac{m}{d}\right)^t \sum_{\mu\vdash m'+t}(p_\mu s_\mu)\sum_{\nu\vdash c'-t}(q_\nu s_\nu)\end{aligned} \tag{85}$$

To prove Theorem B.2.4 is to show—assuming the induction hypotheses for the partitions (m) and λ'—that for every $\rho \vdash c = m' + c'$ and every $t > 0$

$$\deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^t A_tB_t\right) < \deg_d(r_\rho). \tag{$*$}$$

In particular, the leading term of r_ρ and the coefficient of s_ρ in $\frac{1}{e_m}A_0B_0$ (or—in the special case *ii*) of Theorem B.2.4) in $\frac{1}{e_m}p_{(m-1,0)}s_{(m-1,0)}B_0$) agree. Theorem B.2.5 then can be proved by choosing $m = \min(\lambda)$.

We proceed with the following steps.

(A) We show that degree of q_ν ($\nu \vdash c' - t$) depends only on $\pi_2(\nu)$ and that its leading coefficient is always positive.

(B) We describe line segments of $\{\mu \vdash m' + t \mid 0 \le t \le c'\}$ with $p_\mu \neq 0$ along which the degree of p_μ and the sign of its leading coefficient are constant, see Figure 1.

(C) For every $\rho \vdash c$ we define a function f_ρ such that for partitions $\mu \vdash m' + t$ $f_\rho(\mu)$ helps us to compare the d-degrees of coefficients of s_ρ's in the *μ-contributions*

$$\left(\frac{m}{d}\right)^t p_\mu s_\mu B_t = \left(\frac{m}{d}\right)^t p_\mu s_\mu \sum_{\nu\vdash c'-t} q_\nu s_\nu. \tag{86}$$

(D) Comparing these values $f_\rho(\nu)$, we determine monotonicity properties of d-degrees of μ-contributions along the line segments of (B).

(E) We show that from the above monotonicity properties

$$\deg_d(\text{coefficients of } s_\rho \text{ in } A_0B_0) > \deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^t A_tB_t\right) \tag{87}$$

and hence $(*)$ follows for any $\rho \vdash c$ and $t > 0$.

(F) We prove the *ii*) case of Theorem B.2.4.

(G) We conclude with a proof for Theorem B.2.5.

(A) A simple substitution into Jacobi's bialternant formula shows that

$$s_{(k,l)}\Big|_{\substack{a=a+x\\b=b+x}} = \sum_{t=0}^{k+l} x^{c'-t}\sum_{(u,v)\vdash t}\left(\binom{k+1}{u+1}\binom{l}{v} - \binom{k+1}{v}\binom{l}{u+1}\right)s_{(u,v)}. \tag{88}$$

Here, for all the $s_{(u,v)}$'s their coefficients are nonnegative and zero if $u > k$ or $v > l$. Hence, the coefficient of $s_{(u,v)}$ ($(u,v) \vdash c' - t$) in

$$\left[\overline{Y}_{\lambda'}(d)\right]\Big|_{\substack{a=a+x\\b=b+x}} = \sum_{\nu\vdash c'} q_\nu s_\nu\Big|_{\substack{a=a+x\\b=b+x}}$$

is x^t times a linear combination of elements in $\left\{q_{(k,l)}\,\middle|\,(u,v) \vdash c', u \le k \text{ and } v \le l\right\}$ with positive coefficients. Using the positivity property and the monotone decreasing nature of $(\deg_d(q_\nu))_{\nu\vdash c'}$

as in Theorem B.2.6 and Theorem B.2.5 part of the induction hypothesis for λ', we deduce that for every $t = 0, \dots, c'$ and $(k,l) \vdash c' - t$

$$\deg_d\left(q_{(k,l)}\right) = \deg_q\left(q_{(k+t,l)}\right) \text{ and the leading coefficient of } q_{(k,l)} \text{ is positive,} \tag{89}$$

that is, $\deg_d(q_\nu)$ only depends on $\pi_2(\nu)$, therefore

$$\begin{gathered}\deg_d(q_\nu) \text{ is monotone decreasing in } \pi_2(\nu) \text{ and} \\ \text{it's difference for } \nu\text{'s with adjacent } \pi_2(\nu)\text{'s is at most 1.}\end{gathered} \tag{90}$$

(B) Expanding its definition, we can write A_t as

$$A_t = \partial\left(a^t \prod_{i=0}^{m-1}(ia + (d+m-i)b)\right)_t = \sum_{f=1}^{m} e_f \partial\left(a^{t+m-f} b^f\right),$$

for some degree f polynomials $e_f \in \mathbb{Z}[d]$, independent of t, whose leading coefficient is positive. Using

$$\partial\left(a^{t+m-f} b^f\right) = \begin{cases} s_{f-1,m+t-f} & \text{if } \ 2f > m+t \\ 0 & \text{if } \ 2f = m+t \\ -s_{m+t-1-f,f} & \text{if } \ 2f < m+t \end{cases},$$

we get that for any $f = 1, \dots, m$ the Schur polynomials in the A_t terms with coefficient $+e_f$ are $s_{f-1,m-f+t}$, $t < 2f - m$ and the Schur polynomials in the A_t terms with coefficient $-e_f$ are $s_{m+t-1-f,f}$, $t > 2f - m$.

In other words, $\deg_d(p_\mu)$ is constant with positive coefficients along each segment

$$\{(i, j+t) \vdash m' + t |\, t \le i - j\}$$

corresponding to partitions $(i,j) \vdash m'$. We will call these segments *diagonal*. Similarly, $\deg_d(p_\mu)$ is constant with negative coefficients along each line

$$\{(i+t, j) |\, t \ge 0\}$$

corresponding to partitions (i,j), $1 \le j \le m$. We will call these segments *horizontal*, see the diagram below.

Our plan is to compare d-degrees of the coefficients of Schur polynomials in (85) along these segments.

(C) Products of Schur polynomials in two variables can be easily calculated using e.g. Pieri's formula. For $\mu \vdash m' + t$ and $\nu \vdash c' - t$

$$s_\mu s_\nu = \sum_{\rho \in I(\mu,\nu)} s_\rho,$$

where $I(\mu,\nu) = [p(\mu,\nu), P(\mu,\nu)]$ is an interval of partitions $\rho \vdash c$, with endpoints

$$p((i,j),(k,l)) = (i+k, j+l) \text{ and } P((i,j),(k,l)) = (\max(i+l, j+k), \min(i+l, j+k)).$$

The positivity of the leading coefficients of the q_ν's, see (89), implies that if $\rho \vdash c$ and $\mu \vdash m' + t$ are partitions such that $\rho \in \cup_{\nu \vdash c'-t} I(\mu,\nu)$, then s_ρ appears in the expansion of the μ-contribution (86).

As $\deg_d q_\nu$ is monotone decreasing in $\nu \vdash c' - t$ (90), we are interested in the smallest $\nu \vdash c' - t$ such that $\rho \in I(\mu,\nu)$. Therefore, for every $\rho \vdash c$ and $\mu \vdash m' + t$ we define

$$f_\rho(\mu) = \begin{cases} \min\{\nu \vdash c' - t |\, \rho \in I(\mu,\nu)\} & \text{if } \ \rho \in \bigcup_{\nu \vdash c'-t} I(\mu,\nu) \\ \infty & \text{if } \ \rho \notin \bigcup_{\nu \vdash c'-t} I(\mu,\nu) \end{cases}.$$

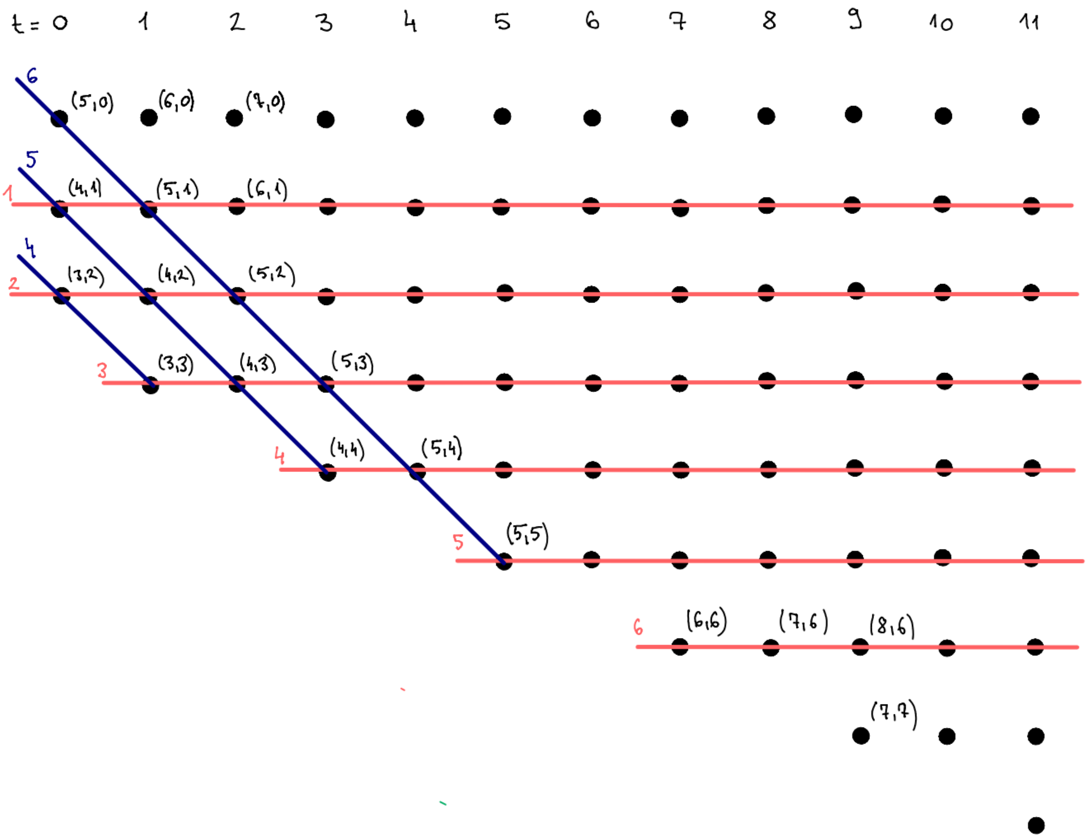


FIGURE 1. Segments with constant d-degree, nonzero coefficients p_μ for $m = 6$, where the leading coefficient positive on the blue diagonal segments and negative on the red horizontal lines, their degrees written on the lines.

If we extend the list of q_ν coefficients with $q_\infty = 0$, then—again by the positivity of the leading coefficients, (89)—the coefficient of s_ρ in (86) and in

$$\left(\frac{m}{d}\right)^t p_\mu s_\mu q_{f_\rho(\mu)} s_{f_\rho(\mu)},$$

have the same d-degree. Here, we define the d-degree of the constant 0 polynomial to be $-\infty$. In particular, the d-degree of the coefficient of s_ρ in (86) only depends on $f_\rho(\mu)$. Therefore

$$\deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^t p_\mu s_\mu B_t\right) = \deg_d(p_\mu) + \deg_d\left(q_{f_\rho(\mu)}\right) - t. \tag{91}$$

Combining this with (90), we get that if $\mu_i \vdash m - t_i$ and $\rho \vdash c$ are partitions such that $\pi_2(f_\rho(\mu_1)) \leq \pi_2(f_\rho(\mu_2)) + f$ for some $f \geq 0$, then $\deg_d(q_{f_\rho(\mu_1)}) + f \geq \deg_d(q_{f_\rho(\mu_2)})$, in other words,

$$\begin{aligned}\deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^{t_1} p_{\mu_1} s_{\mu_1} B_{t_1}\right) - \deg_d(p_{\mu_1}) + f \geq \\ \deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^{t_2} p_{\mu_2} s_{\mu_2} B_{t_2}\right) + (t_2 - t_1) - \deg_d(p_{\mu_2}).\end{aligned} \tag{92}$$

Here, we define $\pi_2(\infty) = \infty$ for the $\rho \notin \bigcup_{\nu \vdash c'-t} I(\mu, \nu)$ case.
(D) Goal of this part is to prove that for every $\rho \vdash c$ the d-degree of coefficients of s_ρ in the μ-contribution

(D/I) along the diagonal segments is strictly monotone decreasing (including $-\infty > -\infty$) and

(D/II) along the horizontal lines is monotone decreasing.

We do this by comparing $f_\rho(\mu)$'s for adjacent μ's along these line segments:

(D/I) Let $\mu_1 = (i, j) \vdash m' + t$ and $\mu_2 = (i, j+1) \vdash m' + (t+1)$ adjacent partitions of a diagonal segment. Then for every $(k, l) \vdash c' - (t+1)$

$$p\left((i,j),(k+1,l)\right) < p\left((i,j+1),(k,l)\right) \text{ and } P\left((i,j),(k+1,l)\right) = P\left((i,j+1),(k,l)\right),$$

hence

$$I\left((i,j),(k+1,l)\right) \supset I\left((i,j+1),(k,l)\right), \tag{93}$$

which in turn—as illustrated by Figure 2 with an example—implies that for every $\rho \vdash c$

$$\pi_2(f_\rho(\mu_1)) \leq \pi_2(f_\rho(\mu_2)).$$

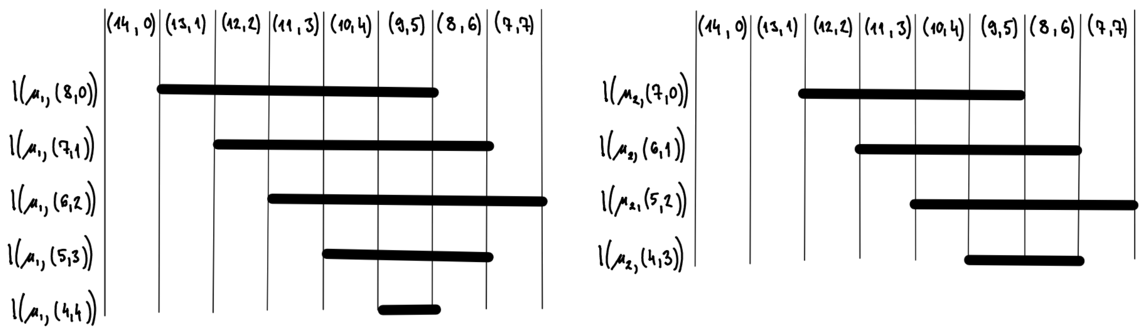


FIGURE 2. Comparison of intervals $I\left(\mu_1,(k+1,l)\right)$ and $I\left(\mu_2,(k,l)\right)$ for adjacent partitions $\mu_1 = (5,1) \vdash m'+1$ and $\mu_2 = (5,2) \vdash m'+2$ of the degree 6 diagonal segment in the $m = 6$, $c' = 9$ case.

Then (92) becomes

$$\deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^t p_{\mu_1} s_{\mu_1} B_t\right) \geq$$
$$\deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^{t+1} p_{\mu_2} s_{\mu_2} B_{t+1}\right) + 1,$$

proving the desired strictly decreasing property.

(D/II) We claim that for $\mu_1 = (i,j) \vdash m' + t$ and $\mu_2 = (i+1, j) \vdash m' + (t+1)$, adjacent partitions of a horizontal segment,

$$\pi_2\left(f_\rho(\mu_1)\right) \leq \pi_2\left(f_\rho(\mu_2)\right) + 1 \tag{94}$$

holds for every $\rho \vdash c$. Then (92) becomes

$$\deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^t p_{\mu_1} s_{\mu_1} B_t\right) + 1 \geq$$
$$\deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^{t+1} p_{\mu_2} s_{\mu_2} B_{t+1}\right) + 1,$$

proving the desired decreasing property.

To prove (94), we will compare the corresponding intervals $I(\mu_1,(k+1,l))$, $(k+1,l)\vdash h := c'-t$ and $I(\mu_2,(k,l))$, $(k,l)\vdash c'-(t+1)=h-1$. As their starting points are equal,

$$p(\mu_1,(k+1,l)) = (i+k+1,j+l) = p(\mu_2,(k,l)), \tag{95}$$

we can move on to their endpoints. More precisely, we are interested in their π_2 projections, which we will denote by

$$g_1(l) := \pi_2(P(\mu_1,(h-l,l))) = \min(i+l,j+h-l), \quad 0\le l\le \left\lfloor\frac{h}{2}\right\rfloor$$

and

$$g_2(l) := \pi_2(P(\mu_2,(h-1-l,l))) = \min(i+1+l,j+h-1-l), \quad 0\le l\le \left\lfloor\frac{h-1}{2}\right\rfloor.$$

Then $g_1(l+1)=g_2(l)$ for every $0\le l<\lfloor h/2\rfloor$. Together with (95), this tells us that for every $0\le l<\lfloor h/2\rfloor$

$$\bigcup_{\nu_1\le(k,l+1)} I(\mu_1,\nu_1) \supset \bigcup_{\nu_2\le(k,l)} I(\mu_2,\nu_2),$$

in other words,

$$\pi_2(f_\rho(\mu_1)) \le \pi_2(f_\rho(\mu_2))+1 \text{ for every } \rho\in \bigcup_{l<\lfloor h/2\rfloor} I(\mu_2,(k,l)). \tag{96}$$

What is left to prove (94) is that the ρ's occuring in (96) are all the ρ's with finite $f_\rho(\mu_2)$, that is

$$\bigcup_{(k,l)\vdash h-1} I(\mu_2,(k,l)) = \bigcup_{l<\lfloor h/2\rfloor} I(\mu_2,(k,l)). \tag{97}$$

We define

$$x_1 := x(\mu_1) := \frac{h+j-i}{2} \text{ and } x_2 := x(\mu_2) := \frac{h-1+j-(i-1)}{2} = x_1-1,$$

elements where the $i+l$, $j+h-l$ arguments of $g_1(l)$ and the $i+1+l$, $j+h-1-l$ arguments of g_2 intersect respectively. At $\lfloor x_i\rfloor$ (and $\lceil x_i\rceil$) g_i reaches its maximum $\lfloor c/2\rfloor$, therefore

$$\lfloor x_2\rfloor < \lfloor x_1\rfloor \le \left\lfloor\frac{h}{2}\right\rfloor \tag{98}$$

shows that even if $\lfloor h-1/2\rfloor = \lfloor h/2\rfloor$, and hence there is an extra interval on the left-hand side of (97), for its endpoint

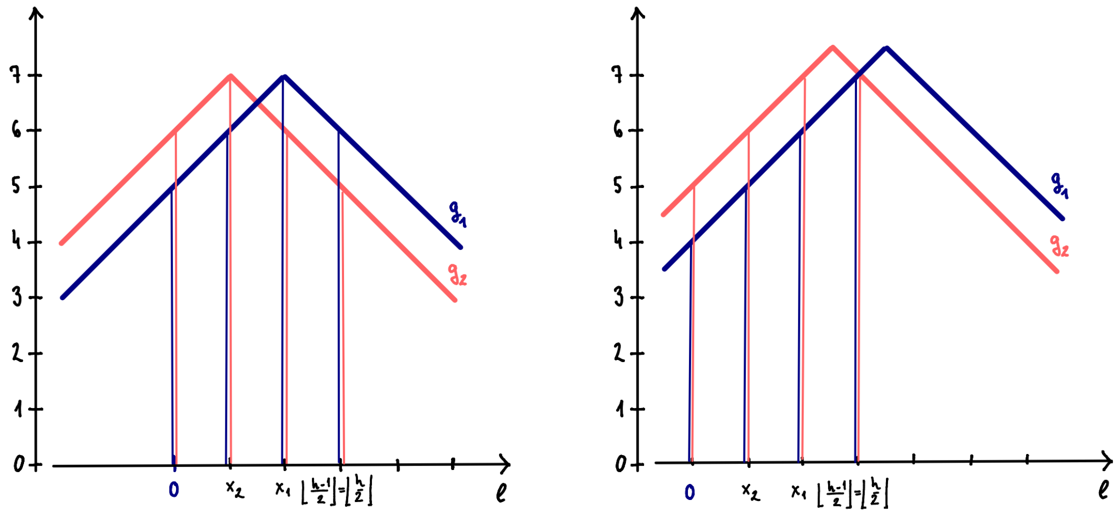
$$g_2\left(\left\lfloor\frac{h-1}{2}\right\rfloor\right) \le g_2\left(\left\lfloor\frac{h-1}{2}\right\rfloor - 1\right),$$

therefore (97) holds.
(E) We will prove (87) by showing—using induction on $t\ge 1$—that for every $\rho\vdash c$ and $\mu\vdash m'+t$

$$\deg_d(\text{coefficients of } s_\rho \text{ in } A_0B_0) > \deg_d\left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^t p_\mu s_\mu B_t\right). \tag{99}$$

First, it is true for $t=1$: for any $\mu\vdash m'+1$ with $p_\mu\ne 0$ we have $\pi_2(\mu)\ge 1$, therefore there exists a diagonal segment along which μ is adjacent to a $\mu_1\vdash m'$ and whose strict monotonocity property implies that

$$\deg_d(\text{coefficients of } s_\rho \text{ in } p_{\mu_1}s_{\mu_1}B_0) > \deg_d\left(\text{coefficients of } s_\rho \text{ in } \frac{m}{d}p_\mu s_\mu B_1\right).$$



$\mu_1 = (5,2) \vdash m'+2$ and $\mu_2 = (6,2) \vdash m'+3$ of the degree 2 horizontal segment in the $m = 6$, $c' = 9$ case

$\mu_1 = (4,4) \vdash m'+2$ and $\mu_2 = (5,4) \vdash m'+3$ of the degree 4 horizontal segment in the $m = 7$, $c' = 9$ case

FIGURE 3. Comparison of endpoints of intervals $P(\mu_1, (k+1, l))$ and $P(\mu_2, (k, l))$ for adjacent partitions μ_1 and μ_2 of horizontal segments.

For any $\mu_1 \vdash m'$, p_{μ_1} has positive leading coefficient. Together with the positivity of the leading coefficients in q_ν, $\nu \vdash c'$, this means that these high d-degree terms cannot cancel each other out, hence the strict inequality remains true for (99).

Now suppose that (99) is true for $t \geq 1$ and let $\mu \vdash m' + t + 1$ such that $p_\mu \neq 0$. Then, except possibly for $\mu = (m, m)$, there is a horizontal or a diagonal segment along which μ is adjacent to some $\mu_1 \vdash m' + t$. The monotonicity property of either of them gives

(100)
$$\deg_d \left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^t p_{\mu_1} s_{\mu_1} B_t \right) \geq \deg_d \left(\text{coefficients of } s_\rho \text{ in } \left(\frac{m}{d}\right)^{t+1} p_\mu s_\mu B_{t+1} \right).$$

For the special case $\mu = (m, m)$, the d-degree of the coefficient of s_ρ in the μ-contribution can be compared to that of $\mu_1 = (m, m-1) \vdash m' + t$ $(t \geq 2)$ using an analysis analogous to (D/I). Only that—as this time $\deg_d(p_{\mu_1}) = \deg_d(p_\mu) - 1$—(92) yields the same non-strict inequality (100), concluding the proof.

(F) We will prove $ii)$ case of Theorem B.2.4 by showing that

(101)
$$\bigcup_{\nu \vdash c'} I\left((m', 0), \nu\right) = \left\{\rho | \, \rho \vdash c\right\},$$

and that for any $\rho \vdash c$ the d-degree of the coefficient of s_ρ in the μ-contribution is strictly monotone decreasing along the $t = 0$ *vertical* segment.

The first claim follows from the hypothesis: $m - 2 \leq c'$ is equivalent to

$$x\left((m', 0)\right) := \frac{c' - m'}{2} > -1.$$

This, together with $x\left((m', 0)\right) \leq c'/2$, ensures that the set of endpoints $\left\{P\left((m', 0), \nu\right) | \, \nu \vdash c'\right\}$ contains the maximum $(\lceil c/2 \rceil, \lfloor c/2 \rfloor)$. As $p\left((m', 0), (c', 0)\right) = (c, 0)$, (101) holds.

Proof of the second claim will be similar to the analysis we have done in part (D/II). More precisely, we will prove that for any $\rho \vdash c$ and $\mu_1 = (i,j)$, $\mu_2 = (i-1, j+1) \vdash c'$ adjacent partitions of the vertical segment

$$\pi_2\left(f_\rho(\mu_1)\right) \leq \pi_2\left(f_\rho(\mu_2)\right). \tag{102}$$

Then (92) becomes

$$\deg_d\left(\text{coefficients of } s_\rho \text{ in } p_{\mu_1} s_{\mu_1} B_0\right) - (m-j) \geq \\ \deg_d\left(\text{coefficients of } s_\rho \text{ in } p_{\mu_2} s_{\mu_2} B_0\right) - (m-(j+1)),$$

showing the strictly monotone decreasing property.

In the comparison of intervals $I\left(\mu_1, (k,l)\right)$ and $I\left(\mu_2, (k,l)\right)$, for their starting points we have

$$p\left(\mu_1, (k,l)\right) < p\left(\mu_2, (k,l)\right).$$

To investigate their endpoints, we again define

$$g_1(l) := \pi_2\left(P\left(\mu_1, (k,l)\right)\right) \text{ and } g_2(l) := \pi_2\left(P\left(\mu_2, (k,l)\right)\right), \quad 0 \leq l \leq \left\lfloor \frac{c'}{2} \right\rfloor,$$

for which $g_1(l) = g_2(l+1)$, $0 \leq l \leq \lfloor c'/2 \rfloor$. These imply that for every $(k,l) \vdash c'$

$$\bigcup_{\nu_1 \leq (k,l)} I\left(\mu_1, \nu_1\right) \supsetneq \bigcup_{\nu_2 \leq (k,l)} I\left(\mu_2, \nu_2\right),$$

therefore (102) holds.

(G) To prove Theorem B.2.5, let us choose $m = \min \lambda$. $m-1 = m' \leq c'$ so we can use case $ii)$ of Theorem B.2.4 which, combined with (91), tells us that for every $\rho \vdash c$

$$\deg_d(r_\rho) = \deg_d(p_\mu) + \deg_d(q_{f_\rho(\mu)}) = m + \deg_d(q_{f_\rho(\mu)}), \tag{103}$$

where, throughout this part, we fix $\mu = (m', 0)$.

If we define

$$\vartheta(\lambda) := \max\left\{ f \middle| \deg_d(r_{(c-v,v)}) = |\lambda|,\ 0 \leq v \leq f \right\},$$

the threshold for which $\deg_d(r_\rho)$ is constant $|\lambda|$. Then it remains to show that

$$\vartheta(\lambda) = \min\left(\left\lfloor \frac{c}{2} \right\rfloor, c - \lambda_1 + 1\right)$$

and that after $(c - \vartheta(\lambda), \vartheta(\lambda))$ every time we increase ρ by one, $deg_d(r_\rho)$ decreases by one.

To describe the term $\deg_d(q_{f_\rho(\mu)})$, we need to look into the function $\rho = (c-h, h) \mapsto f_\rho(\mu)$. Specifically, it's π_2-projection, which—in this $m-2 \leq c'$ case—is

$$\pi_2(f_{(c-h,h)}(\mu)) = \max(0, h - m'). \tag{104}$$

Assuming the induction hypothesis for λ', (103) implies that $\deg_d(r_\rho) = m + |\lambda'| = |\lambda|$ for every ρ such that $f_\rho(\mu) \leq \vartheta(\lambda')$ or, by (104) equivalently, for every ρ in

$$\bigcup_{l \leq \vartheta(\lambda')} I(\mu, (k,l)),$$

whose endpoint has π_2-projection

$$\vartheta(\lambda) = \min\left(\left\lfloor \frac{c}{2} \right\rfloor, m' + \vartheta(\lambda')\right). \tag{105}$$

Again by the induction hypotheses for λ', the behaviour of $\deg_d(r_\rho)$ for $\rho > (c - \vartheta(\lambda), \vartheta(\lambda))$ will be the same as the behaviour of $\deg_d(q_\nu)$ for $\nu > (c' - \vartheta(\lambda'), \vartheta(\lambda'))$.

$\lambda_1 = \lambda'_1$, thus to complete the induction step all we need to check is that

$$\min\left(\left\lfloor \frac{c}{2} \right\rfloor, m' + \vartheta(\lambda')\right) = \min\left(\left\lfloor \frac{c}{2} \right\rfloor, m' + \min\left(\left\lfloor \frac{c'}{2} \right\rfloor, c' - \lambda_1 + 1\right)\right) = \min\left(\left\lfloor \frac{c}{2} \right\rfloor, c - \lambda_1 + 1\right).$$

This follows easily from the observation that adding m' to the inequality

$$\left\lfloor \frac{c'}{2} \right\rfloor \leq c' - \lambda_1 + 1,$$

we get

$$\left\lfloor \frac{c}{2} \right\rfloor = \left\lfloor \frac{c' + m'}{2} \right\rfloor \leq \left\lfloor \frac{c'}{2} \right\rfloor + m' \leq c' - \lambda_1 + 1 + m' = c' - \lambda_1 + 1.$$

□

Bibliography

[AH62] M.F. Atiyah and F. Hirzebruch. The Riemann-Roch theorem for analytic embeddings. *Topology*, 1(2):151–166, 1962.

[Alu06] Paolo Aluffi. Classes de chern des variétés singulières, revisitées. *Comptes Rendus Mathematique*, 342(6):405–410, 2006.

[Alu13] P. Aluffi. Euler characteristics of general linear sections and polynomial Chern classes. *Rend. Circ. Mat. Palermo (2)*, 62(1):3–26, 2013.

[AMSS17] Paolo Aluffi, Leonardo Mihalcea, Joerg Schuermann, and Changjian Su. Shadows of characteristic cycles, verma modules, and positivity of chern-schwartz-macpherson classes of schubert cells. 09 2017.

[BM95] Edward Bierstone and Pierre Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Inventiones Mathematicae*, 128, 09 1995.

[Col86] S. J. Colley. Lines having specified contact with projective varieties. In *Proceedings of the 1984 Vancouver Conference in Algebraic Geometry*, Annual seminar of the Canadian Mathematical Society, pages 47–70. American Mathematical Society, 1986.

[CZ20] C. Ciliberto and M. Zaidenberg. On fano schemes of complete intersections. In *Polynomial Rings and Affine Algebraic Geometry*, pages 1–39. Springer International Publishing, 2020.

[Dol12] Igor V. Dolgachev. *Classical Algebraic Geometry: A Modern View*. Cambridge University Press, 2012.

[EH16] D. Eisenbud and J. Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[Feh21] L. M. Fehér. Motivic Chern classes of cones. In Javier Fernández de Bobadilla, Tamás László, and András Stipsicz, editors, *Singularities and Their Interaction with Geometry and Low Dimensional Topology*, pages 181–205, Cham, 2021. Springer International Publishing.

[FH91] W. Fulton and J. Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics. Springer New York, 1991.

[FNR05] L. M. Fehér, A. Némethi, and R. Rimányi. Degeneracy of 2-forms and 3-forms. *Canad. Math. Bull.*, 48(4):547–560, 2005.

[FNR06] L. M. Fehér, A. Némethi, and R. Rimányi. Coincident root loci of binary forms. *Michigan Math. J.*, 54(2):375–392, 2006.

[FNR12] L. M. Fehér, A. Némethi, and R. Rimányi. Equivariant classes of matrix matroid varieties. *Commentarii Mathematici Helvetici*, 87(4):861–889, 2012.

[FR18] L. M. Fehér and R. Rimányi. Chern-Schwartz-MacPherson classes of degeneracy loci. *Geom. Topol.*, 22(6):3575–3622, 2018.

[FRW18] L. M. Fehér, R. Rimányi, and A. Weber. Characteristic classes of orbit stratifications, the axiomatic approach. In *Schubert Calculus and its applications in combinatorics and representation theory*. Springer, 2018. to appear, `https://arxiv.org/pdf/1811.11467.pdf`.

[FRW21] L. M. Fehér, R. Rimányi, and A Weber. Motivic chern classes and K-theoretic stable envelopes. *Proceedings of the London Mathematical Society*, 122(1):153–189, 2021.

[Ful95] W. Fulton. *Algebraic Topology: A First Course*. Graduate texts in mathematics. World Publishing Corporation, 1995.

[GKPL89] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, and Stanley Liu. Concrete mathematics: a foundation for computer science. *Computers in Physics*, 3(5):106–107, 1989.

[GM88] Mark Goresky and Robert MacPherson. *Stratified Morse theory*, pages 3–22. Springer Berlin Heidelberg, Berlin, Heidelberg, 1988.

[GM08] D. B. Grünberg and P. Moree. Sequences of enumerative geometry: congruences and asymptotics, with an appendix by Don Zagier. *Experimental Mathematics*, 17(4):409 – 426, 2008.

[Hen74] Peter Henrici. *Applied and computational complex analysis, Volume 1*, volume 7 of *Pure and Applied Mathematics*. Wiley-Interscience [John Wiley & Sons], 1974.

[HT84] J. Harris and L. W. Tu. On symmetric and skew-symmetric determinantal varieties. *Topology*, 23:71–84, 1984.

[Kaz06] M. Kazarian. Morin maps and their characteristic classes. `https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.486.8275&rep=rep1&type=pdf`, 2006.

[Kő3] B. Kőműves. Thom polynomials via restriction equations. Master's thesis, Eötvös Lóránt University Budapest, 2003.

[Kir84] F. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry*. Number 31 in Mathematical Notes. Princeton UP, 1984.

[Kle77] S. L. Kleiman. The enumerative theory of singularities. *Uspekhi Matematicheskikh Nauk*, 1977.

[Kle81] S. L. Kleiman. Multiple-point formulas I: Iteration. *Acta Mathematica*, 147:13 – 49, 1981.

[Kle82] S. L. Kleiman. *Multiple Point Formulas for Maps*, pages 237–252. Birkhäuser Boston, Boston, MA, 1982.

[LB82] Patrick Le Barz. Formules multisécantes pour les courbes gauches quelconques. In *Enumerative geometry and classical algebraic geometry (Nice, 1981)*, volume 24 of *Progr. Math.*, pages 165–197. Birkhäuser, Boston, Mass., 1982.

[Mac74] R. D. MacPherson. Chern classes for singular algebraic varieties. *Annals of Mathematics*, 100(2):423–432, 1974.

[Man99] Laurent Manivel. Sur les hypersurfaces contenant des espaces linéaires. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 328(4):307–312, 1999.

[Nic12] Liviu Nicolaescu. *An invitation to Morse theory*. Springer New York, 2012.

[Ohm04] Toru Ohmoto. Equivariant Chern classes of singular algebraic varieties with group actions. *Mathematical Proceedings of the Cambridge Philosophical Society*, 140, 07 2004.

[Ohm16] Toru Ohmoto. Singularities of maps and characteristic classes. volume 68 of *Adv. Stud. Pure Math.*, pages 191–265, 2016.

[Sag99] Bruce Sagan. Why the characteristic polynomial factors. *Bulletin of the American Mathematical Society*, 36(2):113–133, 1999.

[Sch17] Jörg Schürmann. Chern classes and transversality for singular spaces. In José Luis Cisneros-Molina, Dũng Tráng Lê, Mutsuo Oka, and Jawad Snoussi, editors, *Singularities in Geometry, Topology, Foliations and Dynamics*, pages 207–231, Cham, 2017. Springer International Publishing.

[ST22] Hunter Spink and Dennis Tseng. PGL_2-equivariant strata of point configurations in $\mathbb{P}^1$. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 23(2):569–621, 2022.

[Tot99] Burt Totaro. The Chow ring of a classifying space. In *Algebraic K-theory (Seattle, WA, 1997)*, volume 67 of *Proc. Sympos. Pure Math.*, pages 249–281. Amer. Math. Soc., Providence, RI, 1999.

[Web17] A. Weber. Hirzebruch class and Bialynicki-Birula decomposition. *Transformation Groups*, 22(2):537–557, 2017.

[Wlo09] J. Wlodarczyk. Simple constructive weak factorization. In *Algebraic Geometry, Part 2: Seattle 2005*, Proceedings of Symposia in Pure Mathematics, pages 957–1004. American Mathematical Society, 2009.